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Some Hermite-Hadamard Type Inequalities with the Aid of Newly Defined Double Post-Quantum Integrals

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Abstract In the present paper, we first give four post-quantum integrals for functions of two variables, denoted by ${}_{ac}T_{p,q}$, ${}_{a}^{d}T_{p,q}$, ${}_{c}^{b}T_{p,q}$ and ${}^{bd}T_{p,q}$. Afterwards, each of these newly defined integrals is illustrated. Moreover, some new Hermite-Hadamard inequalities are established based on these definitions. We also show the correctness of these inequalities with the aid of some numerical examples.

Keywords Hermite-Hadamard inequality, q-integral, T_q-integral, quantum calculus, co-ordinated convexity

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1. Introduction

Quantum calculus has been the cornerstone of physics and mathematics. In particular, quantum integrals have solved many problems in the literature. After F. H. Jackson described the q-Jackson integral in [12], this topic has attracted the attention of many mathematicians. Quantum calculus and related properties are discussed in [14] by P. Cheung and V. Kac. Tariboon and Ntouyas obtained several q-analogues of classical mathematics topics in [20]. Agarwal described the q-fractional derivative in [1]. Noor et al. created new q-analogues of the inequalities using the q-differentiable convex function in [19]. Tariboon and Notuyas introduced q_a-definite integral in [20]. Alp et al. obtained q_a-Hermite-Hadamard inequalities with convex functions on quantum integral in [5]. Bermudo et al. introduced a new quantum integral concept called q^b-integral and presented the related Hermite-Hadamard type inequalities in [6]. In addition, the authors established new inequalities that include q_a and q^b integrals together. In the paper [18], Latif offered q_{ac}-integral and properties this of integral for two variables functions. The author also worked on the Hermite-Hadamard inequality based on this definition. However, Alp and

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Sarıkaya revised the minor errors in this inequality and established a new inequality in [3]. Budak et al. described new q^b-integrals for two variables in [9]. The authors also proved three different Hermite-Hadamard inequalities through the three new integrals, namely q_a^d, q_c^b and q^{bd}-integrals. Nowadays, there are many studies on integral inequalities and quantum integrals (see, [2], [8], [10], [11], [17], [22]).

On the other hand, using the facts of trpezoid areas in [4], Alp and Sarıkaya investigated the generalized quantum integral, expressed as the ${}_aT_q$-integral. In addition, the Hermite-Hadamard inequality in the case of this definition is also constructed by the authors. In the paper [16], Kara et al. presented the generalized quantum integral stated as the ${}^bT_q$-integral involving areas of the trapezoids. With the help of the given definition of this paper, the researchers obtained the new Hermite-Hadamard inequalities. Moreover, Kara and Budak established new T_q-integrals by two variables in [15]. In addition to these, they were also proved to correspond to four Hermite-Hadamard type inequalities on co-ordinates. Vivas-Cortez et al. considered to the generalized post-quantum integral, called the ${}_aT_{p,q}$-integral in [21]. More precisely, the authors also investigated new Hermite-Hadamard-type inequalities in the case of ${}_aT_{p,q}$-integral. Budak et al. demonstrated a new concept of post quantum integral, namely ${}^bT_{p,q}$-integral, in [7]. They also provided several Hermite-Hadamard inequalities to the case of ${}^bT_{p,q}$-integral by using convex functions. The definitions and theorems mentioned in this paragraph and that used in the article are detailed in the following section.

2. T_q-integrals and $T_{p,q}$-integrals

This section presents the desired definitions and related inequalities.

From the fact of the area of trapezoids, Alp and Sarikaya investigated the generalized quantum integral, which is called the ${}_aT_q$-integral as follows:

Definition 2.1 (see, [4])**.** Suppose that $\Psi : [a,b] \to \mathbb{R}$ is a continuous function. For $\varkappa \in [a,b]$, it follows that

$$\int_a^b \Psi(s)\ {}_ad_q^T s = \frac{(1-q)(b-a)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n \Psi(q^n b + (1-q^n)a) - \Psi(b)\right],$$

where $0<q<1$.

Because of the ${}_aT_q$-integrals, Hermite-Hadamard inequalities are also as follows:

Theorem 2.1 (see, [4])**.** *[${}_aT_q$-Hermite-Hadamard Inequalities] If $\Psi : [a,b] \to \mathbb{R}$ is a convex function on $[a,b]$ and $0<q<1$, then we get the following double inequality*

$$\Psi\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b \Psi(\varkappa)\ {}_ad_q^T \varkappa \le \frac{\Psi(a)+\Psi(b)}{2}. \tag{2.1}$$

By using the area of trapezoids, Kara et al. [16] introduced the following generalized quantum integral which is called ${}^bT_q$-integral.