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On Fractional Hybrid Integral Inequalities via Extended s-Convexity

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Abstract In this study, we introduce a novel hybrid identity that successfully combines Newton-Cotes and Gauss quadratures, enabling us to recover both Simpson's second formula and the left and right Radau 2 point rules, among others. Based on this versatile foundation, we establish some new biparametric fractional integral inequalities for functions whose first derivatives are extended s-convex in the second sense. To support our findings, we present illustrative examples featuring graphical representations and conclude with several practical applications to demonstrate the effectiveness of our results.

Keywords Newton-Cotes inequalities, extended s-convex functions, Gauss-Radau formula, P-functions, hypergeometric function

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1. Introduction

Integral inequalities find extensive applications in diverse fields, including mathematical disciplines such as approximation theory, numerical analysis, and differential equations, as well as other scientific domains such as physics, economics, biology, and engineering.

Furthermore, the theory of convex functions serves as a potent analytical tool, particularly within optimization and inequality theories, which share a close relationship. Various integral inequalities, including Hadamard's, Ostrowski's, Simpson's, and Newton's inequalities, utilize different classes of functions to encompass a wide range of function spaces. Among the widely studied and applied classes, the category of convex functions, along with its variants and generalizations, stands out. It's worth noting that a function ξ is deemed to be extended s-convex in the

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second sense if

$$\xi(\mu f+(1-\mu)g) \ \leq \mu^{s}\xi(f)+(1-\mu)^{s}\xi(g)$$

holds, for all $f,g\in I,\mu\in[0,1]$ and $s\in[-1,1]$ (see [33]). This class recaptures several other classes of convex functions. For $s=1$ we obtain the classical convex ([22]) functions; for $s=0$ we obtain the P-function ([6]); for $s=-1$ we obtain Godunova-Levin function ([12]); for $s\in(0,1]$ we obtain the s-convex functions ([4]); for $s\in[-1,0)$ s-Godunova-Levin function ([7]).

Regarding some papers dealing with integral inequalities via convexity, we refer the reader to [1–3, 5, 8, 18–20, 25–27].

The following inequality is known in the literature as Simpson's second formula inequality (see [11]).

$$\left|\tfrac{1}{8}\left(\xi\left(f\right)+3\xi\left(\tfrac{2f+g}{3}\right)+3\xi\left(\tfrac{f+2g}{3}\right)+\xi\left(g\right)\right)-\tfrac{1}{g-f}\int\limits_{f}^{g}\xi\left(u\right)du\right|\leq\tfrac{(g-f)^{4}}{6480}\left\|\xi^{(4)}\right\|_{\infty},$$

where ξ is four times continuously differentiable mapping on the interval $[f,g]$ and $\left\|\xi^{(4)}\right\|_{\infty}=\sup\limits_{u\in[f,g]}\left|\xi^{(4)}\left(u\right)\right|$.

In [21], Noor et al., established the following results regarding the Simpson's second formula

$$\begin{aligned}&\left|\tfrac{1}{8}\left(\xi\left(f\right)+3\xi\left(\tfrac{2f+g}{3}\right)+3\xi\left(\tfrac{f+2g}{3}\right)+\xi\left(g\right)\right)-\tfrac{1}{g-f}\int\limits_{f}^{g}\xi\left(u\right)du\right|\\&\leq\left(g-f\right)\left(\tfrac{17}{756}\left(\tfrac{973\left|\xi'(f)\right|^{q}+251\left|\xi'(g)\right|^{q}}{1224}\right)^{\frac{1}{q}}\right.\\&\quad\left.+\tfrac{1}{36}\left(\tfrac{\left|\xi'(f)\right|^{q}+\left|\xi'(g)\right|^{q}}{2}\right)^{\frac{1}{q}}+\tfrac{17}{756}\left(\tfrac{251\left|\xi'(f)\right|^{q}+973\left|\xi'(g)\right|^{q}}{1224}\right)^{\frac{1}{q}}\right),\end{aligned}$$

where $|\xi'|^{q}$ is convex with $q\geq1$. Furthermore, they demonstrated the following estimations for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$,

$$\begin{aligned}&\left|\tfrac{1}{8}\left(\xi\left(f\right)+3\xi\left(\tfrac{2f+g}{3}\right)+3\xi\left(\tfrac{f+2g}{3}\right)+\xi\left(g\right)\right)-\tfrac{1}{g-f}\int\limits_{f}^{g}\xi\left(u\right)du\right|\\&\leq\left(g-f\right)\left(\left(\tfrac{3^{p+1}+5^{p+1}}{24^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\tfrac{\left|\xi'(f)\right|^{q}+\left|\xi'\left(\frac{2f+g}{3}\right)\right|^{q}}{6}\right)^{\frac{1}{q}}\right.\\&\quad+\left(\tfrac{2}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\tfrac{\left|\xi'\left(\frac{2f+g}{3}\right)\right|^{q}+\left|\xi'\left(\frac{f+2g}{3}\right)\right|^{q}}{6}\right)^{\frac{1}{q}}\\&\quad\left.+\left(\tfrac{3^{p+1}+5^{p+1}}{24^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\tfrac{\left|\xi'\left(\frac{f+2g}{3}\right)\right|^{q}+\left|\xi'(g)\right|^{q}}{6}\right)^{\frac{1}{q}}\right)\end{aligned}$$