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Bifurcation Analysis of a Discrete Predator-Prey Model with Gompertz Growth and Increased Functional Response

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Abstract This study examines a discrete predator-prey model that employs a Gompertz growth function for the prey and a Holling type I functional response. Initially, the research explores the existence and local stability of fixed points within the system, employing a fundamental lemma. Subsequently, the conditions necessary for the emergence of transcritical and Neimark-Sacker bifurcations of the system are established through the application of the center manifold theorem and bifurcation theory. Finally, numerical simulations are performed to confirm the existence of the Neimark-Sacker bifurcation.

Keywords Discrete predator-prey system, semidiscretization method, Gompertz growth model, transcritical bifurcation, Neimark-Sacker bifurcation

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1. Introduction

Since the pioneering work of Alfred J. Lotka and Vito Volterra in the 1920s, the field of biomathematical modeling has attracted considerable attention from mathematicians and biologists worldwide. Their focus on the intricate dynamics between species and their environments laid the foundation for decades of rapid advancement in the study of predator-prey interactions [1–12].

Recently, Huang and Ruan [13] revisited the classic Gaussian-type predator-prey model, which was given by the following system of differential equations:

$$\begin{cases} \dot{x} = xg(x,k) - yp(x), \\ \dot{y} = y(-d + cq(x)). \end{cases} \tag{1.1}$$

In this model, $x(t)$ and $y(t)$ represent the population densities of the prey and predator at time t, respectively. The function $g(x;k)$ describes the specific growth rate of the prey in the absence of predator. The parameter c represents the efficiency of the predator in converting consumed prey into growth, and d represents the predator mortality rate.

Unlike most studies that employ the logistic function, this paper adopts the Gompertz growth function [14–16], namely, $xg(x,k) = rxln\frac{x}{k}$, where, the positive

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constants r and k denote the intrinsic growth rate of the prey and the carrying capacity of the environment for the prey in the absence of predator, respectively. The Gompertz growth curve is S-shaped, similar to the logistic growth function, but it exhibits a faster growth rate in the early stage, which then gradually slows down. There are solid grounds for using the Gompertz growth rate in the study of predator - prey systems, mainly reflected in aspects such as biology, mathematics, and comparisons with other models. In natural ecosystems, the growth of prey populations is often affected by multiple factors. The Gompertz growth rate can accurately describe the growth pattern of invasive species in a new environment, which is characterized by rapid growth in the initial stage and a slowdown as the population approaches the environmental carrying capacity. This is more consistent with reality than the logistic growth model. Moreover, the actual growth data of many prey populations show non - linear characteristics, and the Gompertz growth rate is more in line with this. For example, the research data on the growth of fish populations in a lake can prove this. The Gompertz growth function has a simple form, which is easy to handle in mathematical analysis and has obvious advantages when calculating the equilibrium points and stability of the predator - prey model. In addition, it is well - compatible with the Holling Type I functional response, and can comprehensively demonstrate the dynamic changes of the predator - prey system.Compared with the logistic growth model, the Gompertz growth model has a faster initial growth rate, which can better reflect the short - term explosive growth of prey in some ecosystems, such as the growth of grass after rain in a grassland ecosystem. Furthermore, it responds more flexibly to environmental changes. When there is a sudden change in resource availability, the prey population modeled by this model can respond more quickly in terms of growth, making it more suitable for studying predator - prey systems in dynamic ecological environments. Four common functional response functions,

$$p(x) = mx, mx/(a+x), mx^2/(ax^2+bx+1), mx/(ax^2+bx+1),$$

corresponding Holling type I, II, III and IV, are listed here to simulate predation. Predator-prey bio-models using the above Holling response function types have been extensively studied. This paper adopts the Holling Type I [21–26], which leads to the following system

$$\begin{cases} \frac{dx}{dt} = xr\ln\frac{k}{x} - axy, \\ \frac{dy}{dt} = acxy - dy. \end{cases} \tag{1.2}$$

The system (1.2) is a complex nonlinear system, and it is almost impossible for us to obtain its exact solutions. Therefore, we consider utilizing computational methods to find approximate solutions. Since computers can only handle a series of discrete points, this motivates us to approximate the continuous system (1.2) with a discrete system. To do this, consider the average rate of change of the system on integer time points

$$\begin{cases} \frac{1}{x(t)}\frac{dx(t)}{dt} = rln(\frac{k}{x([t])}) - ay([t]), \\ \frac{1}{y(t)}\frac{dy(t)}{dt} = acx([t]) - d. \end{cases} \tag{1.3}$$

It is easy to see that the system (1.3) has piecewise constant arguments, and that a solution $(x(t), y(t))$ of the system (1.3) for $t \in [0, +\infty)$ possesses the following

characteristics:
1. on the interval $[0,+\infty)$, $x(t)$ and $y(t)$ are continuous;
2. when $t \in [0,+\infty)$ possibly except for the points $t \in \{0,1,2,3,\cdots\}$, $\frac{dx(t)}{dt}$ and $\frac{dy(t)}{dt}$ exist everywhere. The following system can be obtained by integrating the system (1.3) over the interval $[n,t]$ for any $t \in [n,n+1)$ and $n = 0,1,2,\cdots$

$$\begin{cases} x(t) = x_n e^{r\ln\frac{k}{x_n} - ay_n}(t-n), \\ y(t) = y_n e^{acx_n - d}(t-n), \end{cases} \tag{1.4}$$

where $x_n = x(n)$ and $y_n = y(n)$. Letting $t \to (n+1)^-$ in the system (1.4) produces

$$\begin{cases} x_{n+1} = x_n e^{rln\frac{k}{x_n} - ay_n}, \\ y_{n+1} = y_n e^{acx_n - d}, \end{cases} \tag{1.5}$$

where the parameters (a,c,d,k,r) in the space

$$\Omega = \{(a,c,d,k,r)|a>0, c>0, d>0, k>0, r>0\}.$$

Based on biological plausibility of the system, we select the initial conditions $x_0 \geq 0, y_0 \geq 0$, ensuring that the initial population sizes of both the prey and the predator are nonnegative. Obviously, under the parameter space Ω and the initial conditions, the solutions of the system always remain nonnegative. From an ecological perspective, this is of crucial importance, as population sizes cannot be negative. In an environment with limited resources (responding to related parameter conditions), the size of population are necessarily bounded.

We undertake a comprehensive investigation into the existence and local stability of fixed points for the system (1.5) in Section 2. Following this, we meticulously identify potential bifurcation points, anchoring our analysis on the fixed points derived from the system. Subsequently, we deduce sufficient conditions under which the system could undergo transcritical bifurcation and Neimark-Sacker bifurcation in Section 3. This deduction is facilitated by pivotal lemmas that provide the mathematical groundwork for our conclusions. To validate our theoretical findings, we conduct numerical simulations using computational methods in Section 4. These simulations are followed by some remarks that synthesize our observations and insights in the final section.

