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Study of Certain Navier Problems in Sobolev Space with Weights

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Abstract In this paper, we study the following Navier problem

$$-\text{div}\Big[v_1\mathcal{K}(z,\nabla w)+v_2\mathcal{L}(z,w,\nabla w)\Big]+\Delta\left[\phi_1|\Delta w|^{t-2}\Delta w+\phi_2|\Delta w|^{q-2}\Delta w\right]+ v_3 b(z,w)+v_4|w|^{p-2}w=h(z),$$

Here, $h\in L^{p'}(\mathcal{Q},v_1^{1-p'})$, $\mathcal{K}$, $\mathcal{L}$ and b are Carathéodory functions and ϕ_1,ϕ_2,v_1, v_2, v_3 and v_4 are A_p-weights functions. By using the theory of monotone operators (Browder–Minty Theorem), we demonstrate the existence and uniqueness of weak solution to the above problem.

Keywords Navier problem, degenerate quasilinear elliptic equations, weighted Sobolev spaces, weak solution

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1. Introduction

Nonlinear elliptic equations with perturbation in the sense of singularity and decay are useful problems arising from these differential equations in various applications, including non-Newtonian fluid mechanics, reaction-diffusion difficulties, flows in porous media and hydrology, (we refer to [3, 6, 19] where it is possible to find some examples of applications of degenerate elliptic equations).

In the so-called degenerate partial differential equations, which have different types of singularities in the coefficients, it is natural to find solutions in weighted Sobolev spaces [8–10, 13]. The weightless Sobolev spaces $W^{k,t}(\mathcal{Q})$, in general, appear as solution spaces for parabolic and elliptic partial differential equations. In particular when $t=q=2$ and $\phi_1=\phi_2\equiv 1, v_1=v_3=v_4=0$ and $v_2=1$ we have the equation

$$\Delta^2 w-\sum_{j=1}^{n}D_j\mathcal{L}_j(z,w,\nabla w)=h,$$

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where $\Delta^2 w$ is the biharmonic operator. Many real phenomena, such as radar imaging or incompressible flows, are the subject of mathematical models in which biharmonic equations are found.

There are a lot of examples of weight (see [13]). A well-established class of weights, introduced by B. Muckenhoupt [16] , is the class of A_p-weights (or Muckenhoupt class) . These weights have found many useful applications in harmonic analysis [17].

Our goal in this paper is to show the uniqueness and existence of a weak solution in the weighted Sobolev space. Consider $W_0^{1,t}(\mathcal{Q}, v)$ (see Definition 2.2) for the Navier problem associated with the degenerate elliptic equation

$$\begin{cases} \Delta\Big[\phi_1|\Delta w|^{t-2}\Delta w + \phi_2|\Delta w|^{q-2}\Delta w\Big] - \operatorname{div}\Big[v_1\mathcal{K}(z,\nabla w) + v_2\mathcal{L}(z,w,\nabla w)\Big] & \\ \qquad\qquad + v_3 b(z,w) + v_4|w|^{p-2}w = h & \text{in } \mathcal{Q}, \\ w(z) = \Delta w(z) = 0 & \text{on } \partial\mathcal{Q}, \end{cases} \tag{1.1}$$

where, $\mathcal{Q}$ is a bounded open set in $\mathbb{R}^d$,ϕ_1,ϕ_2, v_1, v_2, v_3 and v_4 are a weight functions, and the functions $\mathcal{L} : \mathcal{Q} \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $\mathcal{K} : \mathcal{Q} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ and $b : \mathcal{Q} \times \mathbb{R} \longrightarrow \mathbb{R}$ are Caratéodory functions that satisfy the growth assumptions, monotonicity and ellipticity conditions. Problems like (1.1) have been studied by many authors in the unweighted and weighted case (see [2, 4, 22]).

The structure of this work is as follows: in Section 2, we give some basic results and some technical lemmas. In Section 3, we specify all the assumptions on $\mathcal{K}$, $\mathcal{L}$, b and we present the notion of weak solution for Problem (1.1). The main results will be proved in Section 4.

2. Preliminaries

To understand our findings, we must first review certain definitions and fundamental aspects which are used during this paper. Full presentations can be found in the monographs by A. Torchinsky [17] and J. Garcia-Cuerva et al. [11].

We will call a locally integrable function v by a weight on $\mathbb{R}^d$ such that $v(z) > 0$ for a.e. $z \in \mathbb{R}^d$. Each weight v gives rise to a measure on the measurable subsets of $\mathbb{R}^d$ by integration. This measure will be denoted v. Thus,

$$v(E) = \int_E v(z)dz \qquad \text{for measurable subset} \quad E \subset \mathbb{R}^d.$$

For $0 < t < \infty$, we denote by $L^t(\mathcal{Q}, v)$ the space of measurable functions v on $\mathcal{Q}$ such that

$$||h||_{L^t(\mathcal{Q},v)} = \left(\int_{\mathcal{Q}} |h|^t v(z)dz\right)^{\frac{1}{t}} < \infty,$$

where h is a weight, and $\mathcal{Q}$, is open in $\mathbb{R}^d$. It is a widely known fact that the space $L^t(\mathcal{Q}, v)$, endowed with this norm is a Banach space. We also have that the dual space of $L^t(\mathcal{Q}, v)$ is the space $L^{t'}(\mathcal{Q}, v^{1-t'})$.

Let us now specify the conditions on the weight v that ensure that the functions in $L^t(\mathcal{Q}, v)$ are locally integrable on $\mathcal{Q}$.