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Monotonicity Analysis of Generalized Discrete Fractional Proportional h-Differences with Applications

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Abstract Monotonicity analysis is an important aspect of fractional mathematics. In this paper, we perform a monotonicity analysis for a generalized class of nabla discrete fractional proportional difference on the $h\mathbb{Z}$ scale of time. We first define the sums and differences of order $0 < \alpha \leq 1$ on the time scale for a general form of nabla fractional along with Riemann-Liouville h-fractional proportional sums and differences. We formulate the Caputo fractional proportional differences and present the relation between them and the fractional proportional differences. Afterward, we introduce and prove the monotonicity results for nabla and Caputo discrete h-fractional proportional differences. Finally, we provide two numerical examples to verify the theoretical results along with a proof for a new version of the fractional proportional difference of the mean value theorem on $h\mathbb{Z}$ as an application.

Keywords Monotonicity analysis, h-fractional proportional difference, Caputo fractional proportional difference, fractional proportional Mean Value Theorem(MVT)

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1. Introduction

Fractional calculus has become an important and active area of research. Many researchers use this topic to model and successfully solve diverse types of problems that appear in science and engineering [1–5]. Fractional calculus extends traditional calculus by allowing derivatives and integrals of non-integer orders, providing

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a powerful tool for modeling complex, real-world phenomena such as memory effects and anomalous diffusion. In differential equations, fractional derivatives offer greater flexibility and accuracy in describing processes with long-range dependencies and hereditary properties, making them essential in fields such as physics, biology, and engineering [6–9]. While continuous fractional calculus is well established, discrete fractional calculus still has high potential in modern applications. Lately, this specific field has been under the spotlight of the research community. Different properties of discrete fractional operators are studied to reveal the potential in various aspects of such a topic [10–13].

One important aspect of discrete fractional calculus is the study of fractional sums and differences with nabla operators. The theory and applications have been extensively considered with new developments over the last few decades [12,14–17]. For example in [18], the Laplace transform on the fractional proportional operators is studied and a generalization of fractional proportional sums and differences is given. Wei et al. [19] consider the series representation for nabla discrete fractional sums and differences. A new discrete fractional solution of the modified Bessel differential equation is introduced in [20]. The monotonicity properties of discrete delta and nabla fractional operators have been an active topic of research recently [21–24]. In [25], the monotonicity properties for nabla fractional sums and differences of order $0 < \alpha \leq 1$ on the time scale $h\mathbb{Z}$, where $0 < h \leq 1$, are studied. Monotonicity results for Riemann-Liouville and Caputo fractional proportional differences on the time scale $\mathbb{Z}$ are presented in [26]. In [27], a new method for negative nabla and delta fractional proportional differences is introduced. Authors of [28] present a comprehensive study on the monotonicity analysis for delta and nabla discrete fractional operators of the Liouville-Caputo family, which directly aligns with the exploration of fractional operators in this work. Additionally, [29] investigates unexpected properties of fractional difference operators, particularly finite and eventual monotonicity, offering insights that are relevant to the current study's focus on operator behavior.

Motivated by the aforementioned work, we present the monotonicity properties for a generalized class of discrete h-fractional proportional differences on the time scale $h\mathbb{Z}$. We start by defining the general fractional sums and differences along with the Riemann-Liouville form. We then move to find and prove the relation between the nabla fractional sums and differences and the Caputo proportional differences. Monotonicity analysis is then conducted for both nabla and Caputo proportional differences. Moreover, we present a new form of the fractional proportional difference of the mean value theorem on $h\mathbb{Z}$ time scale. The work presented in this paper is a direct generalization of results presented in [25] and [26]. In order to validate the theoretical findings, we introduce two numerical examples that are considered direct illustrations of the basic results presented in this work. Additionally, we provide a comprehensive proof of an updated version of the fractional proportional difference of the mean value theorem.

The rest of the paper is organized as follows. In Section 2, basic definitions and preliminary work are introduced. Section 3 includes the main monotonicity results for nabla h-fractional proportional differences. In Section 4, two numerical examples that verify and support the theoretical results are presented and a fractional proportional version of the Mean Value Theorem is developed. Finally, Section 5 concludes the paper.