The following definition and lemma will be used in the sequel. For them, refer to [17–20].

Lemma 1.1. *Let $E(x,y)$ be a fixed point of the system* (1.5) *with multipliers λ_1 and λ_2.*

(*i*) *If $|\lambda_1| < 1$ and $|\lambda_2| < 1$, the fixed point $E(x,y)$ is called sink, so a sink is locally asymptotically stable.*
(*ii*) *If $|\lambda_1| > 1$ and $|\lambda_2| > 1$, the fixed point $E(x,y)$ is called source, so a source is locally asymptotically unstable.*
(*iii*) *If $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$), the fixed point $E(x,y)$ is called saddle.*
(*iv*) *If either $|\lambda_1| = 1$ or $|\lambda_2| = 1$, the fixed point $E(x,y)$ is nonhyperbolic.*

Lemma 1.2. *Let $F(\lambda) = \lambda^2 + B\lambda + C$, where B and C are real constants. Suppose λ_1 and λ_2 are the roots of $F(\lambda) = 0$. Then the following conclusions hold.*

(i) If $F(1) > 0$, then

(i.1) $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $C < 1$;

(i.2) $\lambda_1 = -1$ and $\lambda_2 \neq -1$ if and only if $F(-1) = 0$ and $B \neq 2$;

(i.3) $|\lambda_1| < 1$ and $|\lambda_2| > 1$ if and only if $F(-1) < 0$;

(i.4) $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 0$ and $C > 1$;

(i.5) λ_1 and λ_2 are a pair of conjugate complex roots, and $|\lambda_1| = |\lambda_2| = 1$ if and only if $-2 < B < 2$ and $C = 1$;

(i.6) $\lambda_1 = \lambda_2 = -1$ if and only if $F(-1) = 0$ and $B = 2$.

(ii) If $F(1) = 0$, then the other root λ satisfies $|\lambda| = (<,>)1$ if and only if $|C| = (<,>)1$.

(iii) If $F(1) < 0$, then $F(\lambda) = 0$ has one root in the interval $(1, \infty)$. Moreover,

(iii.1) the other root λ satisfies $\lambda < (=) - 1$ if and only if $F(-1) < (=)0$;

(iii.2) the other root satisfies $-1 < \lambda < 1$ if and only if $F(-1) > 0$.

2. Existence and stability of fixed points

First, we consider the existence of fixed points. We identify each fixed point and then analyze the local stability of each fixed point in the system (1.5) and classify them. The fixed points of the system (1.5) satisfy

$$x = xe^{rln\frac{k}{x} - ay},\ y = ye^{acx-d}.$$

Considering the biological significance of the system (1.5), it is evident that each parameter must be nonnegative, and the system(1.5) has only two nonnegative fixed points $E_1 = (k, 0)$ and $E_2 = (\frac{d}{ac}, \frac{r}{a}ln\frac{ack}{d},)$ provided that $ack > d$. The Jaccobian matrix of the system (1.5) at any fixed point $E(x, y)$ takes the following form

$$J(E) = \begin{bmatrix} (1-r)e^{rln\frac{k}{x} - ay} & -axe^{rln\frac{k}{x} - ay} \\ acye^{acx-d} & e^{acx-d} \end{bmatrix}.$$

The characteristic polynomial of Jaccobian matrix $J(E)$ reads

$$F(\lambda) = \lambda^2 - p\lambda + q,$$

where

$$p = (1-r)e^{rln\frac{k}{x} - ay} + e^{acx-d}, \quad q = (1 - r + a^2cxy)e^{rln\frac{k}{x} - ay + acx - d}.$$

For the stability of fixed points E_1 and E_2, we can easily get the following Theorems 2.1 and 2.2, respectively.

Theorem 2.1. *The following statements about the fixed point $E_1 = (k, 0)$ tof the system (1.5) are true.*

(1) *If $0 < r < 2$ and $ack < d$, the fixed point $E_1(k, 0)$ is a sink.*

(2) *If $r > 2$ and $ack > d$, the fixed point $E_1(k, 0)$ is a source.*

(3) *If $0 < r < 2$ and $ack > d$ or $r > 2$ and $cak < d$, the fixed point $E_1(k, 0)$ is a saddle.*

(4) *If $r = 2$ or $ack = d$, the fixed point $E_1(k, 0)$ is non-hyperbolic.*

Proof. The Jaccobian matrix of the system (1.5) at $E_1 = (k, 0)$ is

$$J(E_1) = \begin{bmatrix} 1-r & -ak \\ 0 & e^{ack-d} \end{bmatrix}.$$

□

Obviously, $\lambda_1 = 1 - r$ and $\lambda_2 = e^{ack-d}$.
(1) When $0 < r < 2$ and $ack < d$, $|\lambda_1| < 1$ and $|\lambda_2| < 1$. Therefore, the fixed point $E_1(k, 0)$ is a stable node, i.e., a sink.
(2) When $r > 2$ and $ack > d$, $|\lambda_1| > 1$ and $|\lambda_2| > 1$, then the fixed point $E_1(k, 0)$ is an unstable node, i.e., a source.
(3) When $0 < r < 2$ and $ack > d$ or $r > 2$ and $cak < d$, $|\lambda_1| < 1$ and $|\lambda_2| > 1$ or $|\lambda_1| > 1$ and $|\lambda_2| < 1$. Hence, the fixed point $E_1(k, 0)$ is a saddle.
(4) When $r = 2$ or $ack = d$, either $|\lambda_1| = 1$ or $|\lambda_2| = 1$. Therefore, the fixed point $E_1(k, 0)$ is non-hyperbolic.
The proof is complete.

Theorem 2.2. *When $d < ack$, $E_2 = (\frac{d}{ac}, \frac{r}{a} ln\frac{ack}{d})$ is a positive fixed point of the system (1.5). Let $c_0 = \frac{d}{ak}e^{\frac{1}{d}}$ and $c_1 = \frac{d}{ak}e^{\frac{2r-4}{dr}}$. Then the following statements are true about the positive fixed point E_2.*
If $0 < r < 4$, then $c_1 < c_0$, so,
1. for $c < c_1$, E_2 is a saddle;
2. for $c = c_1$, E_2 is nonhyperbolic;
3. for $c_1 < c < c_0$, E_2 is a stable node, i.e., a sink;
4. for $c = c_0$, E_2 is nonhyperbolic;
5. for $c > c_0$, E_2 is an unstable node, i.e., a source.
If $r \geq 4$, then $c_1 \geq c_0$, hence,
1. for $c < c_1$, E_2 is a saddle;
2. for $c = c_1$, E_2 is nonhyperbolic;
3. for $c > c_1$, E_2 is an unstable node, i.e., a source.

Proof. The Jaccobian matrix of the system (1.5) at the fixed point E_2 can be simplified as

$$J(E_2) = \begin{bmatrix} 1-r & -\dfrac{d}{c} \\ cr\ln\frac{ack}{d} & 1 \end{bmatrix}.$$

□

The characteristic polynomial of Jacobian matrix $J(E_2)$ reads as

$$F(\lambda) = \lambda^2 - p\lambda + q,$$

where

$$p = 2 - r \quad \text{and} \quad q = 1 - r + drln\frac{ack}{d}.$$

It is easy to get

$$F(1) = drln\frac{ack}{d} \quad \text{and} \quad F(-1) = 4 - 2r + drln\frac{ack}{d}.$$

Obviously, $ack > d$ implies $F(1) > 0$.

$$F(-1) > (=,<)0 \Leftrightarrow c > (=,<)c_1 \quad \text{and} \quad q > (=,<)1 \Leftrightarrow c > (=,<)c_0.$$

Consider the following two cases.
Case 1: $0 < r < 4$**.**
Then $c_1 < c_0$. So,
1. for $c < c_1$, $F(-1) < 0$. By Lemma 1.2 (i.3), E_2 is a saddle.
2. For $c = c_1$, $F(-1) = 0$. Evidently, E_2 is nonhyperbolic.
3. For $c_1 < c < c_0$, $F(-1) > 0$ and $q < 1$. According to Lemma 1.2 (i.1), $|\lambda_1| < 1$ and $|\lambda_2| < 1$, so E_2 is a stable node, i.e., a sink.
4. For $c = c_0$, $F(-1) > 0$ and $q = 1$. By Lemma 1.2 (i.5), λ_1 and λ_2 are a pair of conjugate complex roots with $|\lambda_1| = |\lambda_2| = 1$. So, E_2 is nonhyperbolic.
5. For $c_0 < c$, $F(-1) > 0$ and $q > 1$. In view of Lemma 1.2 (i.4), $|\lambda_1| > 1$ and $|\lambda_2| > 1$, so E_2 is an unstable node, i.e., a source.
Case 2: $r \geq 4$**.**
Then $c_0 \leq c_1$. Thus,
1. for $c < c_1$, $F(-1) < 0$. By Lemma 1.2 (i.3), E_2 is a saddle.
2. For $c = c_1$, $F(-1) = 0$. Obviously, E_2 is nonhyperbolic.
3. For $c > c_1$, $F(-1) > 0$ and $q > 1$. In view of Lemma 1.2 (i.4), $|\lambda_1| > 1$ and $|\lambda_2| > 1$, so E_2 is an unstable node, i.e., a source. The proof is over.

3. Bifurcation discussion

In this section, we are in a position to use the Center Manifold Theorem and bifurcation theory to analyze the local bifurcation problems of the fixed points E_1 and E_2. For related work, refer to [21], [22] and [23].

3.1. For fixed point $E_1 = (k, 0)$

Theorem 2.1 shows that a bifurcation of the system (1.5) at the fixed point E_1 may occur in the space of parameters

$$(a, c, d, k, r) \in S_{E_1} = \{(a, c, d, k, r) \in \Omega | 0 < r < 2 \text{ } or \text{ } r > 2\}.$$

In fact, the following consequence can be derived.

Theorem 3.1. *Set the parameters* $(a, c, d, k, r) \in S_{E_1}$ *and* $d_0 = ack$. *Then the system* (1.5) *undergoes a transcritical bifurcation at the fixed point* E_1 *when the parameter* d *varies in a small neighborhood of the critical value* d_0.

Proof. In order to show the detailed process, we proceed according to the following steps. □

The first step. Let $u_n = x_n - k, v_n = y_n - 0$, which transforms the fixed point $E_1 = (k, 0)$ to the origin $O(0, 0)$, and the system (1.5) to

$$\begin{cases} u_{n+1} = (u_n + k)e^{r \ln \frac{k}{u_n+k} - av_n} - k, \\ v_{n+1} = v_n e^{-d+acu_n+ack}. \end{cases} \tag{3.11}$$

The second step. Giving a small perturbation d^* of the parameter d, i.e., $d^* = d - d_0$, with $0 < |d^*| << 1$, the system (3.11) is perturbed into

$$\begin{cases} u_{n+1} = (u_n + k)e^{r\ln\frac{k}{u_n+k} - av_n} - k, \\ v_{n+1} = v_n e^{-d^* - d_0 + acu_n + ack}. \end{cases} \tag{3.12}$$

Letting $d^*_{n+1} = d^*_n = d^*$, the system (3.12) can be written as

$$\begin{cases} u_{n+1} = (u_n + k)e^{r\ln\frac{k}{u_n+k} - av_n} - k, \\ v_{n+1} = v_n e^{-d^* - d_0 + acu_n + ack}, \\ d^*_{n+1} = d^*_n. \end{cases} \tag{3.13}$$

The third step. Taylor expanding of the system (3.13) at $(u_n, v_n, d^*_n) = (0, 0, 0)$ gets

$$\begin{cases} u_{n+1} = a_{100}u_n + a_{010}v_n + a_{110}u_nv_n + a_{200}u_n^2 + a_{020}v_n^2 \\ \qquad\quad + a_{120}u_nv_n^2 + a_{210}u_n^2v_n + a_{300}u_n^3 + a_{030}v_n^3 + o(\rho_1^3), \\ v_{n+1} = b_{100}u_n + b_{010}v_n + b_{001}d^*_n + b_{200}u_n^2 + b_{020}v_n^2 \\ \qquad\quad + b_{002}{d^*_n}^2 + b_{110}u_nv_n + b_{101}u_nd^*_n + b_{011}v_nd^*_n \\ \qquad\quad + b_{300}u_n^3 + b_{030}v_n^3 + b_{003}{d^*_n}^3 + b_{210}u_n^2v_n \\ \qquad\quad + b_{120}u_nv_n^2 + b_{021}v_n^2d^*_n + b_{201}u_n^2d^*_n + b_{102}u_n{d^*_n}^2 \\ \qquad\quad + b_{012}v_n{d^*_n}^2 + b_{111}u_nv_nd^*_n + o(\rho_1^3), \\ d^*_{n+1} = d^*_n, \end{cases} \tag{3.14}$$

where $\rho_1 = \sqrt{u_n^2 + v_n^2 + {d^*_n}^2}$, $a_{100} = 1 - r, a_{010} = -ak, a_{110} = a(r-1), a_{020} = \frac{a^2k}{2}, a_{200} = \frac{r^2-r}{2k}$,
$a_{120} = \frac{a^2(1-r)}{2}, a_{210} = \frac{ar(1-r)}{2k}, a_{300} = -\frac{1}{k^2}, a_{030} = \frac{-a^3k}{6}$,
$b_{100} = b_{001} = b_{200} = b_{020} = b_{002} = b_{101} = b_{300} = b_{030} = b_{003} = b_{120} = b_{021} = b_{201} = b_{102} = 0, b_{010} = 1, b_{110} = ac, b_{011} = -1, b_{111} = -ac, b_{012} = \frac{1}{2}, b_{210} = \frac{a^2c^2}{2}$.

