

# Bifurcation Analysis of a Discrete Predator-Prey Model with Gompertz Growth and Increased Functional Response

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**Abstract** This study examines a discrete predator-prey model that employs a Gompertz growth function for the prey and a Holling type I functional response. Initially, the research explores the existence and local stability of fixed points within the system, employing a fundamental lemma. Subsequently, the conditions necessary for the emergence of transcritical and Neimark-Sacker bifurcations of the system are established through the application of the center manifold theorem and bifurcation theory. Finally, numerical simulations are performed to confirm the existence of the Neimark-Sacker bifurcation.

**Keywords** Discrete predator-prey system, semidiscretization method, Gompertz growth model, transcritical bifurcation, Neimark-Sacker bifurcation

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## 1. Introduction

Since the pioneering work of Alfred J. Lotka and Vito Volterra in the 1920s, the field of biomathematical modeling has attracted considerable attention from mathematicians and biologists worldwide. Their focus on the intricate dynamics between species and their environments laid the foundation for decades of rapid advancement in the study of predator-prey interactions [1–12].

Recently, Huang and Ruan [13] revisited the classic Gaussian-type predator-prey model, which was given by the following system of differential equations:

$$\begin{cases} \dot{x} = xg(x, k) - yp(x), \\ \dot{y} = y(-d + cq(x)). \end{cases} \quad (1.1)$$

In this model,  $x(t)$  and  $y(t)$  represent the population densities of the prey and predator at time  $t$ , respectively. The function  $g(x; k)$  describes the specific growth rate of the prey in the absence of predator. The parameter  $c$  represents the efficiency of the predator in converting consumed prey into growth, and  $d$  represents the predator mortality rate.

Unlike most studies that employ the logistic function, this paper adopts the Gompertz growth function [14–16], namely,  $xg(x, k) = rx \ln \frac{x}{k}$ , where, the positive

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constants  $r$  and  $k$  denote the intrinsic growth rate of the prey and the carrying capacity of the environment for the prey in the absence of predator, respectively. The Gompertz growth curve is S-shaped, similar to the logistic growth function, but it exhibits a faster growth rate in the early stage, which then gradually slows down. There are solid grounds for using the Gompertz growth rate in the study of predator - prey systems, mainly reflected in aspects such as biology, mathematics, and comparisons with other models. In natural ecosystems, the growth of prey populations is often affected by multiple factors. The Gompertz growth rate can accurately describe the growth pattern of invasive species in a new environment, which is characterized by rapid growth in the initial stage and a slowdown as the population approaches the environmental carrying capacity. This is more consistent with reality than the logistic growth model. Moreover, the actual growth data of many prey populations show non - linear characteristics, and the Gompertz growth rate is more in line with this. For example, the research data on the growth of fish populations in a lake can prove this. The Gompertz growth function has a simple form, which is easy to handle in mathematical analysis and has obvious advantages when calculating the equilibrium points and stability of the predator - prey model. In addition, it is well - compatible with the Holling Type I functional response, and can comprehensively demonstrate the dynamic changes of the predator - prey system. Compared with the logistic growth model, the Gompertz growth model has a faster initial growth rate, which can better reflect the short - term explosive growth of prey in some ecosystems, such as the growth of grass after rain in a grassland ecosystem. Furthermore, it responds more flexibly to environmental changes. When there is a sudden change in resource availability, the prey population modeled by this model can respond more quickly in terms of growth, making it more suitable for studying predator - prey systems in dynamic ecological environments. Four common functional response functions,

$$p(x) = mx, mx/(a + x), mx^2/(ax^2 + bx + 1), mx/(ax^2 + bx + 1),$$

corresponding Holling type I, II, III and IV, are listed here to simulate predation. Predator-prey bio-models using the above Holling response function types have been extensively studied. This paper adopts the Holling Type I [21-26], which leads to the following system

$$\begin{cases} \frac{dx}{dt} = xr \ln \frac{k}{x} - axy, \\ \frac{dy}{dt} = acxy - dy. \end{cases} \tag{1.2}$$

The system (1.2) is a complex nonlinear system, and it is almost impossible for us to obtain its exact solutions. Therefore, we consider utilizing computational methods to find approximate solutions. Since computers can only handle a series of discrete points, this motivates us to approximate the continuous system (1.2) with a discrete system. To do this, consider the average rate of change of the system on integer time points

$$\begin{cases} \frac{1}{x(t)} \frac{dx(t)}{dt} = r \ln \left( \frac{k}{x([t])} \right) - ay([t]), \\ \frac{1}{y(t)} \frac{dy(t)}{dt} = acx([t]) - d. \end{cases} \tag{1.3}$$

It is easy to see that the system (1.3) has piecewise constant arguments, and that a solution  $(x(t), y(t))$  of the system (1.3) for  $t \in [0, +\infty)$  possesses the following