Journal of Nonlinear Modeling and Analysis
Volume 7, Number 6, December 2025, 2213–2223
https://journal.global-sci.org/jnma
DOI:10.12150/jnma.2025.2213

Minimizing the Eigenvalue Ratio for the *p*-Laplacian Operator

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Abstract We focus on the minimization problem of the eigenvalue ratio for the p-Laplacian operator with Robin boundary conditions on an interval $[0, \hat{\pi}]$, where $\hat{\pi} = \frac{2\pi}{p \sin(\pi/p)}$. Using variational techniques and Prüfer-type transformations, we show that the constant weight is not minimizing for the class of concave weights.

Keywords Eigenvalue ratio, p-Laplacian operator, concave weight, Robin boundary conditions

MSC(2010) 34L15, 34B27, 35J60, 34B24.

1. Introduction

The eigenvalue ratio of the first two eigenvalues, $\Lambda[w] = \frac{\lambda_2[w]}{\lambda_1[w]}$, is a cornerstone of spectral analysis, offering profound insights into the dynamic behavior of physical systems such as vibrating strings, membranes, and fluid flows. This ratio quantifies the relative frequencies of fundamental vibrational modes, influencing stability, resonance, and energy distribution in applications spanning acoustics, structural engineering, and non-Newtonian fluid dynamics [6,9,23]. In this paper, we investigate the minimization of $\Lambda[w]$ for the p-Laplacian operator, a nonlinear generalization of the Laplacian defined by $\Delta_p u = (|u'|^{p-2}u')'$ for $p > 1$, under Robin boundary conditions. The p-Laplacian's nonlinearity introduces mathematical richness, distinguishing it from the classical linear case ($p = 2$) and making it a powerful model for complex physical phenomena [7, 16]. Our study focuses on whether constant weights minimize $\Lambda[w]$ among concave weights, extending classical results to the nonlinear setting with flexible boundary conditions.

Throughout this paper, we consider the following problem under Robin boundary conditions.

$$\begin{cases} -\Delta_p u(x) = \lambda w(x)|u(x)|^{p-2}u(x), \quad x \in (0, \hat{\pi}), \\ \mid u'(0) \mid^{p-2} u'(0) = -\alpha \mid u(0) \mid^{p-2} u(0), \\ \mid u'(\hat{\pi}) \mid^{p-2} u'(\hat{\pi}) = \beta \mid u(\hat{\pi}) \mid^{p-2} u(\hat{\pi}). \end{cases} \tag{1.1}$$

Meanwhile $\Delta_p u = (|u'|^{p-2}u')'$ and $\hat{\pi} = \displaystyle\int_0^1 \frac{2dt}{(1-t^p)^{\frac{1}{p}}} = \frac{2\pi}{p \sin\left(\frac{\pi}{p}\right)}$ is the first

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zero of $\sin_p(x)$; here $\sin_p(x)$ is called the generalized sine function (see [16]) which is defined for $x \in [0, \frac{\hat{\pi}}{2}]$ implicitly by the following formula

$$x = \int_0^{\sin_p(x)} \frac{dt}{(1-t^p)^{\frac{1}{p}}}.$$

α and β are two constants and there exists a physical reason to writing with opposite signs (see [25]). Robin conditions model physical scenarios where the boundary interacts with the environment, such as elastic supports or heat exchange, making them highly relevant in engineering and physics [22, 25]. Our objective is to determine the optimal weight w that minimizes $\Lambda[w]$ among concave functions, a question with implications for designing systems with desired vibrational properties.

The study of eigenvalue ratios has a rich history, particularly for linear vibrating string equations ($p = 2$). Keller [21] initiated early investigations into minimizing eigenvalue ratios, establishing foundational results for Sturm-Liouville problems. Huang [15] demonstrated that for Dirichlet boundary conditions, constant weights minimize $\Lambda[w] = 4$ among symmetric single-well densities, while achieving $\Lambda[w] \geq 4$ for concave or symmetric single-barrier densities, with equality only for constant weights. Horváth [14] extended these findings to non-symmetric single-barrier densities, removing symmetry constraints. More recently, Gu and Sun [13] explored the eigenvalue ratio for vibrating strings with single-barrier densities and mixed boundary conditions, showing that boundary variations significantly affect optimal weights. These results highlight the interplay between weight functions and boundary conditions in determining spectral properties, but they are limited to the linear case, where analytical solutions are more tractable.

For the p-Laplacian, the nonlinear nature complicates analysis, requiring advanced tools such as Prüfer-type transformations and variational methods [16, 19]. Cheng et al. [11] made significant progress by studying the p-Laplacian with Dirichlet boundary conditions, proving that constant weights minimize $\Lambda[w]$ for single-barrier densities, generalizing Huang's results to $p > 1$. Their work leveraged the generalized trigonometric functions $\sin_p(x)$ and $\cos_p(x)$, which play a role analogous to sine and cosine in the linear case [18, 20]. However, the p-Laplacian with Robin boundary conditions remains underexplored, as the mixed boundary constraints introduce additional complexity in eigenfunction behavior and eigenvalue dependence on w. Preliminary studies, such as those by Ahrami and El Allali [1, 5], suggest that Robin conditions may alter the optimality of constant weights, but a comprehensive analysis for concave weights is lacking. This gap motivates our investigation, as the nonlinear dynamics and boundary flexibility offer a fertile ground for new mathematical insights.

Our main contribution is to prove that the constant weight does not minimize $\Lambda[w]$ for the p-Laplacian with Robin boundary conditions among concave weights, challenging the intuition from Dirichlet cases. Instead, we show that affine weights of the form $w(x) = ax + b$ with $a \neq 0$ often yield lower eigenvalue ratios, suggesting a dependence on the boundary parameters α and β. This result has practical implications for optimizing vibrational systems, such as musical instruments or structural components, where non-constant density profiles may enhance performance [9, 23]. Methodologically, we employ a variational approach, building on the Feynman-Hellmann formula to analyze eigenvalue perturbations, and leverage technical lemmas on eigenfunction monotonicity to characterize optimal weights [11, 12].