Let

$$J(E_1) = \begin{pmatrix} a_{100} & a_{010} & 0 \\ b_{100} & b_{010} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{i.e.,} \quad J(E_1) = \begin{pmatrix} 1-r & -ak & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, we can derive the three eigenvalues of $J(E_1)$ to be $\lambda_1 = 1 - r, \lambda_{2,3} = 1$, and the corresponding eigenvectors

$$(\xi_1, \eta_1, \varphi_1)^T = (1, 0, 0)^T, (\xi_2, \eta_2, \varphi_2)^T = \left(1, \frac{-r}{ak}, 0\right)^T, (\xi_3, \eta_3, \varphi_3)^T = (0, 0, 1)^T,$$

respectively.

The fourth step. Take $T = \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \\ \varphi_1 & \varphi_2 & \varphi_3 \end{pmatrix}$, namely,

$$T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -\frac{r}{ak} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ then } T^{-1} = \begin{pmatrix} 1 & \frac{ak}{r} & 0 \\ 0 & -\frac{ak}{r} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Taking the following transformation

$$(u_n, v_n, d_n^*)^T = T(X_n, Y_n, \delta_n)^T,$$

the system (3.14) is changed into the following form

$$\begin{cases} X_{n+1} = (1-r)X_n + F(X_n, Y_n, \delta_n) + o(\rho_2^3), \\ Y_{n+1} = Y_n + G(X_n, Y_n, \delta_n) + o(\rho_2^3), \\ \delta_{n+1} = \delta_n, \end{cases} \tag{3.15}$$

where $\rho_2 = \sqrt{X_n^2 + Y_n^2 + \delta_n^2}$.

The fifth step. Suppose on the center manifold

$$X_n = h(Y_n, \delta_n) = h_{20}Y_n^2 + h_{11}Y_n\delta_n + h_{02}\delta_n^2 + o(\rho_3^2),$$

where $\rho_3 = \sqrt{Y_n^2 + \delta_n^2}$. Then, according to

$$X_{n+1} = (1-r)X_n + F(X_n, Y_n, \delta_n) + o(\rho_3^2),$$

$$\begin{aligned} h(Y_{n+1}, \delta_{n+1}) =& h_{20}Y_{n+1}^2 + h_{11}Y_{n+1}\delta_{n+1} + h_{02}\delta_{n+1}^2 + o(\rho_3^2) \\ =& h_{20}(Y_n + G(X_n, Y_n, \delta_n))^2 + h_{11}(Y_n + G(X_n, Y_n, \delta_n))\delta_n \\ &+ h_{02}\delta_n^2 + o(\rho_3^2), \end{aligned}$$

and $X_{n+1} = h(Y_{n+1}, \delta_{n+1})$, we obtain the center manifold equation

$$\begin{aligned} &(1-r)h(Y_n, \delta_n) + F(h(Y_n, \delta_n), Y_n, \delta_n) \\ =& h_{20}(Y_n + G(h(Y_n, \delta_n), Y_n, \delta_n))^2 \\ &+ h_{11}(Y_n + G(h(Y_n, \delta_n), Y_n, \delta_n))\delta_n + h_{02}\delta_n^2 + o(\rho_3^2). \end{aligned}$$

Comparing the corresponding coefficients of terms with the same orders in the above center manifold equation, we get

$$h_{20} = \frac{1}{r} + \frac{1}{2k}, h_{11} = -1, h_{02} = 0.$$

So, the system (3.15) restricted to the center manifold takes as

$$Y_{n+1} = Y_n + G(h(Y_n, \delta_n), Y_n, \delta_n) + o(\rho_3^2) =: f(Y_n, \delta_n).$$

Do not consider terms of degree higher than three in the above expression. Then, we can get

$$Y_{n+1} =: f(Y_n, \delta_n) = Y_n + h_{20} Y_n^2 + h_{11} Y_n \delta_n + o(\rho_3^3).$$

Therefore, one has

$$\begin{aligned} & f(Y_n, \delta_n)|_{(0,0)} = 0, \left.\frac{\partial f}{\partial Y_n}\right|_{(0,0)} = 1, \left.\frac{\partial f}{\partial \delta_n}\right|_{(0,0)} = 0, \\ & \left.\frac{\partial^2 f}{\partial Y_n \partial \delta_n}\right|_{(0,0)} = h_{11} = -1 \neq 0, \\ & \left.\frac{\partial^2 f}{\partial Y_n^2}\right|_{(0,0)} = 2h_{20} = \frac{1}{k} + \frac{2}{r} \neq 0. \end{aligned}$$

According to (21.1.42)-(21.1.46) in the literature [24], all the conditions for the occurrence of the transcritical bifurcation are established. Hence, it is valid for the occurrence of transcritical bifurcation of the system (3.11) at the fixed point E_1.

3.2. For fixed point $E_2 = (\frac{d}{ac}, \frac{r}{a} ln\frac{ack}{d})$

When $c = c_0 = \frac{d}{ak} e^{\frac{1}{d}}$, Theorem 2.3 with Lemma 1.2 ($i.5$) shows that $F(1) > 0$, $F(-1) > 0$, $-2 < p < 2$ and $q = 1$. So λ_1 and λ_2 are a pair of conjugate complex roots with $|\lambda_1| = |\lambda_2| = 1$. At this time we derive that the system (1.5) at the fixed point E_2 can undergo a Neimark-Sacker bifurcation in the space of parameters

$$(a, c, d, r, k) \in S_{E_2} = \{(a, c, d, r, k) \in \Omega | c > \frac{d}{ak}\}.$$

In order to show the process clearly, we carry out the following steps.

