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On Dynamics of Certain Models via Generalized Conformable Fractional Derivative

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Abstract This paper deals with the generalized conformable fractional derivative and certain interesting properties which are not compatible with Riemann-Liouville and Caputo fractional derivatives. The newly defined derivative is more efficient than other conformable fractional derivatives and the nonlocal fractional derivatives from a time perspective. To justify the claim, we provide some direct applications, such as population growth, Newton's body cooling, heat equation and susceptible-infected-removed models. Solutions obtained from models and comparison with respective previous data are demonstrated with the help of graphs or stems.

Keywords Fractional derivative, SECH-fractional derivative, fractional population growth model, fractional body cooling model, fractional heat equation, SIR model

MSC(2010) 26A33, 34A08, 35R11, 92D25.

1. Introduction

The geometry of fractional derivative of a function has been an important open problem in mathematics since 1695. In this context several definitions are proposed, where some are local and some are non-local in nature. Due to the absence of memory, ordinary derivative is considered as local derivative. In contrast, most of the popular fractional derivatives like Riemann-Liouville and Caputo possess a memory factor, so they are called non-local derivatives. Among all the non-local fractional derivatives, Riemann-liouville's definition [4, 19, 20] is the most popular and for order β it is defined as:

$$\left({}_{c}^{RL}D_{t}^{\beta}f\right)(t)=\frac{1}{\Gamma(n-\beta)}\frac{d^{n}}{dt^{n}}\int_{c}^{t}(t-x)^{n-\beta-1}f(x)dx,\qquad t>c,$$

where β is a positive real number and f is the 1^{st} order differentiable function with $n-1\leq\beta<n$, $n\in\mathbb{N}$(the set of all positive integers). Caputo fractional derivative [4, 19, 20] of order β is defined as:

$$\left({}_{c}^{C}D_{t}^{\beta}f\right)(t)=\frac{1}{\Gamma(n-\beta)}\int_{c}^{t}(t-x)^{n-\beta-1}f^{n}(x)dx,\qquad t>c,$$

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where β is a positive real number and f is the n^{th} order differentiable function with $n-1 \leq \beta < n$, $n \in \mathbb{N}$.

In [14] Khalil et al. first introduced the concept of conformable fractional derivative of order β, which is local in nature and is defined as:

$$T_\beta(f)t = \lim_{\epsilon \to 0} \frac{f(t+\epsilon t^{1-\beta}) - f(t)}{\epsilon},$$

where $f : [0,\infty) \to \mathbb{R}$(the set of all real numbers), $\beta \in (0,1)$ and $t > 0$. Moving forward, the geometrical and physical properties of Khalil's conformable derivative were given in [9] and some of the major theorems and applications were given in [5–7, 11, 15, 18]. An extension of this conformable fractional derivative were given in [10] with the dynamical properties of some nonlinear partial differential equations along its solutions. Also, the dynamical properties of a non-linear fractional Schrödinger equation are compared using different known differential operators. The advantages of this derivative are that it exhibits some general properties of calculus which are not compatible with Riemann-Liouville and Caputo fractional derivative. Geometrically this idea is related to tangent vector i.e., we can approximate a tangent vector at a point where the ordinary tangent is not available. Another generalized conformable definition of β order fractional derivative was found in [3, 7], which is defined by:

$$N_F^\beta f(t) = \lim_{\epsilon \to 0} \frac{f(t+\epsilon F(t,\beta)) - f(t)}{\epsilon}, \tag{1.1}$$

where $f : [0,+\infty) \to \mathbb{R}$, F is arbitrary, $\epsilon > 0$,$t > 0$ and $\beta \in (0,1)$. For $F(t,\beta) = \exp(\beta - 1)t$, we get the definition given by Katugampola [12] and denoted by $D^\beta f(t)$. Generalized conformable type derivatives are not a suitable option for a process which involves memory concept. Other than this, they are used in several well-known modeling problems and provide more precise and good outcomes.

In population modelling, there are many popular models which involve different kinds of operators. The exponential population growth model is a well-accepted model for a small time interval. The logistic model is considered when we take environmental or artificial restrictions into the consideration. In [2, 6, 15] exponential population model was given with Khalil's conformable fractional derivative and also the error was depicted graphically.

Another widely interpreted modelling with integer order derivative is Newton's body cooling. With the help of different non-local fractional operators, Newton's body cooling model was well explained in [5, 18], where they compared the theoretical data with the experimental data carried out under different types of liquid and fluid medium. In [7], a simulation based model was presented for Newton's body cooling, where they used Markov chain, Monte Carlo simulations(MCMC), likelihood function and different distribution techniques. The results are compared graphically.

