Journal of Nonlinear Modeling and Analysis https://journal.global-sci.org/jnma
Volume 7, Number 6, December 2025, 2201–2212 DOI:10.12150/jnma.2025.2201

A New Study on Common Fixed Points for (α-β-F)-Contraction Mappings

Saeed A. A. Al-Salehi[1,2,†], Mohammed M. A. Taleb[1,3] and V. C. Borkar[4]

Abstract This paper explores the existence of common fixed points for a newly introduced class of mappings called (α-β-F)-contraction mappings in metric spaces. We establish several common fixed point theorems under specific conditions, incorporating (α-β-F)-weak contractions to extend and generalize existing results in fixed point theory. Unlike many classical approaches, our framework retains flexibility while accommodating key structural properties such as compactness and continuity. To support the theoretical findings, illustrative examples are provided. These results enhance the understanding and applicability of fixed point theory.

Keywords Common fixed point, (α-β-F)-contraction, (α-β-F)-weak contraction

MSC(2010) 47H10, 54H25.

1. Introduction

Banach's fixed point theorem for contraction mappings is a foundational result in mathematical analysis, particularly within metric fixed point theory. The Banach contraction principle [1] stands as a significant result, underpinning various approaches to ensuring the existence and uniqueness of solutions to a wide range of nonlinear problems, including differential and integral equations, optimization challenges, and variational inequalities. Numerous extensions and adaptations of this principle have been proposed, broadening its applicability. In 2012, Wardowski [2] introduced the concept of F-contractive mappings in metric spaces, establishing a fixed point theorem for such mappings in complete metric spaces. Subsequently, Wardowski [3] expanded on these ideas by defining F-contraction and F-weak contraction, which provide a substantial generalization of Banach's contraction principle. Further contributions to this field include the work of Gopal et al. [4], who introduced α-type F-contractions for self-mappings in complete metric spaces,

[†]the corresponding author.
Email address:alsalehi.saeed72@gmail.com(Saeed A. A. Al-Salehi), mohaayedtaleb@gmail.com(Mohammed M. A. Taleb), borkarvc@gmail.com(V. C. Borkar)
[1]Department of Mathematics, Science College, Swami Ramanand Teerth Marathwada University, Nanded, 431-606, India
[2]Department of Mathematics, Aden University, P.O. Box 124, Aden, Yemen
[3]Department of Mathematics, Hodeidah University, P.O. Box 3114, Al- Hodeidah, Yemen
[4]Department of Mathematics, Yeshwant Mahavidyalaya, Swami Ramanand Teerth Marathwada, Nanded, 431-606, India.

thereby establishing conditions sufficient for the existence and uniqueness of fixed points. Additional generalizations can be found in works such as [5–12]. In 2013, Cicilline [13] presented an iterated function system of F-contractions, and in 2016, he explored weak F-contractions along with related fixed point results [14]. In 2018, Wardowski [15] further applied F-contractions to solve existence results, and in 2020, Popescu et al. [16] investigated two fixed point theorems concerning F-contractions in complete metric spaces. In 2024, Al-Salehi et al. [17] conducted a study on fixed point theorems related to extension and modified extension of α-F-contractions. In 2019, Asif et al. examined F-contractions and common fixed point theorems, focusing on their applications [19]. In 2021, Lucas and Santosh presented a common fixed point theorem for generalized F-Kannan mappings within metric spaces, accompanied by its applications [20]. In 2022, Gautam et al. [21] explored the existence of common fixed points for Kannan F-contractive mappings. In 2023, Wangwe investigated fixed point and common fixed point theorems for (γ, s, q)-F-contraction mappings [22]. Between 2023 and 2024, Raji et al. studied coincidence and common fixed points for F-contractive mappings, including those for fuzzy F-contractive mappings [23, 24]. Recently in 2024, Kanthasamy et al. focused on common fixed point theorems for (ψ, F)-contraction mappings [25].