The first step. Take the changes of variables $u_n = x_n - x_0, v_n = y_n - y_0$, which transform the fixed point $E_2 = (x_0, y_0)$ to the origin $O(0,0)$, and the system (1.5) into

$$\begin{cases} u_{n+1} = (u_n + x_0) e^{r \ln \frac{k}{u_n + x_0} - a(v_n + y_0)} - x_0, \\ v_{n+1} = (v_n + y_0) e^{ac(u_n + x_0) - d} - y_0. \end{cases} \tag{3.31}$$

The second step. Give a small perturbation c^* of the parameter c, i.e., $c^* = c - c_0$. Then the perturbation of the system (3.31) can be regarded as

$$\begin{cases} u_{n+1} = (u_n + x_0) e^{r \ln \frac{k}{u_n + x_0} - a(v_n + y_0)}, \\ v_{n+1} = (v_n + y_0) e^{(c^* + c_0) a(u_n + x_0) - d} - y_0. \end{cases} \tag{3.32}$$

The corresponding characteristic equation of the linearized equation of the system (3.32) at the equilibrium point (0,0) can be expressed as

$$F(\lambda) = \lambda^2 - p(c^*)\lambda + q(c^*) = 0,$$

where

$$p(c^*) = 2 - r, \quad \text{and} \quad q(c^*) = 1 - r + drln\frac{ak(c^* + c_0)}{d}.$$

It is easy to derive $p^2(c^*) - 4q(c^*) < 0$ when $c^* = 0$ for $r < 4$, then the two roots of $F(\lambda) = 0$ are as follows

$$\lambda_{1,2}(c^*) == \frac{p(c^*) \pm i\sqrt{4q(c^*) - p^2(c^*)}}{2},$$

moreover

$$(|\lambda_{1,2}(c^*)|)\Big|_{c^*=0} = \sqrt{q(c^*)}\Big|_{c^*=0} = 1,$$

which implies

$$\Big(\frac{d|\lambda_{1,2}(c^*)|}{dc^*}\Big)\Big|_{c^*=0} = \frac{dr}{2c_0} \neq 0.$$

The occurrence of Neimark-Sacker bifurcation requires the following conditions to be satisfied

$$(H.1) \quad \Big(\frac{d|\lambda_{1,2}(c^*)|}{dc^*}\Big)\Big|_{c^*=0} \neq 0;$$

$$(H.2) \quad \lambda_{1,2}^i(0) \neq 1, i = 1, 2, 3, 4.$$

Since $p(c^*)\big|_{c^*=0} = 2 - r$ and $q(c^*)\big|_{c^*=0} = 1$, we have $\lambda_{1,2}(0) = \frac{2-r\pm i\sqrt{r(4-r)}}{2}$, and $\lambda_{1,2}^m(0) \neq 1$, for all $m = 1, 2, 3, 4$. According to [18], all of the conditions are satisfied for Neimark-Sacker bifurcation to occur.

The third step. In order to derive the normal form of the system (3.32), we expand the system (3.32) into power series up to third-order form around the origin to get the following

$$\begin{cases} u_{n+1} = c_{10}u_n + c_{01}v_n + c_{20}u_n^2 + c_{11}u_nv_n + c_{02}v_n^2 \\ \qquad\qquad +c_{30}u_n^3 + c_{21}u_n^2v_n + c_{12}u_nv_n^2 + c_{03}v_n^3 + o(\rho_4^3), \\ v_{n+1} = d_{10}u_n + d_{01}v_n + d_{20}u_n^2 + d_{11}u_nv_n + d_{02}v_n^2 \\ \qquad\qquad +d_{30}u_n^3 + d_{21}u_n^2v_n + d_{12}u_nv_n^2 + d_{03}v_n^3 + o(\rho_4^3), \end{cases} \tag{3.33}$$

where $\rho_4 = \sqrt{u_n^2 + v_n^2}$,

$$c_{10} = 1 - r, c_{01} = -\frac{d}{c}, c_{11} = ar - a,$$

$$c_{20} = \frac{ac(r^2 - r)}{2d}, c_{02} = \frac{ad^2}{2c}, c_{03} = -\frac{a^2d}{6c},$$

$$c_{30} = \frac{a^2c^2r^2}{2d^2} + \frac{a^2c^2r}{2d^2} - \frac{2a^2c^2r + 3a^2c^2r^2 + a^2c^2r^3}{6d^3},$$

$$c_{21} = \frac{a^2cr}{d} - \frac{a^2c(r^2 + r)}{2d}, c_{12} = (1 - r)\frac{a^2}{2},$$

$$d_{10} = cr\ln\frac{ack}{d}, d_{01} = 1, d_{20} = \frac{a^2c^2r\ln\frac{ack}{d}}{2},$$

$$d_{11} = ac, d_{02} = 0, d_{30} = \frac{a^2c^3r\ln\frac{ack}{d}}{6},$$

$$d_{21} = \frac{a^2c^2}{2}, d_{12} = 0, d_{03} = 0.$$

Let

$$J(E_2) = \begin{pmatrix} c_{10} & c_{01} \\ d_{10} & d_{01} \end{pmatrix}, \text{namely}, J(E_2) = \begin{pmatrix} 1-r & -\frac{d}{c} \\ crln\frac{ack}{d} & 1 \end{pmatrix}.$$

It is easy to derive that the two eigenvalues of the matrix $J(E_2)$ are

$$\lambda_{1,2} = A \pm Bi,$$

where $A = \frac{2-r}{2}$, $B = \frac{\sqrt{r(4-r)}}{2}$, with the corresponding eigenvectors $v_{1,2} = \begin{pmatrix} -\frac{d}{c} \\ -\frac{r}{2} \end{pmatrix} \pm i \begin{pmatrix} 0 \\ B \end{pmatrix}$.

Let

$$P = \begin{pmatrix} 0 & -\frac{d}{c} \\ B & -\frac{r}{2} \end{pmatrix}, \quad \text{then}, \quad P^{-1} = \begin{pmatrix} -\frac{c_0 r}{2dB} & \frac{1}{B} \\ -\frac{c_0}{d} & 0 \end{pmatrix}.$$