Heat equation model is one of the most important models in mathematical physics which involves integer order partial differential equations. This concept can be extended to fractional case, which may involve different kinds of partial fractional operators. In [6, 13], heat equation was expressed in terms of Khalil's conformable derivative and its solutions are depicted.

Continuing the applications of conformable fractional derivative, the idea was utilized in SIR model, which provides the data of transmission among susceptible, infected and removed compartments. In [1], they dealt with higher-order models with Caputo fractional derivative with the reproduction number and the change in dynamics of different compartments.

Motivated by the literature, this work is divided into three sections and the study is carried out with SECH-fractional derivative. The first section contains some literature of the present study and some useful definitions. We also provide some theoretical data on different types of modeling problems. Moving forward to next section, we give some properties of SECH-fractional derivative and also provide a comparable graph of different operators for different values of β. The third section contains three important models involving SECH-fractional derivative.

2. Definitions and new results

The unified definition defined in Eqn.(1.1) has been extended with $'\operatorname{sech}'$ function and can be defined as follows:

Definition 2.1. Suppose that the first order derivative of H exists in $\mathbb{R}$. Then the βth-order extensional conformable derivative or βth-order SECH-fractional derivative [10] for H is defined by:

$$ {}^{s}D^{\beta}H(t) = \lim_{h\to 0} \frac{H[t + h\operatorname{sech}(1-\beta)t] - H(t)}{h}, \tag{2.1} $$

where $t \in \mathbb{R}$ and $0 < \beta \leq 1$, $\operatorname{sech}(1-\beta)t \in [0,1]$. Eqn.(2.1) is reduced to

$$ {}^{s}D^{\beta}H(t) = \operatorname{sech}(1-\beta)t\frac{dH(t)}{dt}, \tag{2.2} $$

where $0 < \beta \leq 1$.

Remark 2.1. For $\beta = 1$ formula(2.2) satisfies the results of ordinary derivative of order 1.

Definition 2.2. Suppose that the n^{th}-order derivatives of H exist in $\mathbb{R}$. The β^{th}-order extensional fractional derivative or β^{th}-order SECH-fractional derivative [10] for H is defined by:

$$ {}^{s}D^{\beta}H(t) = \lim_{h\to 0} \frac{H^{n}[t + h\operatorname{sech}(n+1-\beta)t] - H^{n}(t)}{h}, \tag{2.3} $$

where $t \in \mathbb{R}$, $n < \beta \leq n+1$.

Similar to Eqn.(2.1), Eqn.(2.3) can be reduced to

$$ {}^{s}D^{\beta}H(t) = \operatorname{sech}(n+1-\beta)t\frac{d^{n+1}H(t)}{dt^{n+1}}, \tag{2.4} $$

where $n < \beta \leq n+1$. This definition is an improved version of conformable fractional derivative as its independent variable contains the original domain of conformable fractional derivative so this definition is more useful while dealing with the domain of real numbers.

Remark 2.2. For $\beta = n \in \mathbb{N}$, Eqn.(2.4) coincides with n^{th}-order ordinary derivative.

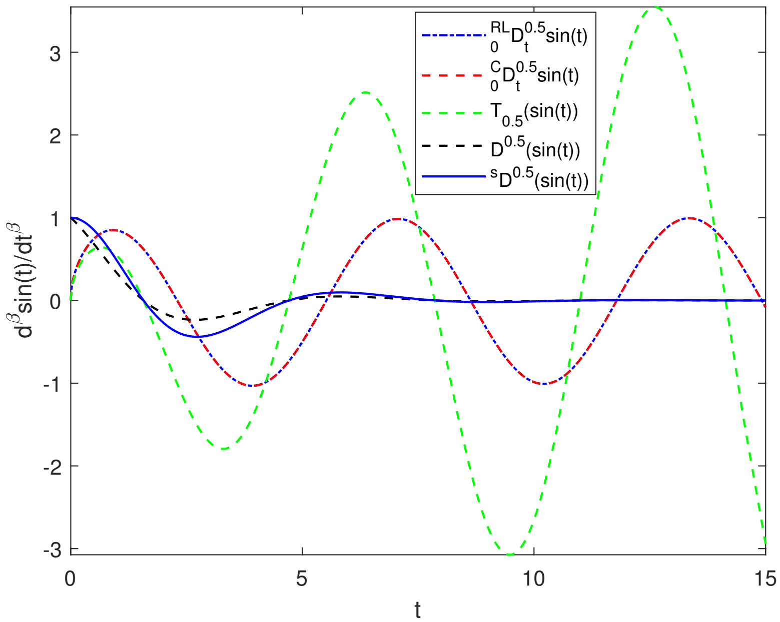

Figure 1. Visualization of different local and non-local fractional derivatives for $\beta = 0.5$ and $f(x) = \sin(x)$ such as Riemann-Liouville fractional derivative, Caputo fractional derivative and three conformable fractional derivatives.

