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A Novel Variant of Milne's Rule Inequalities on Quantum Calculus for Convex Functions with Their Computational Analysis

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Abstract In this investigation, we introduce a novel approach for establishing Milne's type inequalities in the context of quantum calculus for differentiable convex functions. First, we prove a quantum integral identity. We derive numerous new Milne's rule inequalities for quantum differentiable convex functions. These inequalities are relevant in open Newton-Cotes formulas, as they facilitate the determination of bounds for Milne's rule applicable to differentiable convex functions in both classical and q-calculus. In addition, we conduct a computational analysis of these inequalities for convex functions and provide mathematical examples to demonstrate the validity of the newly established results within the framework of q-calculus.

Keywords Milne's inequality, q-calculus, convex functions

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1. Introduction

Convexity, a fundamental mathematical notion derived from ancient Greek philosophy, acquired substantial traction in the late nineteenth century, mainly due to the pioneering work of German mathematician Karl Hermann Amandus Schwarz, who introduced convex functions [13]. Convexity has numerous modern applications in economics, engineering, computer science, and mathematics, particularly in optimization problems and inequalities [19, 29]. Considerable study has demonstrated the strong relationship between convexity theory and integral inequalities, emphasizing their critical roles in differential equations and applied mathematics. This relationship is critical due to the broad range of applications and the significant

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impact of integral inequalities. Exploring numerous inequalities such as Gronwall, Simpson's type, Chebyshev, Jensen, Hölder, Milne, and Hermite-Hadamard (H-H) inequalities enriches the general comprehension of mathematical concepts. For those interested in delving deeper into these inequalities and their practical applications, references [1, 2, 24, 29, 36] provide valuable insights.

The H-H inequality for convex functions is one of several inequalities that can be deduced directly from the applications of convex functions. The H-H inequality, originated by C. Hermite and J. Hadamard, is a cornerstone in the field of convex functions, known for its geometric interpretation and numerous applications [21, 22].

$$\mathsf{F}\left(\frac{\sigma+\rho}{2}\right) \leq \frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}\mathsf{F}(\zeta)d\zeta \leq \frac{\mathsf{F}(\sigma)+\mathsf{F}(\rho)}{2}. \tag{1.1}$$

This inequality has numerous advantages, particularly its widespread application in approximation theory. Its vast applications prompted mathematicians to begin developing it, which resulted in the publication of multiple new results. The trapezoidal and midpoint-type inequalities are reported in [14, 25] by employing the principles of differentiable convexity. Numerous studies have been accomplished over the past twenty years to find new bounds for the inequality on the left and right sides of (1.1). For more information, see [17, 30].

The q-H-H type inequality expands the traditional H-H inequality in mathematical analysis. It offers constraints on convex functions by considering their values at the interval's endpoints. Alp et al. [3] applied quantum calculus techniques to establish a novel variant of the H-H inequality (1.1), as follows:

$$\mathsf{F}\left(\frac{q\sigma+\rho}{1+q}\right) \leq \frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}\mathsf{F}(\zeta)_{\sigma}d_q\zeta \leq \frac{q\mathsf{F}(\sigma)+\mathsf{F}(\rho)}{1+q}.$$

Burmudo et al. [6] proposed a novel formulation specifically designed for q values occurring within the interval $(0,1)$, presented as follows:

$$\mathsf{F}\left(\frac{\sigma+q\rho}{1+q}\right) \leq \frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}\mathsf{F}(\zeta)^{\rho}d_q\zeta \leq \frac{\mathsf{F}(\sigma)+q\mathsf{F}(\rho)}{1+q},$$

and

$$\mathsf{F}\left(\frac{\sigma+\rho}{2}\right) \leq \frac{1}{2(\rho-\sigma)}\left[\int_{\sigma}^{\rho}\mathsf{F}(\zeta)_{\sigma}d_q\zeta+\int_{\sigma}^{\rho}\mathsf{F}(\zeta)^{\rho}d_q\zeta\right] \leq \frac{\mathsf{F}(\sigma)+\mathsf{F}(\rho)}{2}.$$

In recent years, numerous researchers have focused on Milne's type inequality across various categories of mappings. Recognizing the versatility and effectiveness of convexity theory, they utilize it to tackle problems spanning multiple fields of pure and applied mathematics.

Budak et al. [10] revealed fractional versions of Milne-type inequalities for differentiable convex functions. Celik et al. [12] generalized Milne-type inequality for conformable fractional integrals. Also, they discussed different function classes. In [23], Hezenci et al. proposed a tempered fractional version of Milne-type inequality. Demir demonstrated multiple integral inequalities connected to Milne-type integral inequalities concerning the proportional Caputo-hybrid operator [15]. Ali et al. [5] established the error bounds for Milne's formula, a variant of the open Newton-Cotes formulas designed for differentiable convex functions within fractional and classical calculus frameworks.