Make a change of variables

$$(u, v)^T = P(X, Y)^T,$$

then, the system (3.33) is transformed into the following form

$$\begin{cases} X \to \left(1 - \frac{1}{2}r\right) X - BY + \overline{F}(X, Y) + o(\rho_5^3), \\ Y \to BX + \left(1 - \frac{1}{2}r\right) Y + \overline{G}(X, Y) + o(\rho_5^3), \end{cases} \tag{3.34}$$

where

$$\overline{F}(X, Y) = e_{20}X^2 + e_{11}XY + e_{02}Y^2 + e_{30}X^3 + e_{21}X^2Y + e_{12}XY^2 + e_{03}Y^3,$$
$$\overline{G}(X, Y) = f_{20}X^2 + f_{11}XY + f_{02}Y^2 + f_{30}X^3 + f_{21}X^2Y + f_{12}XY^2 + f_{03}Y^3,$$

$$e_{20} = -\frac{c_{02}rB}{2c_{01}}, e_{11} = \frac{2c_{01}^2 d_{11} - c_{01}c_{11}r - c_{02}r^2}{2c_{01}},$$
$$e_{02} = \frac{4c_{01}^2(-c_{20}r + 2c_{01}d_{20} + d_{11}r) - r^2(c_{02}r + 2c_{01}c_{11})}{8c_{01}B}, e_{30} = -\frac{c_{03}B^2 r}{2c_{01}},$$
$$e_{21} = -\frac{(2c_{01}c_{12} + 3c_{03}r)Br}{4c_{01}}, e_{12} = -\frac{c_{01}c_{21}r + c_{12}r^2}{2} - \frac{3c_{03}r^3}{8c_{01}},$$
$$e_{03} = \frac{8c_{01}^3(-c_{30}r + 2c_{01}d_{30}) - r^2(4c_{01}^2c_{21} + c_{03}r^2 + 2c_{01}c_{12}r)}{16c_{01}B},$$
$$f_{20} = \frac{c_{02}}{c_{01}}B^2, f_{11} = c_{11}B + \frac{c_{02}}{c_{01}}Br, f_{02} = c_{01}c_{20} + \frac{1}{2}c_{11}r + \frac{c_{02}}{4c_{01}}r^2,$$

$$f_{30} = \frac{c_{03}}{c_{01}}B^3, f_{21} = c_{12}B^2 + \frac{3c_{03}}{2c_{01}}B^2r, f_{12} = c_{01}c_{21}B + c_{12}Br + \frac{3c_{03}}{4c_{01}}Br^2,$$
$$f_{03} = c_{30}c_{01}^2 + \frac{1}{2}c_{01}c_{21}r + \frac{1}{4}c_{12}r^2 + \frac{c_{03}}{8c_{01}}r^3.$$

Furthermore

$$\overline{F}_{XX} = -\frac{c_{02}Br}{c_{01}}, \overline{F}_{XY} = \frac{2c_{01}^2d_{11} - c_{01}c_{11}r - c_{02}r^2}{2c_{01}}, \overline{F}_{XXX} = -\frac{3c_{03}B^2r}{c_{01}},$$
$$\overline{F}_{YY} = \frac{4c_{01}^2(-c_{20}r + 2c_{01}d_{20} + d_{11}r) - r^2(c_{02}r + 2c_{01}c_{11})}{4c_{01}B},$$
$$\overline{F}_{XXY} = -c_{12}Br - \frac{3c_{03}Br^2}{2c_{01}}, \overline{F}_{XYY} = -c_{01}c_{21}r - c_{12}r^2 - \frac{3c_{03}r^3}{4c_{01}},$$
$$\overline{F}_{YYY} = \frac{3c_{01}^3(-c_{30}r + 2c_{01}d_{30})}{c_{01}B} - \frac{3r^2(4c_{01}^2c_{21} + c_{03}r^2 + 2c_{01}c_{12}r)}{8c_{01}B},$$
$$\overline{G}_{XX} = \frac{2c_{02}B^2}{c_{01}}, \overline{G}_{XY} = c_{11}B + \frac{c_{02}Br}{c_{01}}, \overline{G}_{YY} = 2c_{01}c_{20} + c_{11}r + \frac{c_{02}r^2}{2c_{01}},$$
$$\overline{G}_{XXX} = \frac{6c_{03}B^3}{c_{01}}, \overline{G}_{XXY} = 2c_{12}B^2 + \frac{3c_{03}B^2r}{c_{01}},$$
$$\overline{G}_{XYY} = 2c_{01}c_{21}B + 2c_{12}Br + \frac{3c_{03}Br^2}{2c_{01}},$$
$$\overline{G}_{YYY} = 6c_{30}c_{01}^2 - 3c_{01}c_{21}r + \frac{3}{2}c_{12}r^2 + \frac{3c_{03}r^3}{4c_{01}}.$$

The fourth step. In order to ensure that the system (3.34) has a Neimark-Sacker bifurcation, we need to calculate the discriminating quantity

$$L = -Re\Big(\frac{(1-2\lambda_1)\lambda_2^2}{1-\lambda_1}\zeta_{20}\zeta_{11}\Big) - \frac{1}{2}|\zeta_{11}|^2 - |\zeta_{02}|^2 + Re(\lambda_2\zeta_{21}), \tag{3.35}$$

and L is required not to be zero, where

$$\zeta_{20} = \frac{1}{8}[\overline{F}_{XX} - \overline{F}_{YY} + 2\overline{G}_{XY} + i(\overline{G}_{XX} - \overline{G}_{YY} - 2\overline{F}_{XY})],$$
$$\zeta_{11} = \frac{1}{4}[\overline{F}_{XX} + \overline{F}_{YY} + i(\overline{G}_{XX} + \overline{G}_{YY})],$$
$$\zeta_{02} = \frac{1}{8}[\overline{F}_{XX} - \overline{F}_{YY} - 2\overline{G}_{XY} + i(\overline{G}_{XX} - \overline{G}_{YY} + 2\overline{F}_{XY})],$$
$$\zeta_{21} = \frac{1}{16}[\overline{F}_{XXX} + \overline{F}_{XYY} + \overline{G}_{XXY} + \overline{G}_{YYY} + i(\overline{G}_{XXX} + \overline{G}_{XYY} - \overline{F}_{XXY} - \overline{F}_{YYY})].$$

By calculation we get

$$\zeta_{20} = \frac{1}{8}\Big(- \frac{4c_{01}^2(-c_{20}r + 2c_{01}d_{20} + d_{11}r) - r^2(c_{02}r + 2c_{01}c_{11})}{4c_{01}B}$$

$$+\frac{(2c_{01}c_{11}+c_{02}r)B}{c_{01}}\Big)+\frac{1}{8}\Big(\frac{c_{02}(r^2+4B^2)}{2c_{01}}-2c_{01}(c_{20}+d_{11})\Big)i,$$

$$\zeta_{11}=\frac{1}{4}\Big(\frac{-c_{02}Br}{c_{01}}+\frac{4c_{01}^2(-c_{20}r+2c_{01}d_{20}+d_{11}r)-r^2(c_{02}r+2c_{01}c_{11})}{4c_{01}B}\Big)$$
$$+\frac{1}{4}\Big(\frac{c_{02}(4B^2+r^2)}{2c_{01}}+2c_{01}c_{20}+c_{11}r\Big)i,$$

