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Spreading Speed for Some Cooperative Systems with Nonlocal Diffusion and Free Boundaries, Part 3: Rate of Shifting

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Abstract This is the last part of our work series on a class of cooperative reaction-diffusion systems with free boundaries in one space dimension, where the diffusion terms are nonlocal, given by integral operators involving suitable kernel functions, and some of the equations in the system do not have a diffusion term. Such a system covers various models arising from population biology and epidemiology, including in particular a West Nile virus model [10] and some epidemic models [22, 38], where a "spreading-vanishing" dichotomy is known to govern the long time dynamical behaviour, but the spreading rate was not well understood. In this work series, we develop a systematic approach to determine the spreading profile of the system. In Part 1 [11], we obtained threshold conditions on the kernel functions which decide exactly when the spreading has finite speed c_0, or infinite speed (accelerated spreading), and for the case of finite speed, we determined its value c_0 via semi-wave solutions. In this paper, for some typical classes of kernel functions, we obtain more precise descriptions of the spreading for the finite speed case by revealing the exact rate of shifting of the spreading front from c_0t; the infinite speed case is studied separately in Part 2 [14].

Keywords Free boundary, nonlocal diffusion system, spreading rate

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1. Introduction

We continue our efforts to determine the precise long-time behaviour of cooperative systems with nonlocal diffusion and free boundaries of the following form:

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$$\begin{cases} \partial_t u_i = d_i \mathcal{L}_i[u_i](t,x) + f_i(u_1, u_2, \cdots, u_m), & t > 0,\ x \in (g(t), h(t)),\ 1 \le i \le m_0, \\ \partial_t u_i = f_i(u_1, u_2, \cdots, u_m), & t > 0,\ x \in (g(t), h(t)),\ m_0 < i \le m, \\ u_i(t, g(t)) = u_i(t, h(t)) = 0, & t > 0,\ 1 \le i \le m, \\ g'(t) = -\displaystyle\sum_{i=1}^{m_0} \mu_i \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J_i(x-y) u_i(t,x) \mathrm{d}y \mathrm{d}x, & t > 0, \\ h'(t) = \displaystyle\sum_{i=1}^{m_0} \mu_i \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J_i(x-y) u_i(t,x) \mathrm{d}y \mathrm{d}x, & t > 0, \\ u_i(0,x) = u_{i0}(x), & x \in [-h_0, h_0],\ 1 \le i \le m, \end{cases} \tag{1.1}$$

where $1 \le m_0 \le m$, and for $i \in \{1, ..., m_0\}$,

$$\mathcal{L}_i[v](t,x) := \int_{g(t)}^{h(t)} J_i(x-y) v(t,y) \mathrm{d}y - v(t,x),$$

$$d_i > 0 \text{ and } \mu_i \ge 0 \text{ are constants, with } \sum_{i=1}^{m_0} \mu_i > 0.$$

The initial functions satisfy for $1 \le i \le m$,

$$u_{i0} \in C([-h_0, h_0]), \quad u_{i0}(-h_0) = u_{i0}(h_0) = 0,\ u_{i0}(x) > 0 \text{ in } (-h_0, h_0). \tag{1.2}$$

The kernel functions satisfy, for $J \in \{J_i : 1 \le i \le m_0\}$,

(**J**): $J \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is nonnegative, even, $J(0) > 0$, $\displaystyle\int_{\mathbb{R}} J(x) \mathrm{d}x = 1$.

As in Part 1 [11], we will write $F = (f_1, ..., f_m) \in [C^1(\mathbb{R}^m_+)]^m$ with

$$\mathbb{R}^m_+ := \{x = (x_1, ..., x_m) \in \mathbb{R}^m : x_i \ge 0 \text{ for } i = 1, ..., m\},$$

and use the following notations for vectors in $\mathbb{R}^m$:

(i) For $x = (x_1, \cdots, x_m) \in \mathbb{R}^m$, we simply write $(x_1, \cdots, x_m)$ as (x_i). For $x = (x_i)$, $y = (y_i) \in \mathbb{R}^m$,

$$\begin{array}{lll} x \succeq (\preceq)\ y & \text{means} & x_i \ge (\le)\ y_i \text{ for } 1 \le i \le m, \\ x \succ (\prec)\ y & \text{means} & x \succeq (\preceq)\ y \text{ but } x \ne y, \\ x \succ\!\!\succ (\prec\!\!\prec)\ y & \text{means} & x_i > (<)\ y_i \text{ for } 1 \le i \le m. \end{array}$$

(ii) If $x \preceq y$, then $[x, y] := \{z \in \mathbb{R}^m : x \preceq z \preceq y\}$.

(iii) Hadamard product: For $x = (x_i)$, $y = (y_i) \in \mathbb{R}^m$,

$$x \circ y = (x_i y_i) \in \mathbb{R}^m.$$

(iv) Any $x \in \mathbb{R}^m$ is viewed as a row vector, namely a $1 \times m$ matrix, whose transpose is denoted by x^T.

Our basic assumptions on F are: