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Some Results on Fractional Corrected Euler-Maclaurin-Type Inequalities Related to Various Function Classes

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Abstract In this paper, some corrected Euler-Maclaurin-type inequalities are established by using various function classes involving Riemann-Liouville fractional integrals. We then present our findings using examples and special cases of the theorems that we have discovered. Moreover, we provide several fractional corrected Euler-Maclaurin-type for bounded functions. Additionally, for Lipschitzian functions, we create a few fractional corrected Euler-Maclaurin-type inequalities. Lastly, we provide some fractional corrected Euler-Maclaurin-type inequalities for functions of bounded variation.

Keywords Quadrature formulae, Maclaurin's formula, convex functions, fractional calculus

MSC(2010) 26D07, 26D10, 26D15, 65D32.

1. Introduction

Inequality theory is well-known and still an interesting area of research with a wide range of applications in many fields of mathematics. Convex functions are also significant in the theory of inequality. Afterwards, mathematicians in the mathematical sciences have become interested in fractional calculus due to its fundamental properties and its applications. Because of the importance of fractional calculus, mathematicians have studied a number of fractional integral inequalities.

Dragomir [1] provided an estimate of remainder for Simpson's quadratic formula in the case of bounded variation functions, with applications in the theory of special means. Furthermore, a number of fractional Simpson-type inequalities for functions with convex second derivatives in absolute value were demonstrated in article [2]. Furthermore, in the domain of differentiable convex functions, Budak et al. [3] examined a number of variants of Simpson-type inequalities using generalized fractional integrals. For additional information on Simpson-type inequalities and other characteristics of Riemann-Liouville fractional integrals, readers can see [4, 5] and their references. In the literature, evaluations for three-step quadratic kernels are sometimes referred to as Newton-type results because the three-point Newton-Cotes

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quadrature corresponds to Simpson's second rule. A great deal of research has been done on Newton-type inequalities. For instance, a number of fractional Newton-type inequalities were reported in article [6] for the case of bounded variation functions, and several Newton-type inequalities were proved by using the Riemann-Liouville fractional integrals for the case of differentiable convex functions. Later, various Newton-type integral inequalities were established by Erden et al. in paper [7] for functions whose first derivative is arithmetically-harmonically convex in absolute value at a certain power. According to Sitthiwirattham et al., some fractional Newton-type inequalities for constrained variation functions were given by [8]. Furthermore, Gao and Shi [9] proposed a new convexity-based Newton-type inequality and provided specific applications for specific real function scenarios. Please refer to [10–12] and their references for more information on convex differentiable functions and other Newton-type inequalities.

Dedić et al. [13] created a set of inequalities using the Euler-Maclaurin-type inequalities, and the outcomes were used to produce specific error estimates in the case of the Maclaurin quadrature rules. To establish a set of inequalities, the Euler-Simpson 3/8 formulas were also used. The findings were used to provide some error estimates for the Simpson 3/8 quadrature rules in article [14]. Additionally, several Euler-Maclaurin-type inequalities were stated in [15]. Later, some various corrected Euler-Maclaurin-type inequalities were proved using the Riemann-Liouville fractional integrals in paper [17]. The reader is referred to [18–27] and the references therein for further information on these kinds of inequalities.

This paper uses Riemann-Liouville fractional integrals to derive Corrected Euler-Maclaurin-type inequalities for a variety of function classes. A basic definition of fractional calculus and more research in this area are given in Section 2. We shall demonstrate an integral equality that is necessary to prove the article's primary conclusions, which are presented in Section 3. Moreover, Section 4 provide some corrected Euler-Maclaurin-type inequalities for a number of differentiable convex functions using the Riemann-Liouville fractional integrals. We shall provide some corrected Euler-Maclaurin-type for bounded functions by fractional integrals in Section 5. For Lipschitzian functions, some fractional corrected Euler-Maclaurin-type will be established in Section 6. The corrected Euler-Maclaurin-type will be demonstrated using fractional integrals of bounded variation in Section 7. Moreover, in Section 8 we will offer a number of graphical illustrations to show the accuracy of the recently established inequalities. We will talk about our thoughts on corrected Euler-Maclaurin-type inequalities and how they might affect future directions for study in Section 9.

2. Preliminaries

The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha > 0$ with $a \geq 0$ are presented by

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{b} (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

in that order [28, 29]. Here, f belongs to $L_1[a,b]$ and $\Gamma(\alpha)$ symbolizes the Gamma function that is defined as

$$\Gamma(\alpha) := \int_0^\infty e^{-u} u^{\alpha-1} du.$$

The fractional integral coincides with the classical integral for the case of $\alpha = 1$.

For Simpson's disparities, the following guidelines apply:

i. The formula for Simpson's quadrature, sometimes referred to as Simpson's 1/3 rule, is as follows:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]. \tag{2.1}$$

ii. The Newton-Cotes quadrature formula, which is sometimes called Simpson's 3/8 rule or Simpson's second formula (see [19]), can be stated as follows:

$$\int_a^b f(x)\,dx \frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right]. \tag{2.2}$$

iii. The Maclaurin rule is equivalent to the analogous dual Simpson's 3/8 formula, and it is derived from the Maclaurin formula (see [19]).

$$\int_a^b f(x)\,dx \frac{b-a}{8}\left[3f\left(\frac{5a+b}{6}\right) + 2f\left(\frac{a+b}{2}\right) + 3f\left(\frac{a+5b}{6}\right)\right]. \tag{2.3}$$

Formulae (2.1), (2.2), and (2.3) satisfy for all function f with continuous 4^{th} derivative on $[a,b]$.

The most popular Newton-Cotes quadrature using a three-point Simpson-type inequality is as follows:

Theorem 2.1. *If $f : [a,b] \to \mathbb{R}$ is a four-times differentiable and continuous function on (a,b) and $\left\|f^{(4)}\right\|_\infty = \sup_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty$, then the inequality stated below is true:*

$$\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4.$$

The Simpson's 3/8 rule, a traditional closed-type quadrature rule based on the Simpson's 3/8 inequality, is given by the following

Theorem 2.2. *Let us consider that $f : [a,b] \to \mathbb{R}$ is a four-times differentiable and continuous function on (a,b), and $\left\|f^{(4)}\right\|_\infty = \sup_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty$. Then, one has the inequality*

$$\left|\frac{1}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{1}{6480}\left\|f^{(4)}\right\|_\infty (b-a)^4.$$

Derived from the Maclaurin inequality, the Maclaurin rule is equal to the corresponding dual Simpson's 3/8 formula.

Theorem 2.3. *Suppose that $f:[a,b]\to\mathbb{R}$ is a continuous and four-times differentiable function on (a,b). Suppose also that $\left\|f^{(4)}\right\|_{\infty}=\sup\limits_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$. Then, the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\\&\leq\frac{7}{51840}\left\|f^{(4)}\right\|_{\infty}(b-a)^4.\end{aligned}$$

3. Main result

Lemma 3.1. *Note that $f:[a,b]\to\mathbb{R}$ is a function on (a,b) such that $f'\in L_1[a,b]$. Then, the following equality is valid:*

$$\begin{aligned}&\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]\\&-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\\&=\frac{b-a}{4}\left[I_1+I_2\right],\end{aligned}$$

where

$$\begin{cases}I_1=\int\limits_0^{\frac{1}{3}}t^{\alpha}\left[f'\left(\frac{t}{2}b+\frac{2-t}{2}a\right)-f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right]dt,\\ I_2=\int\limits_{\frac{1}{3}}^{1}\left(t^{\alpha}-\frac{27}{40}\right)\left[f'\left(\frac{t}{2}b+\frac{2-t}{2}a\right)-f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right]dt.\end{cases}$$

Proof. By employing the integration by parts, we have

$$\begin{aligned}I_1&=\int\limits_0^{\frac{1}{3}}t^{\alpha}\left[f'\left(\frac{t}{2}b+\frac{2-t}{2}a\right)-f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right]dt\\&=\frac{2}{b-a}t^{\alpha}\left[f\left(\frac{t}{2}b+\frac{2-t}{2}a\right)+f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right]\Bigg|_0^{\frac{1}{3}}\\&\quad-\frac{2\alpha}{b-a}\int\limits_0^{\frac{1}{3}}t^{\alpha-1}\left[f\left(\frac{t}{2}b+\frac{2-t}{2}a\right)+f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right]dt\\&=\frac{2}{b-a}\left(\frac{1}{3}\right)^{\alpha}\left[f\left(\frac{5a+b}{6}\right)+f\left(\frac{a+5b}{6}\right)\right]-\frac{2c}{(b-a)}\left[f(a)+f(b)\right]\\&\quad-\frac{2\alpha}{b-a}\int\limits_0^{\frac{1}{3}}t^{\alpha-1}\left[f\left(\frac{t}{2}b+\frac{2-t}{2}a\right)+f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right]dt.\end{aligned}\tag{3.1}$$

Based on the provided information, it can be concluded that

$$I_2 = \int_{\frac{1}{3}}^{1} (t^\alpha - 1)\left[f'\left(\frac{t}{2}b + \frac{2-t}{2}a\right) - f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right)\right] dt \tag{3.2}$$
$$= -\frac{2}{b-a}\left(\left(\frac{2}{3}\right)^\alpha - 1\right)\left[f\left(\frac{5a+b}{6}\right) + f\left(\frac{a+5b}{6}\right)\right]$$
$$- \frac{2\alpha}{b-a}\int_{\frac{1}{3}}^{1} t^{\alpha-1}\left[f\left(\frac{t}{2}b + \frac{2-t}{2}a\right) + f\left(\frac{t}{2}a + \frac{2-t}{2}b\right)\right] dt.$$

When (3.1) and (3.2) are combined, we easily have

$$I_1 + I_2 = \frac{1}{20(b-a)}\left[27f\left(\frac{5a+b}{6}\right) + 26f\left(\frac{a+b}{2}\right) + 27f\left(\frac{a+5b}{6}\right)\right] \tag{3.3}$$
$$- \frac{2\alpha}{b-a}\int_{0}^{1} t^{\alpha-1}\left[f\left(\frac{t}{2}b + \frac{2-t}{2}a\right) + f\left(\frac{t}{2}a + \frac{2-t}{2}b\right)\right].$$

If we apply the variable's change $x = \frac{t}{2}b + \frac{2-t}{2}a$ and $y = \frac{t}{2}a + \frac{2-t}{2}b$ for $t \in [0,1]$, then the equality (3.3) can be rewritten as follows

$$I_1 + I_2 = \frac{1}{20\,(b-a)}\left[27f\left(\frac{5a+b}{6}\right) + 26f\left(\frac{a+b}{2}\right) + 27f\left(\frac{a+5b}{6}\right)\right] \tag{3.4}$$
$$- \frac{2^{\alpha-1}\Gamma\,(\alpha+1)}{(b-a)^\alpha}\left[\,J^\alpha_{\frac{a+b}{2}-}f\,(a) + J^\alpha_{\frac{a+b}{2}+}f\,(b)\right].$$

Consequently, multiplying both sides of (3.4) by $\frac{b-a}{4}$ concludes the proof of Lemma 3.1. □

4. Corrected Euler-Maclaurin-type inequalities for convex functions

Theorem 4.1. *Taking into account that Lemma 3.1 holds and the function $|f'|$ is convex on the interval $[a,b]$, the fractional corrected Euler-Maclaurin-type inequality can be established by*

$$\Big|\frac{1}{80}\Big[27f\Big(\frac{5a+b}{6}\Big) + 26f\Big(\frac{a+b}{2}\Big) + 27f\Big(\frac{a+5b}{6}\Big)\Big]$$
$$- \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\Big[J^\alpha_{\frac{a+b}{2}-}f(a) + J^\alpha_{\frac{a+b}{2}+}f(b)\Big]\Big| \tag{4.1}$$
$$\leq \frac{b-a}{4}\big(\Omega_1(\alpha) + \Omega_2(\alpha)\big)\big[|f'(a)| + |f'(b)|\big].$$

Here,

$$\Omega_1\,(\alpha) = \int_{0}^{\frac{1}{3}} t^\alpha dt = \frac{1}{\alpha+1}\left(\frac{1}{3}\right)^{\alpha+1},$$

and

$$\Omega_2(\alpha)=\int\limits_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{27}{40}\right|dt=\begin{cases}\frac{1}{\alpha+1}\left[1-\left(\frac{1}{3}\right)^{\alpha+1}\right]-\frac{9}{20}, & 0<\alpha\leq\frac{\ln\left(\frac{27}{40}\right)}{\ln\left(\frac{1}{3}\right)},\\ \frac{2\alpha}{\alpha+1}\left(\frac{27}{40}\right)^{1+\frac{1}{\alpha}}+\frac{1}{\alpha+1}\left[1+\left(\frac{1}{3}\right)^{\alpha+1}\right]-\frac{9}{10}, & \frac{\ln\left(\frac{27}{40}\right)}{\ln\left(\frac{1}{3}\right)}<\alpha.\end{cases}$$

Proof. Taking into account the absolute value in Lemma 3.1, we may obtain

$$\begin{aligned}&\Big|\frac{1}{80}\Big[27f\Big(\frac{5a+b}{6}\Big)+26f\Big(\frac{a+b}{2}\Big)+27f\Big(\frac{a+5b}{6}\Big)\Big]\\&-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\Big[J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)\Big]\Big|\end{aligned}\tag{4.2}$$

$$\begin{aligned}\leq&\frac{b-a}{4}\Big\{\int_0^{\frac{1}{3}}|t^{\alpha}|\left|f'\big(\tfrac{t}{2}b+\tfrac{2-t}{2}a\big)-f'\big(\tfrac{t}{2}a+\tfrac{2-t}{2}b\big)\right|dt\\&+\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{27}{40}\right|\left|f'\big(\tfrac{t}{2}b+\tfrac{2-t}{2}a\big)-f'\big(\tfrac{t}{2}a+\tfrac{2-t}{2}b\big)\right|dt\Big\}.\end{aligned}$$

Since $|f'|$ is convex, we have

$$\begin{aligned}&\left|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]\right.\\&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\right|\\\leq&\frac{b-a}{4}\left\{\int\limits_0^{\frac{1}{3}}t^{\alpha}\left[\frac{t}{2}|f'(b)|+\frac{2-t}{2}|f'(a)|+\frac{t}{2}|f'(a)|+\frac{2-t}{2}|f'(b)|\right]dt\right.\\&\left.+\int\limits_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{27}{40}\right|\left[\frac{t}{2}|f'(b)|+\frac{2-t}{2}|f'(a)|+\frac{t}{2}|f'(a)|+\frac{2-t}{2}|f'(b)|\right]dt\right\}\\=&\frac{b-a}{4}\left(\Omega_1(\alpha)+\Omega_2(\alpha)\right)\left[|f'(a)|+|f'(b)|\right].\end{aligned}$$

□

Remark 4.1. If we choose $\alpha=1$ in Theorem 4.1, then we can obtain corrected Euler-Maclaurin-type inequality

$$\begin{aligned}&\left|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int\limits_a^b f(t)\,dt\right|\\\leq&\frac{2401}{57600}(b-a)\left[|f'(a)|+|f'(b)|\right],\end{aligned}$$

which is established in paper [16].

Theorem 4.2. *Let the hypotheses of Lemma 3.1 hold and let the function $|f'|^q$, $q>1$ is convex on $[a,b]$. Then, the following corrected Euler-Maclaurin-type inequality holds:*

$$\begin{aligned}&\left|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]\right.\\&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\right| \qquad (4.3)\\ \leq&\frac{b-a}{4}\left\{\left(\frac{1}{\alpha p+1}\left(\frac{1}{3}\right)^{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|f'(b)|^q+11|f'(a)|^q}{36}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+11|f'(b)|^q}{36}\right)^{\frac{1}{q}}\right]\right.\\&\left.+\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{27}{40}\right|^p dt\right)^{\frac{1}{p}}\left[\left(\frac{2|f'(b)|^q+4|f'(a)|^q}{9}\right)^{\frac{1}{q}}+\left(\frac{2|f'(a)|^q+4|f'(b)|^q}{9}\right)^{\frac{1}{q}}\right]\right\}.\end{aligned}$$

Here, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By utilizing Hölder's inequality to (4.2), we have

$$\begin{aligned}&\left|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]\right.\\&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\right|\\ \leq&\frac{b-a}{4}\left\{\left(\int_0^{\frac{1}{3}}|t^{\alpha}|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{3}}\left|f'\left(\frac{t}{2}b+\frac{2-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}\right.\\&+\left(\int_0^{\frac{1}{3}}|t^{\alpha}|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{3}}\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\\&+\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{27}{40}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{3}}^{1}\left|f'\left(\frac{t}{2}b+\frac{2-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}\\&\left.+\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{27}{40}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{3}}^{1}\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

Making use of the convexity $|f'|^q$, we can easily find

$$\begin{aligned}&\left|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]\right.\\&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\right|\\ \leq&\frac{b-a}{4}\left\{\left(\int_0^{\frac{1}{3}}t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{3}}\left(\frac{t}{2}|f'(b)|^q+\frac{2-t}{2}|f'(a)|^q\right)dt\right)^{\frac{1}{q}}\right.\end{aligned}$$

$$+\left(\int_0^{\frac{1}{3}} t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{3}}\left(\frac{t}{2}\left|f'(a)\right|^q+\frac{2-t}{2}\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{27}{40}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{3}}^{1}\left(\frac{t}{2}\left|f'(b)\right|^q+\frac{2-t}{2}\left|f'(a)\right|^q\right)dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{27}{40}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{3}}^{1}\left(\frac{t}{2}\left|f'(a)\right|^q+\frac{2-t}{2}\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\Bigg\}$$

$$=\frac{b-a}{4}\left\{\left(\frac{1}{\alpha p+1}\left(\frac{1}{3}\right)^{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{\left|f'(b)\right|^q+11\left|f'(a)\right|^q}{36}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+11\left|f'(b)\right|^q}{36}\right)^{\frac{1}{q}}\right]\right.$$

$$\left.+\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{27}{40}\right|^p dt\right)^{\frac{1}{p}}\left[\left(\frac{2\left|f'(b)\right|^q+4\left|f'(a)\right|^q}{9}\right)^{\frac{1}{q}}+\left(\frac{2\left|f'(a)\right|^q+4\left|f'(b)\right|^q}{9}\right)^{\frac{1}{q}}\right]\right\}.$$

□

Remark 4.2. Note that $\alpha = 1$ in Theorem 4.2. Then, we readily have

$$\left|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$

$$\leq\frac{b-a}{4}\left\{\left(\frac{1}{p+1}\left(\frac{1}{3}\right)^{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{\left|f'(b)\right|^q+11\left|f'(a)\right|^q}{36}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+11\left|f'(b)\right|^q}{36}\right)^{\frac{1}{q}}\right]\right.$$

$$+\left(\frac{1}{p+1}\left[\left(\frac{13}{40}\right)^{p+1}+\left(\frac{41}{120}\right)^{p+1}\right]\right)^{\frac{1}{p}}$$

$$\left.\times\left[\left(\frac{2\left|f'(b)\right|^q+4\left|f'(a)\right|^q}{9}\right)^{\frac{1}{q}}+\left(\frac{2\left|f'(a)\right|^q+4\left|f'(b)\right|^q}{9}\right)^{\frac{1}{q}}\right]\right\},$$

which is proved in paper [16].

Theorem 4.3. *Assume that Lemma 3.1's assumptions are true and the function* $|f'|^q$, $q \geq 1$ *is convex on* $[a,b]$. *Next, we have the following inequality of the corrected Euler-Maclaurin-type inequality*

$$\left|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]\right.$$
$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\right| \tag{4.4}$$
$$\leq\frac{b-a}{4}\left\{(\Omega_1(\alpha))^{1-\frac{1}{q}}\left[\left[\Omega_3(\alpha)\left|f'(b)\right|^q+\Omega_4(\alpha)\left|f'(a)\right|^q\right]^{\frac{1}{q}}\right.\right.$$
$$\left.+\left[\Omega_3(\alpha)\left|f'(a)\right|^q+\Omega_4(\alpha)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right]$$
$$+(\Omega_2(\alpha))^{1-\frac{1}{q}}\left[\left[\Omega_5(\alpha)\left|f'(b)\right|^q+\Omega_6(\alpha)\left|f'(a)\right|^q\right]^{\frac{1}{q}}\right.$$

$$+\left[\Omega_5(\alpha)\left|f'(a)\right|^q+\Omega_6(\alpha)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\Bigg]\Bigg\}.$$

Here, $\Omega_1(\alpha)$ *and* $\Omega_2(\alpha)$ *are specified in Theorem 4.1 and*

$$\Omega_3(\alpha)=\int_0^{\frac{1}{3}}\frac{t}{2}t^{\alpha}dt=\frac{1}{2}\frac{1}{(\alpha+2)}\left(\frac{1}{3}\right)^{\alpha+2},$$

$$\Omega_4(\alpha)=\int_0^{\frac{1}{3}}\frac{2-t}{2}t^{\alpha}dt=\frac{1}{\alpha+1}\left(\frac{1}{3}\right)^{\alpha+1}-\frac{1}{2}\frac{1}{(\alpha+2)}\left(\frac{1}{3}\right)^{\alpha+2},$$

$$\begin{aligned}\Omega_5(\alpha)&=\int_{\frac{1}{3}}^{1}\frac{t}{2}\left|t^{\alpha}-\frac{27}{40}\right|dt\\&=\begin{cases}\frac{1}{2(\alpha+2)}\left(1-\left(\frac{1}{3}\right)^{\alpha+2}\right)-\frac{3}{20}, & 0<\alpha<\frac{\ln\left(\frac{27}{40}\right)}{\ln\left(\frac{1}{3}\right)},\\ \frac{1}{2(\alpha+2)}\left(1+\left(\frac{1}{3}\right)^{\alpha+2}\right)+\frac{\alpha}{2(\alpha+2)}\left(\frac{27}{40}\right)^{\frac{\alpha+2}{\alpha}}-\frac{3}{16}, & \frac{\ln\left(\frac{27}{40}\right)}{\ln\left(\frac{1}{3}\right)}<\alpha,\end{cases}\end{aligned}$$

$$\begin{aligned}\Omega_6(\alpha)&=\int_{\frac{1}{3}}^{1}\left(\frac{2-t}{2}\right)\left|t^{\alpha}-\frac{27}{40}\right|dt\\&=\begin{cases}\frac{1}{\alpha+1}\left(1-\left(\frac{1}{3}\right)^{\alpha+1}\right)-\frac{1}{2(\alpha+2)}\left(1-\left(\frac{1}{3}\right)^{\alpha+2}\right)-\frac{3}{10}, & 0<\alpha<\frac{\ln\left(\frac{27}{40}\right)}{\ln\left(\frac{1}{3}\right)},\\ \frac{2\alpha}{\alpha+1}\left(\frac{27}{40}\right)^{\frac{\alpha+1}{\alpha}}+\frac{1}{\alpha+1}\left(1+\left(\frac{1}{3}\right)^{\alpha+1}\right)\\ \quad-\frac{1}{2(\alpha+2)}\left(1+\left(\frac{1}{3}\right)^{\alpha+2}\right)-\frac{57}{80}-\frac{\alpha}{2(\alpha+2)}\left(\frac{27}{40}\right)^{\frac{\alpha+2}{\alpha}}, & \frac{\ln\left(\frac{27}{40}\right)}{\ln\left(\frac{1}{3}\right)}<\alpha.\end{cases}\end{aligned}$$

Proof. When we first apply (4.2) to the power-mean inequality, we obtain

$$\begin{aligned}&\left|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]\right.\\&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\right|\\\leq&\frac{b-a}{4}\left\{\left(\int_0^{\frac{1}{3}}\left|t^{\alpha}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{3}}\left|t^{\alpha}\right|\left|f'\left(\frac{t}{2}b+\frac{2-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}\right.\\&+\left(\int_0^{\frac{1}{3}}\left|t^{\alpha}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{3}}\left|t^{\alpha}\right|\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\\&+\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{27}{40}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-1\right|\left|f'\left(\frac{t}{2}b+\frac{2-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}\end{aligned}$$

$$+\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{27}{40}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{27}{40}\right|\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^{q}dt\right)^{\frac{1}{q}}\Bigg\}.$$

Making use of the convexity of $|f'|^q$, it generates

$$\begin{aligned}
&\Bigg|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]\\
&-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\Bigg|\\
\leq&\frac{b-a}{4}\Bigg\{\left(\int_{0}^{\frac{1}{3}}|t^{\alpha}|\,dt\right)^{1-\frac{1}{q}}\left(\int_{0}^{\frac{1}{3}}|t^{\alpha}|\left[\left(\frac{t}{2}\right)|f'(b)|^{q}+\left(\frac{2-t}{2}\right)|f'(a)|^{q}\right]dt\right)^{\frac{1}{q}}\\
&+\left(\int_{0}^{\frac{1}{3}}|t^{\alpha}|\,dt\right)^{1-\frac{1}{q}}\left(\int_{0}^{\frac{1}{3}}|t^{\alpha}|\left[\left(\frac{t}{2}\right)|f'(a)|^{q}+\left(\frac{2-t}{2}\right)|f'(b)|^{q}\right]dt\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{3}}^{1}\left(t^{\alpha}-\frac{27}{40}\right)dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{1}|t^{\alpha}-1|\left[\left(\frac{t}{2}\right)|f'(b)|^{q}+\left(\frac{2-t}{2}\right)|f'(a)|^{q}\right]dt\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{3}}^{1}\left(t^{\alpha}-\frac{27}{40}\right)dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{1}|t^{\alpha}-1|\left[\left(\frac{1+t}{2}\right)|f'(a)|^{q}+\left(\frac{1-t}{2}\right)|f'(b)|^{q}\right]dt\right)^{\frac{1}{q}}\Bigg\}.
\end{aligned}$$

□

Corollary 4.1. *If we assign $\alpha = 1$ in Theorem 4.3, then we have the following corrected Euler-Maclaurin-type inequality*

$$\begin{aligned}
&\left|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\right|\\
\leq&\frac{b-a}{4}\Bigg\{\left(\frac{1601}{14\,400}\right)^{1-\frac{1}{q}}\Bigg[\left[\frac{1961\,791}{20\,736\,000}|f'(b)|^{q}+\frac{343\,649}{20\,736\,000}|f'(a)|^{q}\right]^{\frac{1}{q}}\\
&+\left[\frac{1961\,791}{20\,736\,000}|f'(a)|^{q}+\frac{343\,649}{20\,736\,000}|f'(b)|^{q}\right]^{\frac{1}{q}}\Bigg]\\
&+\left(\frac{1}{18}\right)^{1-\frac{1}{q}}\Bigg[\left[\frac{4}{81}|f'(b)|^{q}+\frac{1}{162}|f'(a)|^{q}\right]^{\frac{1}{q}}\\
&+\left[\frac{4}{81}|f'(a)|^{q}+\frac{1}{162}|f'(b)|^{q}\right]^{\frac{1}{q}}\Bigg]\Bigg\}.
\end{aligned}$$

5. Corrected Euler-Maclaurin-type inequalities for bounded functions

This section uses fractional integrals to address some corrected Euler-Maclaurin-type inequalities for bounded functions.

Theorem 5.1. *Note that the conditions of Lemma 3.1 hold. If there exist* $m, M \in \mathbb{R}$ *such that* $m \leq f'(t) \leq M$ *for* $t \in [a,b]$, *then it follows*

$$\begin{aligned}&\Bigg|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]\\&-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\Bigg|\\&\leq\frac{b-a}{4}\{\Omega_1(\alpha)+\Omega_2(\alpha)\}(M-m).\end{aligned} \tag{5.1}$$

Proof. By using Lemma 3.1, we have

$$\begin{aligned}&\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]\\&-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\\&=\frac{b-a}{4}\Bigg\{\int_0^{\frac{1}{3}}t^{\alpha}\left[f'\left(\frac{t}{2}b+\frac{2-t}{2}a\right)-\frac{m+M}{2}\right]dt\\&+\int_0^{\frac{1}{3}}t^{\alpha}\left[\frac{m+M}{2}-f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right]dt\\&+\int_{\frac{1}{3}}^{1}\left(t^{\alpha}-\frac{27}{40}\right)\left[f'\left(\frac{t}{2}b+\frac{2-t}{2}a\right)-\frac{m+M}{2}\right]dt\\&+\int_{\frac{1}{3}}^{1}\left(t^{\alpha}-\frac{27}{40}\right)\left[\frac{m+M}{2}-f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right]dt\Bigg\}.\end{aligned} \tag{5.2}$$

By virtue of the absolute value of (5.2), we get

$$\begin{aligned}&\Bigg|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]\\&-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f\left(\frac{a+b}{2}\right)+J^{\alpha}_{b-}f\left(\frac{a+b}{2}\right)\right]\Bigg|\\&\leq\frac{b-a}{4}\Bigg\{\int_0^{\frac{1}{3}}|t^{\alpha}|\left|f'\left(\frac{t}{2}b+\frac{2-t}{2}a\right)-\frac{m+M}{2}\right|dt\\&+\int_0^{\frac{1}{3}}|t^{\alpha}|\left|\frac{m+M}{2}-f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt\end{aligned}$$

$$+\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{27}{40}\right|\left|f'\left(\frac{t}{2}b+\frac{2-t}{2}a\right)-\frac{m+M}{2}\right|dt$$

$$\left.+\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{27}{40}\right|\left|\frac{m+M}{2}-f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt\right\}.$$

It is known that $m\leq f'(t)\leq M$ for $t\in[a,b]$. Then, one can obtain

$$\left|f'\left(\frac{t}{2}b+\frac{2-t}{2}a\right)-\frac{m+M}{2}\right|\leq\frac{M-m}{2}, \tag{5.3}$$

$$\left|\frac{m+M}{2}-f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|\leq\frac{M-m}{2}. \tag{5.4}$$

With the aid of (5.3) and (5.4), we get

$$\begin{aligned}&\left|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]\right.\\&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\frac{a+b}{2}-}^{\alpha}f(a)+J_{\frac{a+b}{2}+}^{\alpha}f(b)\right]\right|\\&\leq\frac{b-a}{4}(M-m)\left\{\int_{0}^{\frac{1}{3}}t^{\alpha}dt+\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{27}{40}\right|dt\right\}\\&=\frac{b-a}{4}\{\Omega_{1}(\alpha)+\Omega_{2}(\alpha)\}(M-m).\end{aligned}$$

□

Corollary 5.1. *If we select $\alpha=1$ in Theorem 5.1, then we get*

$$\begin{aligned}&\left|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_{a}^{b}f(t)dt\right|\\&\leq\frac{2401}{57\,600}(b-a)(M-m).\end{aligned}$$

Corollary 5.2. *Assume that the Theorem 5.1 holds. If there exists $M\in\mathbb{R}^{+}$ such that $|f'(t)|\leq M$ for all $t\in[a,b]$, then we have*

$$\begin{aligned}&\left|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]\right.\\&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\frac{a+b}{2}-}^{\alpha}f(a)+J_{\frac{a+b}{2}+}^{\alpha}f(b)\right]\right|\\&\leq\frac{b-a}{2}\{\Omega_{1}(\alpha)+\Omega_{2}(\alpha)\}M.\end{aligned}$$

Corollary 5.3. *Let us think about $\alpha=1$ in Corollary 5.2. Then, the following inequality holds:*

$$\left|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_{a}^{b}f(t)dt\right|$$

$$\leq \frac{2401\,(b-a)}{28\,800} M.$$

6. Euler-Maclaurin-type inequalities for Lipschitzian functions

Some fractional corrected Euler-Maclaurin-type inequalities for Lipschitzian functions are provided in this section.