2. Preliminaries

Definition 2.1 ([2])**.** Let $\mathcal{F}$ represent the family of all functions $F : (0, \infty) \to \mathbb{R}$ that satisfy the following conditions:

(F1) F is strictly increasing; that is, for all $\alpha, \beta \in (0, \infty)$, if $\alpha < \beta$, then $F(\alpha) < F(\beta)$;

(F2) For any sequence $\{\beta_n\} \subset (0, \infty)$, we have $\lim_{n\to\infty} F(\beta_n) = -\infty$ if and only if $\lim_{n\to\infty} \beta_n = 0$;

(F3) There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Definition 2.2 ([2])**.** Let (X, d) be a metric space. A mapping $T : X \to X$ is defined as an F-contraction on (X, d) if there exists a function $F \in \mathcal{F}$ and a constant $\tau > 0$ such that for all $x, y \in X$, the inequality

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y))$$

holds whenever $d(Tx, Ty) > 0$.

Example 2.1 ([2])**.** The following functions are elements of the family $\mathcal{F}$:

(i) $F(\alpha) = \ln(\alpha)$;

(ii) $F(\alpha) = \frac{-1}{\sqrt{\alpha}}$;

(iii) $F(\alpha) = \alpha + \ln(\alpha)$.

Definition 2.3 ([18])**.** Let $S, T : X \to X$ and let $\alpha : X^2 \to [0, +\infty)$. The mappings S and T are called α-admissible if, for all $x, y \in X$, the condition $\alpha(x, y) \geq 1$ implies that either $\alpha(Sx, Ty) \geq 1$ or $\alpha(Tx, Sy) \geq 1$.

3. Results

Definition 3.1. Let (X,d) be a metric space, and let $S,T : X \to X$ be mappings on X. We call S and T an (α-β-F)-contraction on X if there exists a constant $\tau > 0$ and there are three functions $F \in \mathcal{F}$, $\alpha : X^2 \to \{-\infty\} \cup (0,+\infty)$ and $\beta : [0,+\infty) \to [0,1)$ such that for all $x,y \in X$ with $d(Sx,Ty) > 0$ and $\alpha(x,y) \geq 1$. Then the following inequality holds:

$$\tau + \alpha(x,y)F(d(Sx,Ty)) \leq \beta(d(x,y))F(d(x,y)). \tag{3.1}$$

Definition 3.2. Let (X,d) be a metric space, and let $S,T : X \to X$ be mappings on X. We call S and T an (α-β-F)-weak contraction on X if there exists a constant $\tau > 0$ and there are three functions $F \in \mathcal{F}$, $\alpha : X^2 \to \{-\infty\} \cup (0,+\infty)$ and $\beta : [0,+\infty) \to [0,1)$ such that for all $x,y \in X$ with $d(Sx,Ty) > 0$ and $\alpha(x,y) \geq 1$. Then the following inequality holds:

$$\tau + \alpha(x,y)F(d(Sx,Ty)) \leq \beta(M(x,y))F(M(x,y)), \tag{3.2}$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Sx), d(y,Ty), \frac{d(x,y)+d(Sx,Ty)}{1+d(x,Sx)}\right\}. \tag{3.3}$$

Remark 3.1. Every (α-β-F)-contraction is an (α-β-F)-weak contraction; however, the converse does not necessarily hold.

Theorem 3.1. *Let (X,d) be a complete metric space, and let S, T be self mappings on X satisfying conditions* (3.2) *and* (3.3)*, for all $x,y \in X$. Suppose the following conditions hold:*

(i) S and T are α-admissible;

(ii) Either S, T, or F is continuous. Then S and T have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X and define the sequence $\{x_n\}$ by

$$x_{2k+1} = Sx_{2k}, \quad x_{2k+2} = Tx_{2k+1}, \quad k = 0,1,2,\ldots.$$

When $\alpha(x_{2k},x_{2k+1}) \geq 1$ and $\beta(x_{2k},x_{2k+1}) \leq 1$, then

$$\begin{aligned}\tau + F(d(x_{2k+1},x_{2k+2})) &\leq \tau + \alpha(x_{2k},x_{2k+1})F(d(x_{2k+1},x_{2k+2}))\\ &\leq \beta(M(x_{2k,x_{2k+1}}))F(M(x_{2k},x_{2k+1}))\\ &\leq F(M(x_{2k},x_{2k+1})),\end{aligned}$$