$$\zeta_{02}=\frac{1}{8}\Big(-\frac{4c_{01}^2(-c_{20}r+2c_{01}d_{20}+d_{11}r)-r^2(c_{02}r+2c_{01}c_{11})}{4c_{01}B}$$
$$-\frac{(3c_{02}r+2c_{01}c_{11})B}{c_{01}}\Big)+\frac{1}{4}\Big(\frac{c_{02}(4B^2-3r^2)}{4c_{01}}-c_{11}r+c_{01}(d_{11}-c_{20})\Big)i,$$

$$\zeta_{21}=\frac{1}{16}\Big(2c_{01}(3c_{30}c_{01}+c_{21}r)+c_{12}(\frac{1}{2}r^2+2B^2)\Big)+\frac{1}{16}\Big(B(2c_{01}c_{21}+3c_{12}r)$$
$$+\frac{3c_{03}B(r^2+2B^2)}{c_{01}}+\frac{3r^2(4c_{01}^2c_{21}+c_{03}r^2-2c_{01}c_{12}r)}{8c_{01}B}$$
$$-\frac{3c_{01}^3(-c_{30}r+2c_{01}d_{30})}{c_{01}B}\Big)i.$$

Summarizing the above analysis, one has the following result.

Theorem 3.2. *Assume the parameters a, c, d, k, r in the space*

$$S_2=\{(a,c,d,r,k)\in S_{E_2}|0<r<4\}.$$

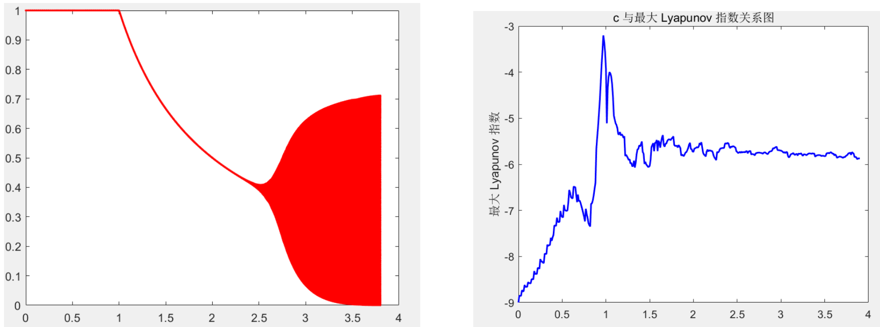
Let $c_0=\frac{d}{ak}e^{\frac{1}{d}}$ and L be defined as in (3.35). *If $L\neq 0$ holds and the parameter a varies in the small neighborhood of c_0, then the system* (1.5) *at the fixed point E_2 undergoes a Neimark-Sacker bifurcation. In addition, if $L<$ (or $>$)0, then an attracting (or a repelling) invariant closed curve bifurcates from the fixed point E_2 for $c<$ (or $>$)c_0.*

4. Numerical simulation

In this section, we use the bifurcation diagrams, phase portraits and Lyapunov exponents of the system (1.5) to illustrate our theoretical results and further reveal some new dynamical behaviors to occur as the parameters vary by Matlab software.

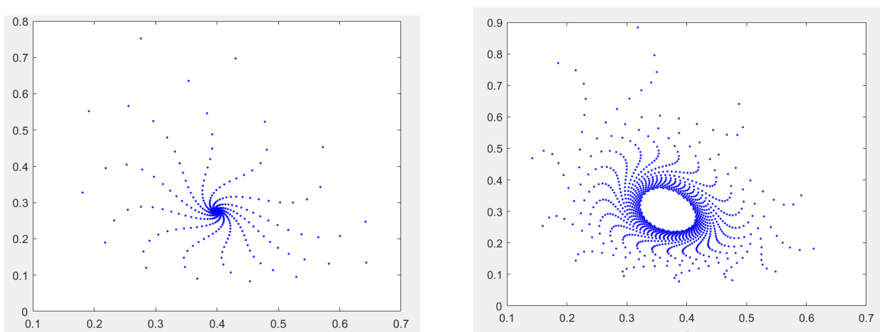
Fix the parameter values $a=1, d=1, k=1$, r=0.3, let $c\in(0,3.8)$ and take the initial values $(x_0,y_0)=(0.25,1),(0.25,0.75)$ in Fig.2 and Fig.3 respectively. Figure 1(a) shows the bifurcation diagram of (c,x)-plane, from which the fixed point E_2 is stable when $c<c_0=2.718281$. Moreover, the fixed point E_2 is unstable when $c>c_0$. Hence, a Neimark-Sacker bifurcation occurs at the fixed point $E_2=(1,0.3)$ when $c=c_0$, whose multipliers are $\lambda_{1,2}=0.85\pm0.52678269i$ with $|\lambda_{1,2}|=1$.

The corresponding maximum Lyapunov exponent diagram of the system(1.5) is plotted in Figure 1(b), from which we can easily see that the maximal Lyapunov exponents are always negative when the parameter $c\in(0,3.8)$. Figures 2(a)-(h) and Figures 3(a)-(d) show that the dynamical properties of the fixed point E_2 change from stable to unstable as the value of the parameter a decreases and there is an occurrence of invariant closed curve around E_2 when $c=c_0$, which agrees with the result of Theorem 3.2.



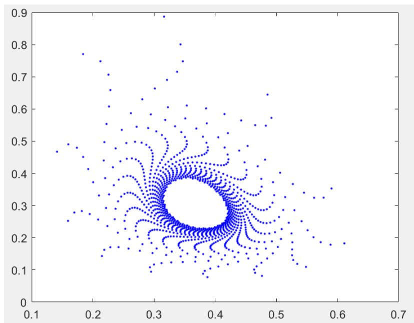
(a) $c \in (0, 3.8)$ (b) $c \in (0, 3.8)$

Figure 1. Bifurcation of the system (1.5) in (c, x)–plane and maximal Lyapunov exponent .

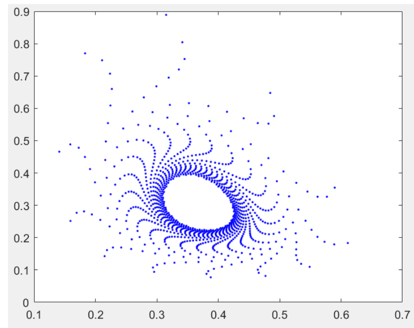


(a) $c = 2.5$ (b) $c = 2.71$

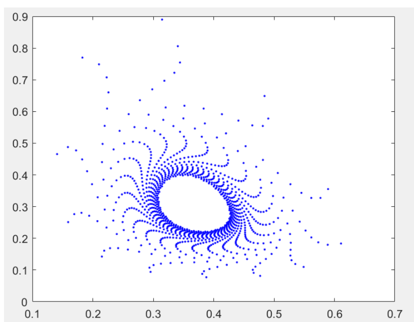
From the phase portraits in Figs 2 and 3, we infer the stability of E_2. Figures 2 (e)-(h) show that the closed curve is stable outside, while Figures 3 (a)-(d) indicate that the closed curve is stable inside the fixed point E_2 as long as the assumptions of Theorem 3.2 hold.