Theorem 6.1. *Assume that Lemma 3.1's assumptions are true. If f' is a L-Lipschitzian function on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}
&\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right] \\
&-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)\right] \\
\leq &\frac{(b-a)^2}{4} L\left\{\Omega_7(\alpha)+\Omega_8(\alpha)\right\}.
\end{aligned}$$

Here,

$$\Omega_7(\alpha)=\int_0^{\frac{1}{3}} t^{\alpha}(1-t)\,dt=\frac{1}{\alpha+1}\left(\frac{1}{3}\right)^{\alpha+1}-\frac{1}{\alpha+2}\left(\frac{1}{3}\right)^{\alpha+2}$$

and

$$\begin{aligned}
\Omega_8(\alpha)&=\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{27}{40}\right|(1-t)\,dt \\
&=\begin{cases}
\frac{1}{\alpha+1}\left(1-\left(\frac{1}{3}\right)^{\alpha+1}\right)-\frac{1}{\alpha+2}\left(1-\left(\frac{1}{3}\right)^{\alpha+2}\right)-\frac{3}{20}, & 0<\alpha<\frac{\ln\left(\frac{27}{40}\right)}{\ln\left(\frac{1}{3}\right)}, \\[2ex]
\frac{2\alpha}{\alpha+1}\left(\frac{27}{40}\right)^{\frac{\alpha+1}{\alpha}}-\frac{\alpha}{\alpha+2}\left(\frac{27}{40}\right)^{\frac{\alpha+2}{\alpha}}+\frac{1}{\alpha+1}\left(1+\left(\frac{1}{3}\right)^{\alpha+1}\right) & \frac{\ln\left(\frac{27}{40}\right)}{\ln\left(\frac{1}{3}\right)}<\alpha. \\
-\frac{1}{\alpha+2}\left(1+\left(\frac{1}{3}\right)^{\alpha+2}\right)-\frac{21}{40}, &
\end{cases}
\end{aligned}$$

Proof. With the aid of Lemma 3.1 and since f' is L-Lipschitzian function, we have

$$\begin{aligned}
&\left|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]\right. \\
&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\right| \\
\leq &\frac{b-a}{4}\left\{\int_0^{\frac{1}{3}}\left|t^{\alpha}\right|\left|f'\left(\frac{t}{2}b+\frac{2-t}{2}a\right)-f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt\right. \\
&\left.+\int_{\frac{1}{3}}^{1}\left|t^{\alpha}-\frac{27}{40}\right|\left|f'\left(\frac{t}{2}b+\frac{2-t}{2}a\right)-f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt\right\}
\end{aligned}$$

$$\leq \frac{b-a}{4}\left\{ \int_{0}^{\frac{1}{3}} |t^{\alpha}| L(1-t)(b-a)\,dt + \int_{\frac{1}{3}}^{1} \left| t^{\alpha} - \frac{27}{40} \right| L(1-t)(b-a)\,dt \right\}$$
$$= \frac{(b-a)^2}{4} L \left\{ \Omega_7(\alpha) + \Omega_8(\alpha) \right\}.$$

□

Corollary 6.1. *Consider $\alpha = 1$ in Theorem 6.1. Then, the following corrected Euler-Maclaurin-type inequalities holds:*

$$\left| \frac{1}{80}\left[27 f\left(\frac{5a+b}{6}\right) + 26 f\left(\frac{a+b}{2}\right) + 27 f\left(\frac{a+5b}{6}\right) \right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J^{\alpha}_{\frac{a+b}{2}-} f(a) + J^{\alpha}_{\frac{a+b}{2}+} f(b) \right] \right|$$
$$\leq \frac{420\,919}{20\,736\,000} (b-a)^2 L.$$

7. Corrected Euler-Maclaurin-type inequalities for functions of bounded variation

In this section, we represent some fractional corrected Euler-Maclaurin-type inequalities for functions of bounded variation.

Theorem 7.1. *Consider that $f : [a,b] \to \mathbb{R}$ is a function of bounded variation on $[a,b]$. Then, we have*

$$\frac{1}{80}\left[27 f\left(\frac{5a+b}{6}\right) + 26 f\left(\frac{a+b}{2}\right) + 27 f\left(\frac{a+5b}{6}\right) \right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J^{\alpha}_{\frac{a+b}{2}-} f(a) + J^{\alpha}_{\frac{a+b}{2}+} f(b) \right]$$
$$\leq \frac{1}{2} \max\left\{ \left(\frac{1}{3}\right)^{\alpha}, \frac{13}{40}, \left| \left(\frac{1}{3}\right)^{\alpha} - \frac{27}{40} \right| \right\} \bigvee_{a}^{b}(f).$$

Here, $\bigvee_{a}^{b}(f)$ denotes the total variation of f on $[a,b]$.

Proof. Define the function $K_{\alpha}(x)$ by

$$K_{\alpha}(x) = \begin{cases} (x-a)^{\alpha}, & a \leq x < \frac{5a+b}{6}, \\ (x-a)^{\alpha} - \frac{27}{40}\left(\frac{b-a}{2}\right)^{\alpha}, & \frac{5a+b}{6} \leq x < \frac{a+b}{2}, \\ \frac{27}{40}\left(\frac{b-a}{2}\right)^{\alpha} - (b-x)^{\alpha}, & \frac{a+b}{2} \leq x < \frac{a+5b}{6}, \\ -(b-x)^{\alpha}, & \frac{a+5b}{6} \leq x \leq b. \end{cases}$$

With the aid of the integrating by parts, we have

$$\int_a^b K_\alpha(x)df(x) \tag{7.1}$$

$$= \int_a^{\frac{5a+b}{6}} (x-a)^\alpha\, df(x) + \int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - \frac{27}{40}\left(\frac{b-a}{2}\right)^\alpha\right] df(x)$$

$$+ \int_{\frac{a+b}{2}}^{\frac{a+5b}{6}} \left[\frac{27}{40}\left(\frac{b-a}{2}\right)^\alpha - (b-x)^\alpha\right] df(x) + \int_{\frac{a+5b}{6}}^{b} \left[-(b-x)^\alpha\right] df(x)$$

$$= (x-a)^\alpha f(x)\Big|_a^{\frac{5a+b}{6}} - \alpha \int_a^{\frac{5a+b}{6}} (x-a)^{\alpha-1} f(x)dx$$

$$+ \left[(x-a)^\alpha - \frac{27}{40}\left(\frac{b-a}{2}\right)^\alpha\right] f(x)\Big|_{\frac{5a+b}{6}}^{\frac{a+b}{2}} - \alpha \int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} (x-a)^{\alpha-1} f(x)dx$$

$$+ \left[\frac{27}{40}\left(\frac{b-a}{2}\right)^\alpha - (b-x)^\alpha\right] f(x)\Big|_{\frac{a+b}{2}}^{\frac{a+5b}{6}} - \alpha \int_{\frac{a+b}{2}}^{\frac{a+5b}{6}} (b-x)^{\alpha-1} f(x)dx$$

$$+ \left[-(b-x)^\alpha\right] f(x)\Big|_{\frac{a+5b}{6}}^{b} - \alpha \int_{\frac{a+5b}{6}}^{b} (b-x)^{\alpha-1} f(x)dx$$

$$= \frac{27}{40}\left(\frac{b-a}{2}\right)^\alpha f\left(\frac{5a+b}{6}\right) + \frac{27}{40}\left(\frac{b-a}{2}\right)^\alpha f\left(\frac{a+5b}{6}\right)$$

$$+ \frac{13}{20}\left(\frac{b-a}{2}\right)^\alpha f\left(\frac{a+b}{2}\right) - \alpha \int_a^{\frac{a+b}{2}} (x-a)^{\alpha-1} f(x)dx - \alpha \int_{\frac{a+b}{2}}^{b} (b-x)^{\alpha-1} f(x)dx$$

$$= \frac{(b-a)^\alpha}{2^{\alpha-1}} \frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right) + 26f\left(\frac{a+b}{2}\right) + 27f\left(\frac{a+5b}{6}\right)\right]$$

$$- \Gamma(\alpha+1)\left[J^\alpha_{\frac{a+b}{2}-} f(a) + J^\alpha_{\frac{a+b}{2}+} f(b)\right].$$

