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Three-Dimensional Polynomial Differential Systems with an Isolated Compact Invariant Algebraic Surface*

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Abstract The aim of this paper is to characterize the simplest three-dimensional polynomial differential system having an equilibrium and a 2-dimensional orientable smooth compact manifold with genus $g \leq 1$ in $\mathbb{R}^3$, where the 2-dimensional orientable smooth compact manifold is sphere $\mathbb{E}^2$ or torus $\mathbb{T}^2$. We first look for the smallest degree of polynomial differential systems with both an equilibrium and an isolated compact invariant algebraic surface $\mathbb{E}^2$ or $\mathbb{T}^2$. It is shown that the smallest degree of the system depends on the relative position between the equilibrium and the compact invariant algebraic surface in $\mathbb{R}^3$. Furthermore, the sufficient and necessary algebraic conditions are given for the smallest order three-dimensional polynomial differential system having both an equilibrium and an isolated compact invariant algebraic surface. Lastly, we discuss the influence of the coexistence of an isolated compact invariant algebraic surface and an equilibrium on dynamics of the three-dimensional polynomial differential system.

Keywords Three-dimensional, polynomial differential systems, isolated, invariant, compact algebraic surface

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1. Introduction

Three-dimensional polynomial differential systems are widely used as some approximations of mathematical models from physics, biology, chemistry and egineering, for instance, Lorenz system [14, 15], Kolmogorov system [1, 2, 9] and Chua system [3–5], whose dynamics plays an important role in understanding complex nonlinear phenomena such as chaos, strange attractors and turbulence. The occurrence of complex phenomena is related to some invariant sets of the three-dimensional polynomial differential system. The invariant set is usually composed of equilibrium points, compact nontrivial orbits and some noncompact orbits whose limit sets are either the equilibrium points or the compact orbits of the system. A natural question is raised: can an invariant set of three-dimensional polynomial differential systems become an isolated 2-dimensional compact invariant manifold embedded

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in $\mathbb{R}^3$? The definition of an invariant manifold can be found in [7]. To discuss the question, we consider the following three-dimensional polynomial differential systems

$$\frac{dx_i}{dt} = f_i(x_1, x_2, x_3), \ i = 1, 2, 3, \tag{1.1}$$

where $f_i(x_1, x_2, x_3)$ is a polynomial in the variables x_1, x_2 and x_3 with degree m_i, denoted by $f_i \in \mathbb{R}[x_1, x_2, x_3]$, $i = 1, 2, 3$. Here $\mathbb{R}[x_1, x_2, x_3]$ is the ring of the polynomials in the variables x_1, x_2 and x_3 with coefficients in $\mathbb{R}$. We say that

$$n = \max_{i=1,2,3} \{\deg f_i(x_1, x_2, x_3)\} = \max \{m_1, m_2, m_3\}$$

is *the order (or degree)* of system (1.1). The existence of invariant algebraic surfaces for system (1.1) or a kind of system (1.1) (e.g. Kolmogorov system) and its dynamics have been studied by many mathematicians, see [8, 10–13, 16, 17] and references therein.

In the paper, we are interested in the case that system (1.1) has at least one equilibrium in $\mathbb{R}^3$. Without loss of generality, it can be assumed that the equilibrium is at the origin $O(0, 0, 0)$. Then the n-order system (1.1) with an equilibrium at $O(0, 0, 0)$ can be rewritten as

$$\frac{dx_i}{dt} = f_i(x_1, x_2, x_3) = \sum_{k=1}^{n} f_i^{(k)}(x_1, x_2, x_3), \ i = 1, 2, 3, \tag{1.2}$$

where $f_i^{(k)}(x_1, x_2, x_3)$ is a homogeneous polynomial in the variables x_1, x_2 and x_3 with degree k, $1 \leq k \leq n$ and $i = 1, 2, 3$. Hence, the vector field $(f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3), f_3(x_1, x_2, x_3))$ associated with system (1.2) has no constant terms.

Let M be a smooth closed orientable surface of genus g in $\mathbb{R}^3$. Then the simplest 2-dimensional orientable smooth compact manifolds with genus $g \leq 1$ in $\mathbb{R}^3$ are ellipsoid and tori. Note that the 2-dimensional ellipsoid surface can be transformed into a unit sphere in $\mathbb{R}^3$ by an affine transformation. Inspired by Llibre et al. [13], we consider if the n-order system (1.2) has an invariant sphere with the following form

$$\mathbb{E}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1 - a)^2 + (x_2 - b)^2 + (x_3 - c)^2 = 1, a, b, c \in \mathbb{R}\},$$

and an invariant torus in the form

$$\mathbb{T}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1^2 + x_2^2 - r^2)^2 + x_3^2 = 1, r > 0\}.$$

Our aim is to look for the smallest degree of system (1.2) such that system (1.2) has an isolated invariant sphere $\mathbb{E}^2$ (or torus $\mathbb{T}^2$), and give the sufficient and necessary algebraic conditions for system (1.2) with the smallest degree having an isolated invariant sphere $\mathbb{E}^2$ (or torus $\mathbb{T}^2$). We say $\mathbb{E}^2$ ($\mathbb{T}^2$) is *invariant* for system (1.2) if the orbit of system (1.2) passing through any a point in $\mathbb{E}^2$ ($\mathbb{T}^2$, resp.) is completely contained in $\mathbb{E}^2$ ($\mathbb{T}^2$, resp.). Note that $\mathbb{E}^2$ and $\mathbb{T}^2$ are quadratic and quartic algebraic surfaces, respectively. One of the important tools used to study the invariance of algebraic surfaces for polynomial differential systems is Darboux theory founded by Darboux in [6]. Assume that $H(x_1, x_2, x_3)$ is a real polynomial with degree m, $m \geq 1$. Then the algebraic surface $H(x_1, x_2, x_3) = 0$ is invariant for the n-order