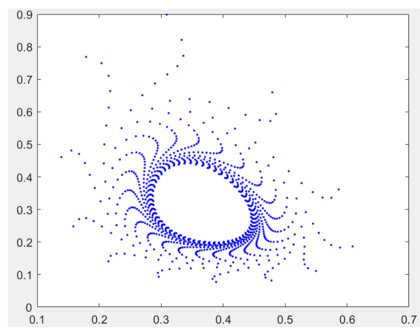
(c) $c = 2.714$



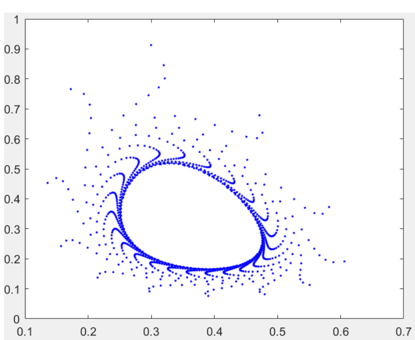
(d) $c = 2.717$



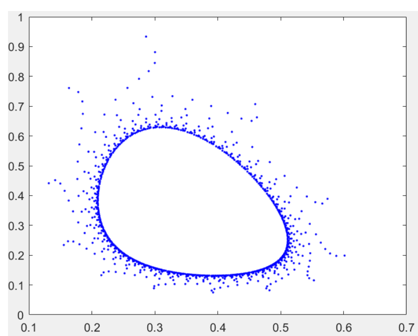
(e) $c = 2.7182$



(f) $c = 2.72$

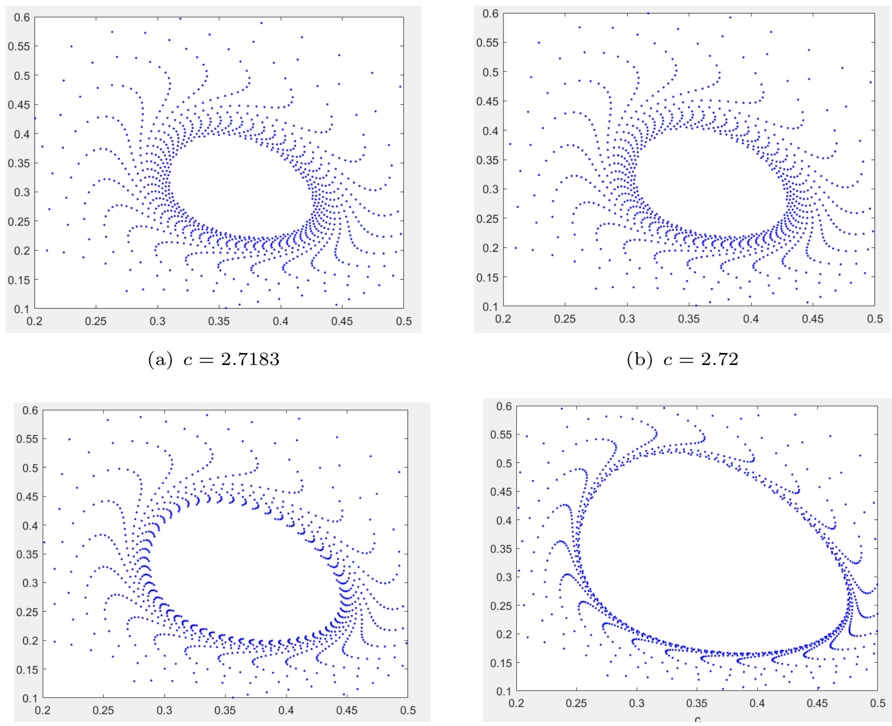


(g) $c = 2.73$



(h) $c = 2.75$

Figure 2. Phase portraits for the system (1.5) with $r = 0.3, d = 1, a = 1, k = 1$ and different $c$when the initial value $(x_0, y_0) = (0.25, 1)$.



(a) $c = 2.7183$ (b) $c = 2.72$

(c) $a = 2.73$ (d) $c = 2.75$

Figure 3. Phase portraits for the system (1.5) $r = 0.3, d = 1, a = 1, k = 1$ and different c when the initial value $(x_0, y_0) = (0.25, 0.8)$.

5. Conclusion

In this paper, we discuss the dynamical behaviors of a predator-prey model (1.5) with Gompertz growth and the Holling type I functional response. Under given parametric conditions, we completely show the existence and stability of two non-negative equilibria $E_1 = (k, 0)$ and $E_2 = (\frac{d}{ac}, \frac{r}{a} ln \frac{ack}{d})$. Then we derive the sufficient conditions for transcritical bifurcation and Neimark-Sacker bifurcation to occur. Meanwhile, it is clear that the positive equilibrium $E_2 = (x_0, y_0)$ is asymptotically stable when $c < c_0 = \frac{d}{ak} e^{\frac{1}{d}}$ and unstable when $c > c_0$. Hence, the system (1.5) undergoes a bifurcation which has been shown to be a Neimark-Sacker bifurcation when the parameter c goes through the critical value c_0. Finally, numerical simulations not only confirm the theoretical analysis results, but also find some new properties of the system (1.5). Especially, the occurrence of Neimark-Sacker bifurcation implies that the predator and prey in the system may coexist under appropriate conditions.

In recent years, research on discrete predator - prey systems has been continuously expanding in terms of model structure, incorporating factors such as time delays, spatial heterogeneity, and stochastic disturbances, etc. For example, a time delay model may exhibit complex dynamical changes. Our model in this paper, which employs the Gompertz growth function and Holling Type I functional response, focuses on the growth of prey and predation behavior at low densities, providing a unique perspective for research, and also offering a fundamental case for the theoretical development of this field. Currently, the theoretical analysis for such systems is moving towards higher dimensional systems and more complex dynamics. With the development of computer technology, numerical simulation has become a crucial part in the research of discrete predator - prey systems. Some studies adopt high - performance computing, parallel computing, and advanced visualization methods, etc. Although our research uses traditional methods, it demonstrates the characteristics of the system, laying a foundation for subsequent optimization of simulations based on advanced technologies.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contribute equally and significantly in writing this paper. All authors read and approve the final manuscript.

Data Availability Statements

There are no applicable data associated with this manuscript.

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