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Correspondence Between Renormalized and Entropy Solutions to the Parabolic Initial-Boundary Value Problem Involving Variable Exponents and Measure Data

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Abstract We study the initial-boundary value parabolic problem involving variable exponent under the generalized Leray-Lions conditions. We clarify the definitions of entropy and renormalized solutions to such parabolic problems, and we establish the equivalence between these definitions of entropy and renormalized solutions to the parabolic problems with the Leray-Lions operator and with measure data.

Keywords Capacity, diffuse measure data, entropy solution, exponential Lebesgue space, parabolic equation, renormalized solution, soft measure, variable Laplacian

MSC(2010) 35K20, 35K55, 35D99, 35J70.

1. Introduction

We consider the parabolic problem with measure data

$$\frac{\partial b\left(u\right)}{\partial t}-div\left(a\left(x,\ t,\ u,\ \nabla u\right)\right)+d\left(x\right)\left|u\right|^{p(x)-2}u=\mu,\quad \mu\in \mathrm{M}_0\left(Q\right), \tag{1.1}$$

$$b\left(u\right)\left(x,\ 0\right)=b\left(\varphi\left(x\right)\right),\quad x\in\Omega, \tag{1.2}$$

$$u|_{\partial\Omega\times(0,\ \infty)}=0, \tag{1.3}$$

where $Q\overset{def}{\equiv}\Omega\times(0,\ T)$, $(x,\ t)\in Q$, $\Omega\subseteq R^n$, $n\geq 3$ is a smooth Lipschitz domain and $\partial\Omega$ is a Lipschitz boundary of an open set Ω; $p\in P^{\log}(\Omega)$ is a log-Holder continuous function such that $1<p_m=\inf\{p(x),x\in\Omega\}$ and $p_S=\sup\{p(x),x\in\Omega\}<\infty$. We assume the function φ is measurable and such that $b(\varphi)$ belongs to $L^1(\Omega)$.

We consider the parabolic equation (1.1) under generalized Leray-Lions conditions. This type of problem was studied in weighted Orlicz–Sobolev spaces in [11]. In the case of constant exponents, the existence result was obtained for the obstacle parabolic problem associated with the equation with a Leray-Lions operator [18] and some a priori estimates were obtained in [1]. The existence of renormalized solutions for nonlinear elliptic equations with variable exponents with L^1 data was studied in [5], and the existence and uniqueness of the renormalized solution for parabolic

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equations under the Leray-Lions conditions [15] were established for constant exponents in [7]. The solvability of parabolic problem with L^1 data was investigated in [3], and for elliptic equations in [4, 5]. The concept of a renormalized solution was introduced by R.J. Perna, and P.L. Lions [15, 21], who studied the existence and uniqueness of solutions for some class of parabolic problems treated in [21].

Recently, results on the solvability of nonlinear parabolic equations with both singularities and unbounded lower-order terms were obtained by T.T. Dang, and G. Orlandi [6]. The authors did not impose the coercivity assumption on the main differential operator of convection-diffusion type (non-atomic measure), and the equation has the convective term with an unbounded coefficient in the Lorentz class. L. Zhao and S. Zheng studied the local Besov regularity of to the elliptic variational inequality with double-phase Orlicz growth. They proved that the fractional differentiability of the differential is reflected by additional differentiability assumptions on the obstacle term and the external force, under some regularity on the coefficient [27, 28]. M. Bendahmane and P. Wittbold [4] investigated the existence and uniqueness of the renormalized solution to a parabolic problem with L^1 data by employing some results of abstract semigroup theory to show the solvability of the parabolic problem with the singular right-hand side. J.X. Yin, J.K. Li, Y.Y. Ke investigated positive solutions of the variable exponent Laplacian equation by applying the Krasnoselskii fixed point theorem on the cone in [24]. Q. H. Zhang showed the existence of solutions to a problem with a variable exponent operator under the Caratheodory conditions. The author used the strict monotony condition to deal with the limit of the approximate solutions [15,26]. The monotone operators were studied by J.L. Lions [15].

As an example of the problem (1.1)-(1.3), we can consider a variable Laplacian problem

$$\frac{\partial b(u)}{\partial t} - \Delta_{p(\cdot)} u = \mu, \quad \mu \in \mathrm{M}_0(Q), \tag{1.4}$$

$$b(u)(x,\ 0) = b(\varphi(x)), \quad x \in \Omega, \tag{1.5}$$

$$u|_{\partial\Omega\times(0,\ \infty)} = 0. \tag{1.6}$$

By changing the unknown $v = b(u)$ and $\Psi = b^{-1}$, we obtain the generalized porous medium operator [1] $\partial_t v - \Delta_{p(\cdot)}\Psi(v)$ with a strictly increasing function Ψ. This type of problem often appears in models that describe the properties of fluids and processes of diffusion.

The presence of measure data presents additional complications. The class of problems covered by our conditions is Leray-Lions operators [15, 21] in divergent form $A(u) = -\nabla \cdot a(x,\ u,\ \nabla u)$. The presence of a measure datum μ compels us to work in the framework of entropy and renormalized solutions. The natural condition of the measure data is that these measures do not charge the sets of null capacity.

In the present paper, we establish that a function u is an entropy solution to the initial boundary parabolic problem (1.1)-(1.3) under the Leray-Lions conditions if and only if this function u is a renormalized solution to the same initial boundary parabolic problem (1.1)-(1.3) under the same conditions.