In other words, we have

$$\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right) + 26f\left(\frac{a+b}{2}\right) + 27f\left(\frac{a+5b}{6}\right)\right]$$

$$- \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\frac{a+b}{2}-} f(a) + J^\alpha_{\frac{a+b}{2}+} f(b)\right]$$

$$= \frac{2^{\alpha-1}}{(b-a)^\alpha} \int_a^b K_\alpha(x)df(x).$$

It is known that if $f, \mathfrak{G} : [a,b] \to \mathbb{R}$ are such that $\mathfrak{G}$ is continuous on $[a,b]$ and f is of bounded variation on $[a,b]$, then $\int_a^b \mathfrak{G}(t)df(t)$ exists and

$$\left| \int_a^b \mathfrak{G}(t)df(t) \right| \leq \sup_{t\in[a,b]} |g(t)| \bigvee_a^b (f). \tag{7.2}$$

By using (7.2), it yields

$$\begin{aligned}
&\left| \frac{1}{80} \left[27f\left(\frac{5a+b}{6}\right) + 26f\left(\frac{a+b}{2}\right) + 27f\left(\frac{a+5b}{6}\right) \right] \right. \\
&\left. - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J^{\alpha}_{\frac{a+b}{2}-} f(a) + J^{\alpha}_{\frac{a+b}{2}+} f(b) \right] \right| \\
=& \frac{2^{\alpha-1}}{(b-a)^{\alpha}} \left| \int_a^b K_{\alpha}(x) df(x) \right| \\
\leq& \frac{2^{\alpha-1}}{(b-a)^{\alpha}} \left\{ \left| \int_a^{\frac{5a+b}{6}} (x-a)^{\alpha} df(x) \right| \right. \\
&+ \left| \int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} \left[(x-a)^{\alpha} - \frac{27}{40}\left(\frac{b-a}{2}\right)^{\alpha} \right] df(x) \right| \\
&+ \left| \int_{\frac{a+b}{2}}^{\frac{a+5b}{6}} \left[\frac{27}{40}\left(\frac{b-a}{2}\right)^{\alpha} - (b-x)^{\alpha} \right] df(x) \right| \\
&\left. + \left| \int_{\frac{a+5b}{6}}^{b} \left[-(b-x)^{\alpha} \right] df(x) \right| \right\} \\
\leq& \frac{2^{\alpha-1}}{(b-a)^{\alpha}} \left\{ \sup_{x\in\left[a,\frac{5a+b}{6}\right]} \left| (x-a)^{\alpha} \right| \bigvee_a^{\frac{5a+b}{6}} (f) \right. \\
&+ \sup_{x\in\left[\frac{5a+b}{6},\frac{a+b}{2}\right]} \left| (x-a)^{\alpha} - \frac{27}{40}\left(\frac{b-a}{2}\right)^{\alpha} \right| \bigvee_{\frac{5a+b}{6}}^{\frac{a+b}{2}} (f) \\
&+ \sup_{x\in\left[\frac{a+b}{2},\frac{a+5b}{6}\right]} \left| \frac{27}{40}\left(\frac{b-a}{2}\right)^{\alpha} - (b-x)^{\alpha} \right| \bigvee_{\frac{a+b}{2}}^{\frac{a+5b}{6}} (f) \\
&\left. + \sup_{x\in\left[\frac{a+5b}{6},b\right]} \left| -(b-x)^{\alpha} \right| \bigvee_{\frac{a+5b}{6}}^{b} (f) \right\}
\end{aligned}$$

$$
\begin{aligned}
=&\frac{2^{\alpha-1}}{(b-a)^{\alpha}}\left\{\left(\frac{b-a}{6}\right)^{\alpha}\bigvee_{a}^{\frac{5a+b}{6}}(f)\right.\\
&+\max\left\{\frac{13}{40}\left(\frac{b-a}{2}\right)^{\alpha},\left|\left(\frac{b-a}{6}\right)^{\alpha}-\frac{27}{40}\left(\frac{b-a}{2}\right)^{\alpha}\right|\right\}\bigvee_{\frac{5a+b}{6}}^{\frac{a+b}{2}}(f)\\
&+\max\left\{\left|\frac{27}{40}\left(\frac{b-a}{2}\right)^{\alpha}-\left(\frac{b-a}{6}\right)^{\alpha}\right|,\frac{13}{40}\left(\frac{b-a}{2}\right)^{\alpha}\right\}\bigvee_{\frac{a+b}{2}}^{\frac{a+5b}{6}}(f)\\
&\left.+\left(\frac{b-a}{6}\right)^{\alpha}\bigvee_{\frac{a+5b}{6}}^{b}(f)\right\}\\
=&\frac{1}{2}\left\{\left(\frac{1}{3}\right)^{\alpha}\bigvee_{a}^{\frac{5a+b}{6}}(f)+\max\left\{\frac{13}{40},\left|\left(\frac{1}{3}\right)^{\alpha}-\frac{27}{40}\right|\right\}\bigvee_{\frac{5a+b}{6}}^{\frac{a+b}{2}}(f)\right.\\
&\left.+\max\left\{\left|\left(\frac{1}{3}\right)^{\alpha}-\frac{27}{40}\right|,\frac{13}{40}\right\}\bigvee_{\frac{a+b}{2}}^{\frac{a+5b}{6}}(f)+\left(\frac{1}{3}\right)^{\alpha}\bigvee_{\frac{a+5b}{6}}^{b}(f)\right\}\\
\leq&\frac{1}{2}\max\left\{\left(\frac{1}{3}\right)^{\alpha},\frac{13}{40},\left|\left(\frac{1}{3}\right)^{\alpha}-\frac{27}{40}\right|\right\}\bigvee_{a}^{b}(f).
\end{aligned}
$$

□

Corollary 7.1. *Let us consider $\alpha = 1$ in Theorem 7.1. Then, the following inequality holds:*

$$
\begin{aligned}
&\left|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_{a}^{b}f(t)dt\right|\\
\leq&\frac{41}{240}\bigvee_{a}^{b}(f).
\end{aligned}
$$

8. Examples

Example 8.1. If a function $f : [a, b] = [0, 2] \to \mathbb{R}$ is defined by $f(x) = x^2$ with $\alpha \in (0, 5]$, then the left-hand side of (4.1) coincides with

$$
\begin{aligned}
&\left|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]\right.\\
&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\right|\\
=&\left|\frac{1}{80}\left[27f\left(\frac{1}{3}\right)+26f(1)+27f\left(\frac{5}{3}\right)\right]-\frac{\Gamma(\alpha+1)}{2}\left[J^{\alpha}_{1-}f(0)+J^{\alpha}_{1+}f(2)\right]\right|\\
=&\left|\frac{13}{10}-\frac{\Gamma(\alpha+1)}{2}\left[\frac{1}{\Gamma(\alpha)}\left(\int_{0}^{1}t^{\alpha+1}dt+\int_{1}^{2}(2-t)^{\alpha-1}t^{2}dt\right)\right]\right|
\end{aligned}
$$

$$= \left| \frac{13}{10} - \frac{\alpha}{2} \left[\frac{2}{\alpha+2} - \frac{4}{\alpha+1} + \frac{1}{\alpha} \right] \right|. \tag{8.1}$$