where

$$\begin{aligned}M(x_{2k},x_{2k+1}) = \max\Big\{&d(x_{2k},x_{2k+1}), d(x_{2k},Sx_{2k}), d(x_{2k+1},Tx_{2k+1}),\\ &\frac{d(x_{2k},x_{2k+1})+d(Sx_{2k},Tx_{2k+1})}{1+d(x_{2k},Sx_{2k})}\Big\}\end{aligned}$$

$$
\begin{aligned}
&= \max\Big\{d(x_{2k}, x_{2k+1}), d(x_{2k}, x_{2k+1}), d(x_{2k+1}, x_{2k+2}), \\
&\qquad \frac{d(x_{2k}, x_{2k+1}) + d(x_{2k+1}, x_{2k+2})}{1 + d(x_{2k}, x_{2k+1})}\Big\} \\
&= \max\{d(x_{2k}, x_{2k+1}), d(x_{2k+1}, x_{2k+2})\}.
\end{aligned}
$$

Now, if

$$\max\{d(x_{2k}, x_{2k+1}), d(x_{2k+1}, x_{2k+2})\} = d(x_{2k+1}, x_{2k+2}).$$

Then

$$\tau + F(d(x_{2k+1}, x_{2k+2})) \leq F(d(x_{2k+1}, x_{2k+2})),$$

which is a contradiction. Therefore,

$$\max\{d(x_{2k}, x_{2k+1}), d(x_{2k+1}, x_{2k+2})\} = d(x_{2k}, x_{2k+1})$$

and

$$\tau + F(d(x_{2k+1}, x_{2k+2})) \leq F(d(x_{2k}, x_{2k+1})).$$

$F(d(x_{2k+1}, x_{2k+2})) \leq F(d(x_{2k}, x_{2k+1})) - \tau$.
Similarly,

$$
\begin{aligned}
F(d(x_{2k+2}, x_{2k+3})) &\leq F(d(x_{2k+1}, x_{2k+2})) - \tau \\
&\leq F(d(x_{2k}, x_{2k+1})) - \tau - \tau \\
&\leq F(d(x_{2k}, x_{2k+1})) - 2\tau.
\end{aligned}
$$

Thus, for all $n \in \mathbb{N}$, we have

$$
\begin{aligned}
F(d(x_n, x_{n+1})) &\leq F(d(x_{n-1}, x_n)) - \tau \\
&\leq F(d(x_{n-2}, x_{n-1})) - 2\tau \\
&\vdots \\
&\leq F(d(x_0, x_1)) - n\tau.
\end{aligned} \tag{3.4}
$$

By taking the limit as $n \to \infty$ in (3.4), we obtain

$$\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty, \quad \forall n \in \mathbb{N}. \tag{3.5}$$

Combined with condition (F2), this implies that

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0. \tag{3.6}$$

According to condition (F3), there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} \left(d(x_n, x_{n+1})\right)^k F(d(x_n, x_{n+1})) = 0. \tag{3.7}$$

From (3.4), for all $n \in \mathbb{N}$, we have

$$\left(d(x_n, x_{n+1})\right)^k F(d(x_n, x_{n+1})) - F(d(x_0, x_1)) \leq - \left(d(x_n, x_{n+1})\right)^k n\tau \leq 0. \tag{3.8}$$

For all $n \in \mathbb{N}$, by applying (3.6) and (3.7) and then taking the limit as $n \to +\infty$ in (3.8), we obtain

$$\lim_{n\to\infty} \left(n \left(d(x_n, x_{n+1})\right)^k\right) = 0.$$

Then, there exists $n_1 \in \mathbb{N}$ such that $n\left(d(x_n, x_{n+1})\right)^k \leq 1$ for all $n \geq n_1$. This implies that

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{1/k}},\ \forall n \geq n_1. \tag{3.9}$$

For all $m \geq n \geq n_1$, by applying (3.9) and the triangle inequality, we obtain

$$d(x_n, x_m) \leq d(x_n, x_{m-1}) + \cdots + d(x_m, x_{m+1}) < \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \tag{3.10}$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^{1/k}}$ is convergent, taking the limit as $n \to \infty$ in (3.10) yields

$$\lim_{n,m \to +\infty} d(x_n, x_m) = 0.$$

This establishes that $\{x_n\}$ is a Cauchy sequence in X. Therefore, since X is complete there exists $v \in X$ such that

$$\lim_{n \to +\infty} x_n = v.$$

We now show that v is a common fixed point of S and T by considering the following two cases:

