Journal of Nonlinear Modeling and Analysis
Volume 7, Number 5, October 2025, 1859–1869
https://journal.global-sci.org/jnma
DOI:10.12150/jnma.2025.1859

On Local and Nonlocal Robin Boundary Value Problem with Critical Nonlinearity

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Abstract In this paper, using the Mountain Pass Theorem, we present results on compactness and the existence of solutions for a class of local and nonlocal p-Laplacian equations involving Robin boundary conditions, with critical nonlinearity and a small perturbation.

Keywords p-Laplacian, Robin problem, critical exponent

MSC(2010) 35J30, 35J60, 35J92.

1. Introduction

This paper deals with the following elliptic problem:

$$\begin{gathered} K\left(\int_\Omega |\nabla u|^p\,dx+\beta\int_{\partial\Omega}|u|^p d\sigma_x\right)\Delta_p u=\gamma a(x)|u|^{q-2}u+|u|^{p^*-2}u+g(x)\quad \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial u}+\beta|u|^{p-2}u=0\quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain with smooth boundary, $p^*=Np/(N-p)$ is the critical Sobolev exponent, $1<p<N$, γ is a positive parameter and $a\in L^{p^*/(p^*-q)}(\Omega)$, $g\in L^{p'}(\Omega)$, with $\frac{1}{p}+\frac{1}{p'}=1$ and $p^*=\frac{Np}{N-p}$.

Here the functional K verifies (K_1) $K:\ (0,+\infty)\to(0,+\infty)$ continuous and $k_0=\inf\limits_{s>0}K(s)>0$.

The problem (1.1) is called nonlocal because of the presence of the term $K(.)$, so it is no longer a pointwise identity. This leads us to some mathematical difficulties which make the study of such a class of problems particularly interesting.

In fact, equations such as (1.1) received more attention after Lions [11] proposed an abstract framework to the problem. Some important and interesting results can be found, for example, in [21].

The critical exponent case poses a significant challenge due to the absence of compactness, rendering standard arguments ineffective. To our knowledge, only few results have studied the elliptic problems featuring critical exponents. Among these references, some of the most noteworthy include [3, 8, 9, 12–15, 18] and their associated literature. However, drawing inspiration from these seminal works from which we will draw certain insights, our aim is to generalize and partially extend

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corresponding results to accommodate cases where $p \neq 2$ and involve a perturbation g.

First, we deal with the case of a local problem: Suppose that the operator $K = Id$ and $1 < q < p < N$; then we can state the following compactness note.

Theorem 1.1. *There exists a constant $L > 0$ depending on p, q, and N such that ϕ_γ satisfies the Palais-Smale condition in the interval I^γ :*

$$I^\gamma = (-\infty, \frac{1}{2N} S^{\frac{N}{2p}} - L\gamma^{\frac{p^*}{p^*-q}}),$$

for every $\gamma > 0$ with g small enough with respect to the norm $\|.\|_*$.

Now, for the non-local case, we make the following assumption:

$$(K_2)\ \widehat{K}(t) \geq K(t)t \text{ for } t > 0 \text{ , with } \widehat{K}(t) = \int_0^t K(s)ds.$$

Accordingly, we can report our main result.

Theorem 1.2. *Under the hypotheses $(K_1), (K_2)$ and $q \in (p, p^*)$, there exists $\gamma^* > 0$, such that problem (1.1) has at least a nontrivial solution for all $\gamma \geq \gamma^*$, provided g is small enough in the norm $\|.\|_*$ of $(W^{1,p}(\Omega))^*$.*

The existence of solutions for problem (1.1) remains largely uncharted territory within the realm of variational methods. As in our forthcoming paper, problem (1.1) can be construed as a Schrödinger equation entwined with a non-local term. The interplay between this nonlocal term and the critical nonlinearity prevents us from using the variational methods in a standard way. Establishing new estimates adjusted to Kirchhoff equations, which entail the utilization of Palais–Smale sequences, is imperative for our endeavor. Let us point out that although the idea was used before for other problems, adapting the procedure to our problem is not trivial at all, owing to the appearance of the non-local term and Robin boundary condition.

In [16], the authors presented a bifurcation-type theorem that describes the dependence of the set of positive solutions for a Robin problem with a concave-convex term.

The paper [10] addressed a nonlinear Robin problem driven by the (p, q)-Laplacian in addition to an indefinite potential term. It is shown that, under minimal conditions on the nonlinearity, the problem admits a nodal solution.

In [7], El Khalil investigates the existence of at least one nondecreasing sequence of positive eigenvalues by applying minimax arguments on a C^1-manifold.

Regarding the eigenvalues of the (p-q)-Laplacian with homogeneous Dirichlet boundary conditions, the author in [19] established the existence of two nontrivial (weak) solutions.

Throughout this paper, we consider the C^1-functional energy:

$$\begin{aligned}\Phi_\gamma(u) =& \frac{1}{p}\widehat{K}\left(\int_\Omega |\nabla u|^p dx + \beta \int_{\partial\Omega} |u|^p \, d\sigma_x\right) - \frac{\gamma}{q}\int_\Omega a(x)|u|^q dx - \frac{1}{p^*}\int_\Omega |u|^{p^*} dx \\ &- \int_\Omega g(x)u dx.\end{aligned}$$

Note that

$$\Phi'_\gamma(u).v = K(\| u \|^p)\Big(\int_\Omega \|\nabla u|^{p-2}\nabla u.\nabla v dx + \int_{\partial\Omega} |u|^{p-2}uv \, d\sigma_x\Big)$$