The right-hand side of (4.1) becomes to

$$2\left(\Omega_1(\alpha) + \Omega_2(\alpha)\right) = \begin{cases} \frac{2}{\alpha+1} - \frac{9}{10}, & 0 < \alpha \leq \frac{\ln\left(\frac{27}{40}\right)}{\ln\left(\frac{1}{3}\right)}, \\ \frac{4\alpha}{\alpha+1}\left(\frac{27}{40}\right)^{1+\frac{1}{\alpha}} + \frac{2}{\alpha+1}\left[1 + 2\left(\frac{1}{3}\right)^{\alpha+1}\right] - \frac{9}{5}, & \frac{\ln\left(\frac{27}{40}\right)}{\ln\left(\frac{1}{3}\right)} < \alpha \leq 5. \end{cases}$$

Finally, we get

$$\begin{cases} \left| \frac{13}{10} - \frac{\alpha}{2} \left[\frac{2}{\alpha+2} - \frac{4}{\alpha+1} + \frac{1}{\alpha} \right] \right| \leq \frac{2}{\alpha+1} - \frac{9}{10}, & 0 < \alpha \leq \frac{\ln\left(\frac{27}{40}\right)}{\ln\left(\frac{1}{3}\right)}, \\ \left| \frac{13}{10} - \frac{\alpha}{2} \left[\frac{2}{\alpha+2} - \frac{4}{\alpha+1} + \frac{1}{\alpha} \right] \right| \\ \leq \frac{4\alpha}{\alpha+1}\left(\frac{27}{40}\right)^{1+\frac{1}{\alpha}} + \frac{2}{\alpha+1}\left[1 + 2\left(\frac{1}{3}\right)^{\alpha+1}\right] - \frac{9}{5}, & \frac{\ln\left(\frac{27}{40}\right)}{\ln\left(\frac{1}{3}\right)} < \alpha \leq 5. \end{cases}$$

The left-hand side of (4.1) in Example 8.1 is regularly below the right-hand side of (4.1), as seen in Figure 1 and Figure 2, for all values of $\alpha \in (0,5]$.

Example 8.2. Note that a function $f : [a,b] = [0,2] \to \mathbb{R}$ given by $f(x) = x^2$. From Theorem 4.2 with $\alpha \in (0,5]$ and $p = q = 2$, the left-hand side of (4.3) reduces to equality (8.1) and the right hand-side of (4.3) equals to

$$\begin{aligned} &\frac{b-a}{4} \left\{ \left(\frac{1}{\alpha p+1} \left(\frac{1}{3}\right)^{\alpha p+1} \right)^{\frac{1}{p}} \left[\left(\frac{|f'(b)|^q + 11|f'(a)|^q}{36} \right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q + 11|f'(b)|^q}{36} \right)^{\frac{1}{q}} \right] \right. \\ &\left. + \left(\int_{\frac{1}{3}}^{1} \left| t^\alpha - \frac{27}{40} \right|^p dt \right)^{\frac{1}{p}} \left[\left(\frac{2|f'(b)|^q + 4|f'(a)|^q}{9} \right)^{\frac{1}{q}} + \left(\frac{2|f'(a)|^q + 4|f'(b)|^q}{9} \right)^{\frac{1}{q}} \right] \right\} \\ =&\frac{1}{2} \left\{ \left(\frac{1}{2\alpha+1} \left(\frac{1}{3}\right)^{2\alpha+1} \right)^{\frac{1}{2}} \left[\left(\frac{|f'(2)|^2}{36} \right)^{\frac{1}{2}} + \left(\frac{11|f'(2)|^2}{36} \right)^{\frac{1}{q}} \right] \right. \\ &\left. + \left(\int_{\frac{1}{3}}^{1} \left(t^\alpha - \frac{27}{40} \right)^2 dt \right)^{\frac{1}{2}} \left[\left(\frac{2|f'(2)|^2}{9} \right)^{\frac{1}{2}} + \left(\frac{4|f'(2)|^2}{9} \right)^{\frac{1}{2}} \right] \right\} \\ =&\frac{1}{3} \left(\frac{1}{2\alpha+1} \left(\frac{1}{3}\right)^{2\alpha+1} \right)^{\frac{1}{2}} \left(1 + \sqrt{11}\right) \\ &+ \frac{2}{3} \left(\frac{1}{2\alpha+1} \left(1 - \left(\frac{1}{3}\right)^{2\alpha+1} \right) - \frac{27}{20(\alpha+1)} \left(1 - \left(\frac{1}{3}\right)^{\alpha+1} \right) + \frac{243}{800} \right)^{\frac{1}{2}} \left(2 + \sqrt{2}\right). \end{aligned}$$

It is easy to confirm that the left-hand side of (4.3) in Example 8.2 is always lower than the right-hand side of (4.3) in Figure 3 for all values of $\alpha \in (0,5]$ using MATLAB software.

Example 8.3. Note that a function $f : [a,b] = [0,2] \to \mathbb{R}$ is presented by $f(x) = x^2$. From Theorem 4.3 with $\alpha \in (0,5]$ and $q = 2$, the left-hand side of (4.4) coincides with equality (8.1) and the right hand-side of (4.4) becomes to

$$2\left(\Omega_1(\alpha)\right)^{\frac{1}{2}} \left[\left[\Omega_3(\alpha)\right]^{\frac{1}{2}} + \left[\Omega_4(\alpha)\right]^{\frac{1}{2}} \right] + 2\left(\Omega_2(\alpha)\right)^{\frac{1}{2}} \left[\left[\Omega_5(\alpha)\right]^{\frac{1}{2}} + \left[\Omega_6(\alpha)\right]^{\frac{1}{2}} \right].$$

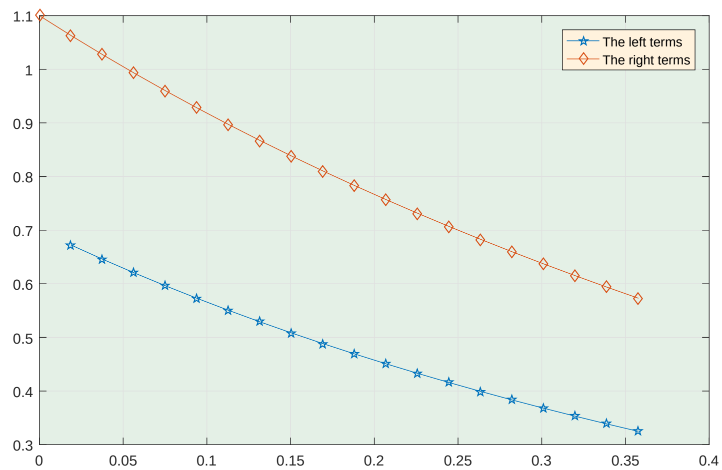

Figure 1. Graph on the interval $0 < \alpha \leq \frac{\ln\left(\frac{27}{40}\right)}{\ln\left(\frac{1}{3}\right)}$.

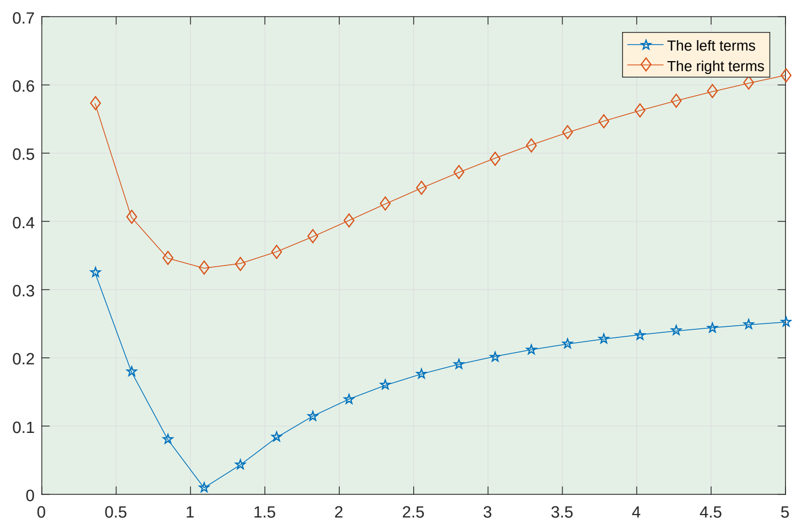

Figure 2. Graph on the interval $\frac{\ln\left(\frac{27}{40}\right)}{\ln\left(\frac{1}{3}\right)} < \alpha \leq 5$.