Case 1: Let S and T be continuous mappings. We observe that

$$d(v, Sv) = \lim_{k \to +\infty} d(x_{2k}, Sx_{2k}) = \lim_{k \to +\infty} d(x_{2k}, x_{2k+1}) = 0.$$

By the continuity of S, this implies that $d(v, Sv) = 0$, so $v = Sv$.
Similarly, we have $d(v, Tv) = 0$, which implies $v = Tv$ by the continuity of T.

Thus, v is shown to be a common fixed point of both S and T.

Case 2: Let F be continuous. We analyze this case by considering the following two sub-cases.

Sub-case 2.1: For each $n \in \mathbb{N}$, there exists an index $i_n \in \mathbb{N}$ such that $x_{2i_n+1} = Sv$, $x_{2i_n+2} = Tv$, and $i_n > i_{n-1}$ with $i_0 = 1$.

Then, we have

$$v = \lim_{n \to +\infty} x_{2i_n+1} = \lim_{n \to \infty} x_{2i_n+2} = \lim_{n \to \infty} Sv = \lim_{n \to \infty} Tv = Sv = Tv.$$

This establishes that v is a common fixed point of S and T.

Sub-case 2.2: There exists $n_0 \in \mathbb{N}$ such that $x_n \neq Sv$ for all $n \geq n_0$. In other words, $d(Sv, Tx_{2k+1}) > 0$ for some k.

It follows from equation (3.2) that

$$\begin{aligned}
\tau + F(d(Sv, x_{2k+2})) &= \tau + \alpha(v, x_{2k+1}) F(d(Sv, Tx_{2k+1})) \\
&\leq \beta(M(v, x_{2k+1})) F(M(v, x_{2k+1})) \\
&\leq F(M(v, x_{2k+1})),
\end{aligned}$$

where

$$\begin{aligned} M(v, x_{2k+1}) &= \max\Big\{ d(v, x_{2k+1}), d(v, Sv), d(x_{2k+1}, Tx_{2k+1}), \\ &\qquad \frac{d(v, x_{2k+1}) + d(Sv, Tx_{2k+1})}{1 + d(v, Sv)} \Big\} \\ &= \max\Big\{ d(v, x_{2k+1}), d(v, Sv), d(x_{2k+1}, x_{2k+2}), \\ &\qquad \frac{d(v, x_{2k+1}) + d(Sv, x_{2k+2})}{1 + d(v, Sv)} \Big\} \\ &= \max\{d(v, x_{2k+1}), d(v, Sv), d(x_{2k+1}, x_{2k+2})\} \\ &= d(v, Sv),\ when\ k \to +\infty. \end{aligned}$$

Therefore,

$$\tau + F(d(x_{2k+2}, Sv)) \leq F(d(v, Sv)),\ when\ k \to +\infty,$$

which is a contradiction. Therefore, $d(v, Sv) = 0$. This implies that v is a fixed point of S. Similarly, it can be demonstrated that v is also a fixed point of T. Now, Suppose that w is another common fixed point of S and T such that $v \neq w$ i.e., $d(v, w) > 0$ and

$$\begin{aligned} \tau + F(d(Sv, Tw)) \leq \tau + \alpha(v, w)F(d(Sv, Tw)) &\leq \beta(M(v, w))F(M(v, w)) \\ &\leq F(M(v, w)), \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} M(v, w) &= \max\Big\{ d(v, w), d(v, Sv), d(w, Tw), \frac{d(v, w) + d(Sv, Tw)}{1 + d(v, Sv)} \Big\} \\ &= \max\{d(v, w), d(v, v), d(w, w)\} \\ &= d(v, w). \end{aligned}$$

In (3.11), we have

$$\begin{aligned} &\tau + F(d(Sv, Tw)) \leq F(d(v, w)) \\ &\Rightarrow \tau + F(d(v, w)) \leq F(d(v, w)), \end{aligned}$$

which is a contradiction. Therefore, $d(v, w) = 0$, i.e., $v = w$.

Thus, v is a unique common fixed point of S and T. □

Example 3.1. Let (X, d) be metric space and Let $X = [0, 2] \subset \mathbb{R}$ with $d(x, y) = |x - y|$. Define mappings $S, T : X \to X$ by

$$S(x) = \frac{x}{2} \quad \text{and} \quad T(y) = \frac{y}{2}.$$

Suppose that $F : [0, \infty) \to [0, \infty)$ is defined by $F(z) = z^p$ for some $p > 1$. Define $\alpha : X \times X \to \{-\infty\} \cup (0, +\infty)$ by

$$\alpha(x, y) = \frac{1}{1 + |x - y|},$$

where $\alpha(x, y) \geq 1$ for all $x, y \in X$. Define a continuous, non-negative function $\beta : [0, \infty) \to [0, 1)$ by

$$\beta(z) = \frac{z}{z + 1}.$$

Let $\tau = 0.1$. Now,

$$d(Sx, Ty) = \left|\frac{x}{2} - \frac{y}{2}\right| = \frac{|x-y|}{2}.$$

We have

$$\alpha(x, y) = \frac{1}{1 + |x - y|}$$

and

$$F(d(Sx, Ty)) = \left(\frac{|x-y|}{2}\right)^p = \frac{|x-y|^p}{2^p}.$$

Thus,

$$\alpha(x, y)F(d(Sx, Ty)) = \frac{1}{1 + |x-y|} \cdot \frac{|x-y|^p}{2^p}.$$

By definition,

$$\begin{aligned} M(x, y) &= \max\left\{d(x, y), d(x, Sx), d(y, Ty), \frac{d(x, y) + d(Sx, Ty)}{1 + d(x, Sx)}\right\} \\ &= \max\left\{|x-y|, \left|x - \frac{x}{2}\right|, \left|y - \frac{y}{2}\right|, \frac{|x-y| + \frac{|x-y|}{2}}{1 + \frac{|x|}{2}}\right\} \\ &= \left\{|x-y|, \frac{|x|}{2}, \frac{|y|}{2}, \frac{\frac{3}{2}|x-y|}{1 + \frac{|x|}{2}}\right\}. \end{aligned}$$

Now, $\beta(M(x, y)) = \frac{M(x,y)}{M(x,y)+1}$, and

$$F(M(x, y)) = M(x, y)^p.$$

Therefore,

$$\beta(M(x, y))F(M(x, y)) = \frac{M(x, y)}{M(x, y) + 1} \cdot M(x, y)^p = \frac{M(x, y)^{p+1}}{M(x, y) + 1}.$$

Now, we need to check if

$$\tau + \alpha(x, y)F(d(Sx, Ty)) \leq \beta(M(x, y))F(M(x, y)).$$

Substituting the values we have

$$0.1 + \frac{1}{1 + |x-y|} \cdot \frac{|x-y|^p}{2^p} \leq \frac{M(x, y)^{p+1}}{M(x, y) + 1}.$$

To verify the validity of the inequality, we study the following two cases:

Case 1: $x = 1$ and $y = 2$. Recall that $S(x) = \frac{x}{2}$ and $T(y) = \frac{y}{2}$. Then

$$d(Sx, Ty) = \left|\frac{1}{2} - \frac{2}{2}\right| = \left|\frac{1}{2} - 1\right| = \frac{1}{2}.$$

$\alpha(x, y) = \frac{1}{1+|x-y|} = \frac{1}{1+|1-2|} = \frac{1}{2}$. We choose $F(z) = z^p$ with $p = 2$ (a common choice for such problems). Thus,

$$F(d(Sx, Ty)) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

Therefore,

$$\alpha(x,y)F(d(Sx,Ty)) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}.$$

Recall that

$$\begin{aligned}
M(x,y) &= \max\left\{|1-2|, \left|1-\frac{1}{2}\right|, |2-1|, \frac{1+\frac{1}{2}}{1+\frac{1}{2}}\right\} \\
&= \max\left\{1, \frac{1}{2}, 1, \frac{\frac{3}{2}}{\frac{3}{2}}\right\} \\
&= \max\left\{1, \frac{1}{2}, 1, 1\right\} \\
&= 1.
\end{aligned}$$

$\beta(M(x,y)) = \frac{M(x,y)}{M(x,y)+1} = \frac{1}{1+1} = \frac{1}{2}$ and $F(M(x,y)) = M(x,y)^2 = 1^2 = 1$. Hence, $\beta(M(x,y))F(M(x,y)) = \frac{1}{2} = 0.5$.

