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Entanglement of Several Classical and Dynamic Estimates with Unified Approach on Time Scales

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Abstract In this research article, we present several generalizations of Qi's inequality on time scales. We establish dynamic versions of Callebaut's inequality and Cauchy-Schwarz's inequality on time scales. To establish our results, we apply the diamond-alpha integral and the time scale Δ or ∇-Riemann-Liouville type fractional integrals. Our findings unify and extend discrete, continuous and quantum analogues.

Keywords Time scales, fractional Riemann-Liouville integrals, Qi's, Callebaut's and Cauchy-Schwarz's inequalities

MSC(2010) 26D15, 26E70, 34N05, 26A33.

1. Introduction

The calculus of time scales was initiated by Stefan Hilger [13]. A time scale is an arbitrary nonempty closed subset of the real numbers. This hybrid theory is also widely applied on dynamic inequalities, see [2, 15–20, 23, 24]. The basic ideas concerning the calculus of time scales are given in [7, 8].

The following Qi's inequality is proved in [12].

Let $r \geq 1$ and Φ be a nonnegative continuous function on $[\xi, \omega]$ such that $0 < \Phi(\lambda) \leq r(\omega-\xi)^{-1}$. Then we have the following inequality

$$\left(\int_{\xi}^{\omega} \Phi(\lambda)d\lambda\right)^{r} \leq \frac{r^r}{e^r}\exp\left(\int_{\xi}^{\omega} \Phi(\lambda)d\lambda\right) \leq \frac{r^{2r}}{(\omega-\xi)^{1+r}}\int_{\xi}^{\omega} \Phi^{-r}(\lambda)d\lambda. \tag{1.1}$$

The following Callebaut's inequality is given in [11].

Let $x_k > 0$, $y_k > 0$ and $w_k \geq 0$ for any $k \in \{1, 2, \ldots, n\}$ with $\sum\limits_{k=1}^{n} w_k = 1$. If there exist the constants m, $M > 0$ such that $0 < m \leq \frac{x_k}{y_k} \leq M < \infty$ for any $k \in \{1, 2, \ldots, n\}$, then

$$\begin{aligned}&\sum_{k=1}^{n} w_k x_k^{2(1-v)} y_k^{2v} \sum_{k=1}^{n} w_k x_k^{2v} y_k^{2(1-v)}\\ \leq &\sum_{k=1}^{n} w_k x_k^2 \sum_{k=1}^{n} w_k y_k^2\end{aligned}$$

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$$\leq K^{\delta}\left(\left(\frac{M}{m}\right)^{2}\right)\sum_{k=1}^{n} w_k x_k^{2(1-v)} y_k^{2v} \sum_{k=1}^{n} w_k x_k^{2v} y_k^{2(1-v)}, \tag{1.2}$$

for any $v \in [0,1]$ and $\delta = \max\{1-v, v\}$.

The following Qi's inequality is proved in [12].

Let $0 < p < q \leq 1$, $r > 0$ and Υ, Φ be measurable nonnegative functions on $[\xi, \omega]$ such that $\int_{\xi}^{\omega} \Upsilon(\gamma)\Phi^{q}(\gamma)d\gamma < \infty$. Then we have the following inequality

$$\left[\left(\int_{\xi}^{\omega} \Upsilon(\gamma)\Phi^{p}(\gamma)d\gamma\right)^{\frac{1}{p}}\right]^{r} \leq \frac{r^r}{e^r}\left(\int_{\xi}^{\omega} \Upsilon(\gamma)d\gamma\right)^{\frac{r}{p}-\frac{r}{q}} \exp\left(\int_{\xi}^{\omega} \Upsilon(\gamma)\Phi^{q}(\gamma)d\gamma\right)^{\frac{1}{q}}. \tag{1.3}$$

We shall unify and extend (1.1) and (1.2) in the calculus of time scales by applying the diamond-alpha integral. We shall also unify and extend (1.3) in the fractional calculus of time scales.

2. Preliminaries

Now we present a short introduction to the diamond-α derivative as given in [1,21].

Let $\mathbb{T}$ be a time scale and $\Phi(\lambda)$ be differentiable on $\mathbb{T}$ in the Δ and ∇ senses. For $\lambda \in \mathbb{T}$, the diamond-α dynamic derivative $\Phi^{\diamond_\alpha}(\lambda)$ is defined by

$$\Phi^{\diamond_\alpha}(\lambda) = \alpha\Phi^{\Delta}(\lambda) + (1-\alpha)\Phi^{\nabla}(\lambda), \quad 0 \leq \alpha \leq 1.$$

Thus Φ is diamond-α differentiable if and only if Φ is Δ and ∇ differentiable.

The following definition is given in [21].

Let $\xi, \kappa \in \mathbb{T}$ and $\Phi : \mathbb{T} \to \mathbb{R}$. Then the diamond-$\alpha$ integral from ξ to κ of Φ is defined by

$$\int_{\xi}^{\kappa} \Phi(\lambda) \diamond_\alpha \lambda = \alpha \int_{\xi}^{\kappa} \Phi(\lambda)\Delta\lambda + (1-\alpha)\int_{\xi}^{\kappa} \Phi(\lambda)\nabla\lambda, \quad 0 \leq \alpha \leq 1, \tag{2.1}$$

provided that there exist delta and nabla integrals of Φ on $\mathbb{T}$.

The following inequality is given in [6,22].

Let $r > 0$ and $z > 0$. Then the following inequality is valid:

$$z^r \leq \frac{r^r}{e^r}e^z. \tag{2.2}$$

The following well-known Young's inequality holds:

For $\Omega, \chi > 0$ and $v \in [0,1]$, we have

$$\Omega^{1-v}\chi^{v} \leq (1-v)\Omega + v\chi. \tag{2.3}$$

Kantorovich's ratio is defined by

$$K(h) := \frac{(h+1)^2}{4h},$$

where $h > 0$.

The following inequality is given in [14].