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Some Bounds for the Steiner-Harary Index of a Graph

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Abstract The Steiner distance for the set $S \subseteq V(G)$ would simply be the number of edges in the minimal subtree connecting them and is denoted as $d_G(S)$. The Steiner-Harary index is $SH_k(G)$, defined as the sum of the reciprocal of the Steiner distance for all subsets with k vertices in G. In this article, we calculate the exact value of $SH_k(G)$ for specific graphs and establish new best possible lower and upper bounds and characterization. Furthermore, we explore the relationship between $SH_k(G)$ and other graph indices based on Steiner distance.

Keywords Harary index, Steiner index, Steiner-Harary index

MSC(2010) 05C05, 05C07, 05C12, 05C38.

1. Introduction

The graphs considered in this paper are undirected, simple, finite, and connected. The graph $G = (V, E)$ has p-vertices and q-edges, where $V = V(G)$ and $E = E(G)$ represent the vertex and edge collections, respectively. The degree of a vertex v_i is defined as the number of vertices adjacent to it and is denoted by $d_G(v_i)$. If a vertex is adjacent to only one edge, it is called a pendant vertex. The distance between two vertices in a graph is given by $d_G(v_i, v_j)$, the shortest path length between v_i and v_j. The greatest distance between any two vertices in a graph G is called the diameter of the graph and is denoted by $diam(G)$. For undefined notations in this paper, we refer to [2, 8].

In 1947, Harold Wiener [18] first introduced the distance-based graph invariant, revealing correlations between the molecular structure of paraffins and their boiling points. The Wiener index, denoted by $W(G)$, is defined by:

$$W(G) = \sum_{v_i, v_j \subseteq V(G)} d_G(v_i, v_j).$$

In 1989, Chartrand [4] introduced the Steiner distance of a connected graph G and is denoted by $d_G(S)$, where $S \subseteq V(G)$ and $2 \leq |S| \leq p$. In 2016, Li, et al. [9]

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introduced the Steiner k-Wiener index, which is defined by:

$$SW_k(G) = \sum_{S \subseteq V(G), |S|=k} d_G(S).$$

In 2016, Furtula, et al. [6] introduced the k-center Steiner Harary index or the Steiner Harary k-index or simply Steiner-Harary index, which is defined by:

$$SH_k(G) = \sum_{S \subseteq V(G), |S|=k} \frac{1}{d_G(S)}.$$

In 2018, Tratnik [17] introduced the Steiner k-hyper Wiener index, defined by:

$$SWW_k(G) = \sum_{S \subseteq V(G)} \left[d_G(S) + d_G(S)^2 \right].$$

The use of graphical indices based on degree, distance, and Steiner distance has been extensively studied. For their history, applications, and mathematical properties, see [1, 3, 5, 7, 10, 12–16] and the references cited therein.

2. Specific families of graphs

Proposition 2.1. *[11] Let G be a specific families of graph with $2 \le k \le p$.*

(i) If K_p is a complete graph, then

$$SH_k(K_p) = \binom{p}{k} \frac{1}{k-1}.$$

(ii) If P_p is a path, then

$$SH_k(P_p) = \sum_{s=0}^{p-k} \binom{k+s-2}{k-2} (p-k-s+1) \frac{1}{k+s-1}.$$

(iii) If S_p is a star, then

$$SH_k(S_p) = \binom{p-1}{k} \frac{1}{k} + \binom{p-1}{k-1} \frac{1}{k-1}.$$

(iv) If $K_{m,n}$ is a complete bipartite graph with $1 \le m \le n$, then

$$SH_k(K_{m,n}) = \begin{cases} \frac{1}{k-1}\binom{p}{k} - \frac{1}{k(k-1)}\binom{m}{k} - \frac{1}{k(k-1)}\binom{n}{k}, & if 1 \le k \le m; \\ \frac{1}{k-1}\binom{p}{k} - \frac{1}{k(k-1)}\binom{n}{k}, & if m \le k \le n; \\ \frac{1}{k-1}\binom{p}{k}, & if n \le k \le p. \end{cases}$$

Proposition 2.2. *If W_p is a wheel and $S \subseteq V(W_p)$ with $p > 4$, then*

$$\begin{aligned} SH_k(W_p) = &\left[\binom{p-1}{k-1} + (p-1) \right] \frac{1}{k-1} \\ &+ \left[\binom{p-1}{k} - (p-1) \right] \frac{1}{k}. \end{aligned}$$