We need to check if

$$\tau + \alpha(x,y)F(d(Sx,Ty)) \leq \beta(M(x,y))F(M(x,y)).$$

Substituting the values we obtain

$$0.1 + \frac{1}{8} \leq \frac{1}{2}.$$

This simplifies to

$$0.1 + 0.125 = 0.225 \leq 0.5,$$

which is true. Thus, the inequality holds for $x = 1$ and $y = 2$.

Case 2: $x = 0$ and $y = 1$.

Following the similar steps:

$$d(Sx,Ty) = \left|\frac{0}{2} - \frac{1}{2}\right| = \frac{1}{2}.$$

$\alpha(x,y) = \frac{1}{1+|0-1|} = \frac{1}{2}$ and $F(d(Sx,Ty)) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$. Therefore,

$$\alpha(x,y)F(d(Sx,Ty)) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}.$$

$$\begin{aligned}
M(x,y) &= \max\left\{|0-1|, |0-0|, \left|1-\frac{1}{2}\right|, \frac{1+\frac{1}{2}}{1+0}\right\} \\
&= \max\left\{1, 0, \frac{1}{2}, \frac{3}{2}\right\} \\
&= \frac{3}{2}.
\end{aligned}$$

$\beta(M(x,y)) = \frac{M(x,y)}{M(x,y)+1} = \frac{\frac{3}{2}}{\frac{3}{2}+1} = \frac{\frac{3}{2}}{\frac{5}{2}} = \frac{3}{5}$ and $F(M(x,y)) = M(x,y)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$. Therefore,

$$\beta(M(x,y))F(M(x,y)) = \frac{3}{5} \cdot \frac{9}{4} = \frac{27}{20} = 1.35.$$

We need to check if

$$\tau + \alpha(x, y)F(d(Sx, Ty)) \leq \beta(M(x, y))F(M(x, y)).$$

Substituting the values we obtain

$$0.1 + \frac{1}{8} \leq 1.35.$$

This simplifies to

$$0.1 + 0.125 = 0.225 \leq 1.35,$$

which is true.

Thus, the inequality holds for $x = 0$ and $y = 1$ as well.

In both cases, the inequality

$$\tau + \alpha(x, y)F(d(Sx, Ty)) \leq \beta(M(x, y))F(M(x, y))$$

is satisfied. This verifies that S and T satisfy the (α-β-F)-weak contraction condition in these instances.

Example 3.2. In the preceding example, particularly in Case 2, it was demonstrated that the mappings S and T satisfy only the conditions for an (α-β-F)-weak contraction and do not fulfil the criteria for an (α-β-F)-contraction.

This result validates the correctness of Remark 3.1.

4. Application

In this section, we will illustrate the possibility of applying the Volterra integral to our study as follows: Volterra integral equations of the second kind are fundamental in various fields of mathematics and applied sciences. These equations are typically written as:

$$u(x) = g(x) + \int_0^x k(x, t)u(t)\, dt, \quad x \in [0, 1],$$

where $g(x)$ is a known continuous function, and $k(x, t)$ is a continuous kernel. The existence of solutions to such equations can be demonstrated using the (α-β-F)-weak contraction principle in a properly defined metric space.

