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# Double Phase Phenomena in Navier Boundary Problems with Degenerated $(p(.), q(.))$-Operators

Abdessamad El Katit[1,†], Abdelrachid El Amrouss[2] and Fouad Kissi[2]

**Abstract** In this paper, we are interested in some results of the existence of multiple solutions for Navier boundary value problem involving degenerated $(p(.), q(.))$-Biharmonic and $(p(.), q(.))$-Laplacian operators. Our approach is based on variational method and critical point theory.

**Keywords** Weighted variable exponent Lebesgue-Sobolev spaces, degenerated $(p(.), q(.))$-Biharmonic operator, Navier boundary problem

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## 1. Introduction

In this paper, we consider the following nonlinear boundary value problem

$$\begin{cases} \Delta\left(\omega(x)(|\Delta u|^{p(x)-2}\Delta u + |\Delta u|^{q(x)-2}\Delta u)\right) - \mathcal{L}(u) = \lambda f(x,u) & in\ \Omega, \\ u = \Delta u = 0 & on\ \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega$ is a smooth and bounded domain of $\mathbb{R}^N$, the variable exponents $p, q : \mathbb{R}^N \to (1, \infty)$ are continuous functions with $1 < p(x) < q(x)$, the weight $\omega$ is non-negative locally integrable function on $\Omega$, $\lambda$ is a real parameter and $f(x, s)$ is continuous on $\bar{\Omega} \times \mathbb{R}$ and

$$\mathcal{L}(u) := \operatorname{div}\left(\omega(x)(|\nabla u|^{p(x)-2}\nabla u + |\nabla u|^{q(x)-2}\nabla u)\right).$$

In the last few years, elliptic equations with variable exponents have been widely performed and have got a considerable amount of attention . They have contributed to the progress in elasticity theory and electrorheological fluids dynamics (see [6,34]).

Problems involving $p$-Laplacian and $(p, q)$-Laplacian operators in bounded or unbounded domains have been studied by many authors, for instance [1, 3, 14, 20–22, 29, 30, 32]. However, very few works have concerned $p(x)$-Laplacian and $p(x)$-Biharmonic type problems with singular weights i.e, with not bounded weights or not separated from zero in $\Omega$ (see [11, 23, 24, 28]). In that case, the above operators are called degenerated operators. We note that similar degeneracy can be physically connected with the equilibrium of continuous anisotropic media [7].

The study of double phase phenomena in partial differential equations is crucial because it extends the understanding of complex systems where material properties

[†]the corresponding author.
Email address: abdessamad.elkatit@ump.ac.ma(A. El katit), elamrouss@hotmail.com(A. R. El Amrouss), kissifouad@hotmail.com(F. Kissi)
[1]Department of mathematics, Mohamed first University, El qods, 60050, Morocco.
[2]LaMao, Faculty of sciences, El qods, 60050, Morocco

exhibit phase transitions, such as in elasticity or fluid dynamics. In such systems, the governing equations have coefficients that vary according to two different phases. The double phase framework is also significant in understanding models where energy densities switch between two distinct behaviors.

In the constant case, the authors in [9] have treated the following bifurcation problem involving degenerated p-Laplacian

$$\begin{cases} -\operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u\right) = \lambda b(x)|u|^{p-2}u + f(\lambda, x, u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Also, for weight $w$ satisfying the Muckenhoupt condition, the following fourth order elliptic equation has been treated

$$\begin{cases} \Delta\left(\omega(x)(|\Delta u|^{p-2}\Delta u + |\Delta u|^{q-2}\Delta u)\right) \\ -\operatorname{div}\left(\omega(x)(|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u)\right) = f(x) - div(G(x)) \text{ in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $f \in L^{p'}(\Omega, \omega^{-1/(p-1)})$ and $G \in \left[L^{p'}(\Omega, \omega^{-1/(q-1)})\right]^N$ (see [5]). The author has shown the existence of unique solution in the weighted Sobolev space $W^{2,p}(\Omega,\omega) \cap W_0^{1,p}(\Omega,\omega)$ .

In the variable case, the Dirichlet problem involving degenerated p(x)-Laplacian

$$\begin{cases} -\operatorname{div}\left(w(x)|\nabla u|^{p(x)-2}\nabla u\right) = \mu g(x)|u|^{p(x)-2}u + f(\lambda, x, u, \nabla u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

has been studied (see [25]), where $w, w^{-1/(p(x)-1)}$ are locally integrable functions on $\Omega$ and $w$ satisfies the hypothesis $(\Im)$ mentioned in (2.4).

The novelty of this paper is in extending the problem (1.3) using critical point theory (see [2, 10]), which requires a particular kind of weight depending on the variable exponent $q$, and a more complicated non-linearity. For this reason further analysis has to be realized.

The problem (1.1) is stated in the weighted Sobolev space $X = W^{2,q(x)}(\Omega,\omega) \cap W_0^{1,q(x)}(\Omega,\omega)$. A function $u \in X$ is a weak solution to (1.1) if and only if for every $v$ in X, we have

$$\begin{aligned} &\int_\Omega \omega(x)(|\Delta u|^{p(x)-2}\Delta u\Delta v + |\Delta u|^{q(x)-2}\Delta u\Delta v)\,dx \\ &+ \int_\Omega \omega(x)(|\nabla u|^{p(x)-2}\nabla u\nabla v + |\nabla u|^{q(x)-2}\nabla u\nabla v) \\ =&\lambda \int_\Omega f(x,u)v dx. \end{aligned}$$

In order to prove the existence and multiplicity of solutions for the problem (1.1), we assume that the weight $w$ belongs to the class $\tilde{A}_{q(.)}(\Omega)$ defined in (2.3), and satisfies the hypothesis $(\Im)$ mentioned in (2.4). Furthermore, we consider the following conditions on $f$