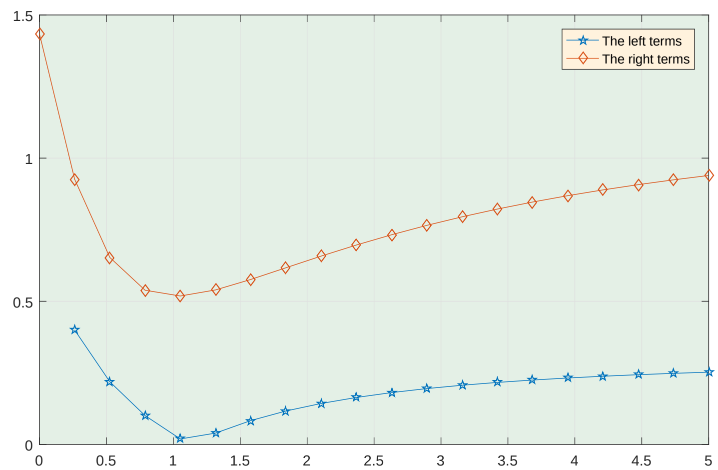

Figure 3. The graph of both sides of (4.3) in Example 8.2, dependent on α, has been computed and plotted using MATLAB.

Finally, we get the inequality

$$\begin{aligned}&\left|\frac{13}{10}-\frac{\alpha}{2}\left[\frac{2}{\alpha+2}-\frac{4}{\alpha+1}+\frac{1}{\alpha}\right]\right|\\ \leq&2\left(\Omega_1(\alpha)\right)^{\frac{1}{2}}\left[\left[\Omega_3(\alpha)\right]^{\frac{1}{2}}+\left[\Omega_4(\alpha)\right]^{\frac{1}{2}}\right]\\ &+2\left(\Omega_2(\alpha)\right)^{\frac{1}{2}}\left[\left[\Omega_5(\alpha)\right]^{\frac{1}{2}}+\left[\Omega_6(\alpha)\right]^{\frac{1}{2}}\right].\end{aligned}$$

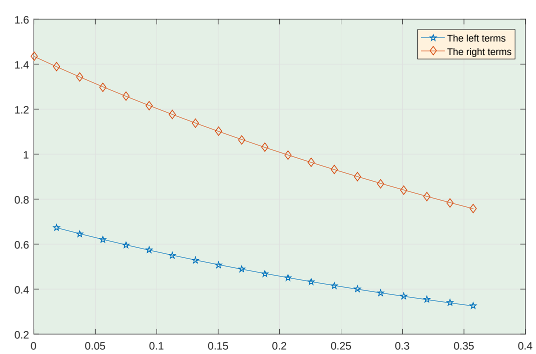

Figure 4. Graph on the interval $0<\alpha\leq\frac{\ln\left(\frac{27}{40}\right)}{\ln\left(\frac{1}{3}\right)}$.

Using MATLAB software, Figures 4 and 5 illustrate that in Example 8.3, the left-hand side of (4.4) consistently remains lower than the right-hand side.

Example 8.4. Note that a function $f:[a,b]=[0,2]\to\mathbb{R}$ is presented by $f(x)=x^2$. From Theorem 5.1 with $\alpha\in(0,5]$ and $0\leq f'(t)\leq 4$, the left-hand side of (5.1)

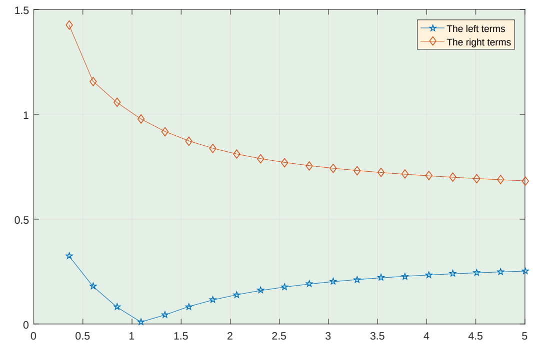

Figure 5. Graph on the interval $\frac{\ln\left(\frac{27}{40}\right)}{\ln\left(\frac{1}{3}\right)} < \alpha \leq 5$.

coincides with equality (8.1) and the right hand-side of (5.1) is

$$\begin{cases} \frac{2}{\alpha+1} - \frac{9}{10}, & 0 < \alpha \leq \frac{\ln\left(\frac{27}{40}\right)}{\ln\left(\frac{1}{3}\right)}, \\ \\ \frac{4\alpha}{\alpha+1}\left(\frac{27}{40}\right)^{1+\frac{1}{\alpha}} + \frac{2}{\alpha+1}\left[1+2\left(\frac{1}{3}\right)^{\alpha+1}\right] - \frac{9}{5}, & \frac{\ln\left(\frac{27}{40}\right)}{\ln\left(\frac{1}{3}\right)} < \alpha \leq 5. \end{cases}$$

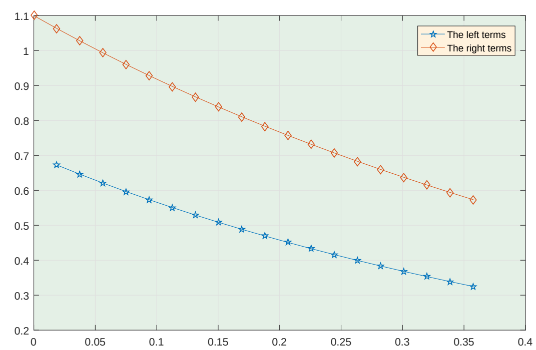

Figure 6. Graph on the interval $0 < \alpha \leq \frac{\ln\left(\frac{27}{40}\right)}{\ln\left(\frac{1}{3}\right)}$.

It is evident from Figure 6 and Figure 7 that the left-hand side of equality (5.1) in Example 8.4 consistently remains below the right-hand side for all values of $\alpha \in (0, 5]$.

9. Conclusion

The purpose of this study is to use Riemann-Liouville fractional integrals to obtain corrected Euler-Maclaurin-type inequalities for pertaining function classes. Firstly,

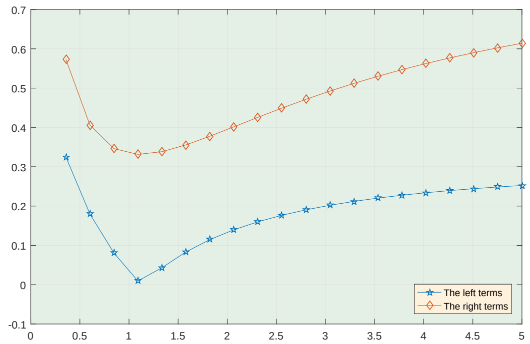

Figure 7. Graph on the interval $\frac{\ln\left(\frac{27}{40}\right)}{\ln\left(\frac{1}{3}\right)} < \alpha \leq 5$.

we provide an integral equality which is necessary to prove the article's primary conclusions. For differentiable convex functions, various corrected Euler-Maclaurin-type inequalities are studied using the Riemann-Liouville fractional integrals. Additionally, we provide other graph-based examples to demonstrate the accuracy of our findings. Furthermore, we provide certain corrected Euler-Maclaurin-type fractional integrals for limited functions. Additionally, some fractional corrected Euler-Maclaurin-type inequalities for Lipschitzian functions are taken into consideration. Finally, fractional integrals of bounded variation are used to verify Euler-Maclaurin-type inequalities.

Our findings about corrected Euler-Maclaurin-type inequalities by conformable fractional integrals may pave the way for new directions in this area of study in subsequent publications. Our results can be further developed or extended by using different fractional integral operators or convex function classes. Moreover, some corrected Euler-Maclaurin-type inequalities for different function classes can be obtained using the quantum calculus.

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