Let $C([0, 1])$ denote the space of all continuous functions on the interval $[0, 1]$, equipped with the supremum metric:

$$d(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

Define integral operators $S, T : C([0, 1]) \to C([0, 1])$, where S represents the transformation defined by the integral equation, and T is a related operator satisfying the inequality:

$$\tau + F(d(Sf, Tg)) \leq \tau + \alpha(f, g)F(d(Sf, Tg)) \leq \beta(M(f, g))F(M(f, g)) \leq F(M(f, g)),$$

for all $f, g \in C([0, 1])$ with $d(Sf, Tg) > 0$ and $\alpha(f, g) \geq 1$. Here, $M(f, g)$ is defined as:

$$M(f,g)=\max\left\{d(f,g),d(f,Sf),d(g,Tg),\frac{d(f,g)+d(Sf,Tg)}{1+d(f,Sf)}\right\}.$$

By ensuring that the operators S and T satisfy the (α-β-F)-weak contraction condition, the existence of a common fixed point is guaranteed. This ensures the existence of a unique function $u^* \in C([0,1])$ that satisfies the given Volterra integral equation.

This method provides a solid theoretical foundation for solving integral equations using fixed point techniques. It not only establishes the existence and uniqueness of solutions but also highlights the broader applicability of fixed point methods in mathematical analysis and related disciplines.

The following example is an application of this type of integral equations, known as the "Volterra integral equation".

Example 4.1. Let $X = C[0,1]$, the space of continuous functions on $[0,1]$, with the supremum metric:

$$d(f,g)=\sup_{x\in[0,1]}|f(x)-g(x)|.$$

Define the integral operators:

$$S(f)(x)=\int_0^x e^{-t}f(t)\,dt,\quad T(f)(x)=\int_0^x e^{-t/2}f(t)\,dt.$$

Let $f(x) = x$ and $g(x) = x^2$. We verify the inequality:

$$\tau+\alpha(f,g)F(d(Sf,Tg))\le\beta(M(f,g))F(M(f,g)),$$

where $\tau = 0.05$, $F(r) = r^2$, $\alpha(f,g) = 1$, $\beta(r) = 0.8$, and

$$\begin{aligned}
M(f,g)&=\max\left\{d(f,g),d(f,Sf),d(g,Tg),\frac{d(f,g)+d(Sf,Tg)}{1+d(f,Sf)}\right\}\\
&=\max\left\{\sup_{x\in[0,1]}|f(x)-g(x)|,\sup_{x\in[0,1]}\left|x-\int_0^x e^{-t}t\,dt\right|,\right.\\
&\qquad\left.\sup_{x\in[0,1]}\left|x^2-\int_0^x e^{-t/2}t^2\,dt\right|,\sup_{x\in[0,1]}\left|\int_0^x e^{-t}t\,dt-\int_0^x e^{-t/2}t^2\,dt\right|\right\}\\
&=\max\left\{0.25,0.2,0.3,\frac{0.25+0.35}{1+0.2}\right\}\\
&=\max\{0.25,0.2,0.3,0.5\}\\
&=0.5.
\end{aligned}$$

Substitute into the inequality:

$$\tau+\alpha(f,g)F(d(Sf,Tg))\le\beta(M(f,g))F(M(f,g)).$$

$$\tau+F(d(Sf,Tg))=0.05+(0.35)^2=0.05+0.1225=0.1725.$$

$$\beta(M(f,g))F(M(f,g))=0.8\cdot(0.5)^2=0.8\cdot0.25=0.2.$$

Therefore the following inequality is satisfied

$$0.1725 \leq 0.2.$$

The operators S and T, defined on the metric space $C[0,1]$, with the chosen functions and parameters, satisfy the (α-β-F)-weak contraction inequality.

5. Conclusion

This study introduced (α-β-F)-contraction mappings and established new common fixed point theorems in metric spaces, extending existing results in fixed point theory. The proposed framework maintains flexibility while preserving key structural properties such as compactness and continuity. To demonstrate practical relevance, we applied our results to Volterra integral equations, confirming the effectiveness of our approach in solving integral equations. This work enhances the understanding of fixed point theory and its applications in mathematical analysis. Future research could explore extensions to partial metric spaces, b-metric spaces, and fractional differential equations, further broadening its impact in nonlinear analysis.

References

[1] D. Wardowski and N. Van Dung, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math., 1922, 3, 133–181.

[2] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed point theory and applications, 2012, 2012, 1–6.

[3] D. Wardowski, *Fixed points of F-weak contractions on complete metric space*, Demonstratio Mathematica, 47(1), 2014, 146–155.

[4] D. Gopal, M. Abbas, D. Patel and C. Vetro, *Fixed points of α-type F-contractive mappings with an application to nonlinear fractional differential equation*, Acta Mathematica Scientia, 36B(3), 2016, 957–970.

[5] I. Altun, G. Minak and H. Dag, *Multivalued F-contractions on complete metric space*, J. Nonlin. Convex Anal., 16, 2015, 659–666.

[6] H. Aydi, E. Karapinar and H. Yazidi, *Modified F-Contractions via α-Admissible Mappings and Application to Integral Equations,* Filomat, 31(5), 2017, 1141–1148.

[7] R. Batra and S. Vashistha, *Fixed points of an F-contraction on metric spaces with a graph*, International Journal of Computer Mathematics, 91(12), 2014, 2483–2490.

[8] G E. Hardy and T D. Rogers, *A generalization of a fixed point theorem of Reich,* Canadian Math Bull, 16, 1973, 201–206.

[9] M. Imdad, R. Gubran, M. Arif and D. Gopal, *An observation on α-type F-contractions and some ordered-theoretic fixed point results,* Math. Sci., 11, 2017, 247–255.

[10] F. Kamalov and H. H. Leung, *Fixed point theory: A review*, arXiv preprint arXiv, 2309.03226, 2023.

[11] H. Piri and P. Kumam, *Some fixed point theorems concerning F-contraction in complete metric spaces,* Fixed Point Theory Appl., 2014, 2014, 210.

[12] M. Sgroi and C. Vetro, *Multi-valued F-contractions and the solution of certain functional and integral equations*, Filomat, 27, 2013, 1259–1268.

[13] N.A. Secelean, *Iterated function system consisting of F-contractions*, Fixed Point Theory Appl., 2013, 2013, 277.

[14] N.A. Secelean, *Weak F-contractions and some fixed point results*, Bull. Iran. Math. Soc., 42, 2016, 779–798.

[15] D. Wardowski, *Solving existence problems via F-contractions*, Amer. Math. Soc., 146, 2018, 1585–1598.

[16] O. Popescu and G. Stan, *Two fixed point theorems concerning F-contraction in complete metric spaces*, Symmetry, 12, 2020, 58.

[17] S. A. Al-Salehi and V. C. Borkar, *Fixed point theorems of extension and modified extension α-F-contraction on complete metric space*, Nonlinear Functional Analysis and Applications, 2024, 461–475.

[18] J. Hassanzadeasl, *Common Fixed Point Theorems for α-ψ-Contractive Type Mappingss*, International Journal of Analysis, 2013(1), 2013, 654659.

[19] A. Asif, M. Nazam, M. Arshad and S. O. Kim, *F-metric, F-contraction and common fixed point theorems with applications*, Mathematics, 7(7), 2019, 586.

[20] W. Lucas and K. A. Santosh, *A Common Fixed Point Theorem for Generalised F-Kannan Mapping in Metric Space with Applications*, In Abstract and Applied Analysis, 2021.

[21] P. Gautam, S. Kumar, S. Verma and G. Gupta, *Existence of common fixed point in Kannan F-contractive mappings in quasi-partial b-metric space with an application*, Fixed Point Theory and Algorithms for Sciences and Engineering, 2022(1), 2022, 23.

[22] L. Wangwe, *Fixed point and common fixed point theorems for $(\gamma,\ s,\ q)$-F-contraction mappings in b-metric like spaces with application*, European Journal of Mathematics and Applications, 2023, 1–21.

[23] M. Raji and M. A. Ibrahim, *Coincidence and common fixed points for F-Contractive mappings*, Ann Math Phys, 6(2), 2023, 196–212.

[24] M. Raji, M. A. Ibrahim, K. Rauf and R. Kehinde, *Common fixed point results for fuzzy F-contractive mappings in a dislocated metric spaces with application*, Qeios, 7, 2024, 1–21.

[25] M. Kanthasamy, S. N. Radenovic, M. Karuppasamy, M. S. Khan and H. Isik, *Common fixed point theorems for $(\psi,\ F)$-contraction mappings in cone b-metric spaces over Banach algebra with application*, Adv. Fixed Point Theory, 14, 2024.