The ordinary integer order derivative is local in nature but most of the fractional derivatives like Riemann-Liouville, Caputo are non-local in nature. This is the most basic and most important property of fractional derivative which provides an advantage to deal with the problems from nature, which are generally dynamic. The conformable fractional derivative is the opposite of Riemann-Liouville and Caputo fractional derivative in nature, as they are local fractional derivatives, so the SECH-fractional derivative is not a fractional derivative in true sense yet. More preciously, Riemann-Liouville and Caputo fractional derivatives possess a memory factor, i.e., the future state is totally dependent upon the past states, so they are called non-local derivatives, but the "SECH-fractional derivative" is evaluated on an infinitesimally small neighborhood over a point, and it fails to capture all the past states of that point, so the "SECH-fractional derivative" is local in nature. Habitually, we still call it SECH-fractional derivative. To enhance the concept, in Fig.1 we provide a visual depiction of the SECH-conformable, Khalil's conformable, exponential conformable , Riemann-Liouville and Caputo fractional derivatives with different orders, while with Fig.2 depicts the local property of SECH-conformable fractional derivative of $\sin(t)$ with different orders. The behavior of the conformable SECH and exponential conformable fractional derivatives approximately equally occur on a short time scale and their graphical properties coincide on a long time scale. From Fig.2 we observe that, on a large time scale the behaviour of conformable SECH-fractional derivative is independent of the order of the fractional derivative, so our objective is to observe all the models on a short time scale.

Theorem 2.1. *If $H : \mathbb{R} \to \mathbb{R}$ is βth-order differentiable function at $t_0 \in \mathbb{R}$, then*

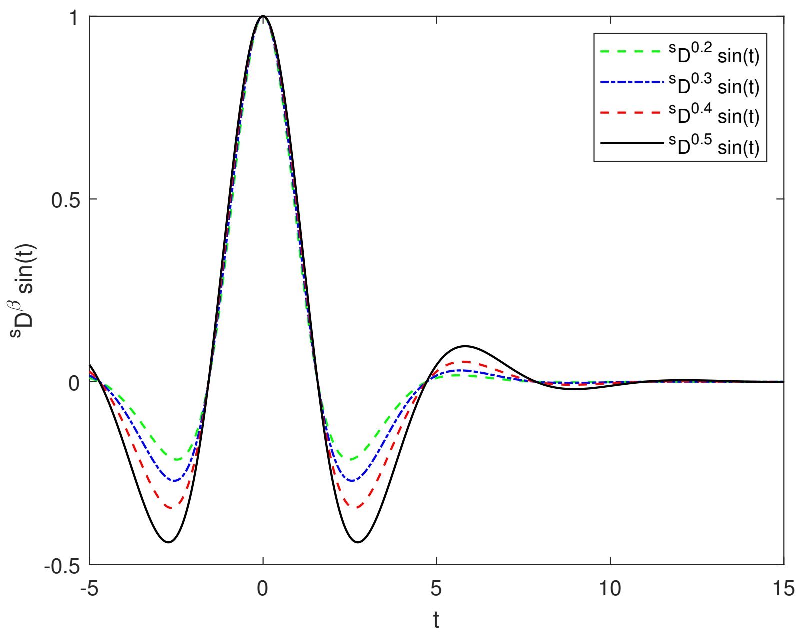

Figure 2. Visual depiction of ${}^sD^\beta \sin(t)$ for $\beta = 0.2, 0.3, 0.4, 0.5$.

H is continuous at t_0.

Proof. The proof can be done in the similar way as done in [6]. □

Theorem 2.2. *Let H and G be βth-order differentiable functions. Then,*

1. ${}^sD^\beta[aH + bH] = a{}^sD^\beta[H] + b{}^sD^\beta[H]$.
2. ${}^sD^\beta[t^n] = [nt^{n-1}]\,\mathrm{sech}(1-\beta)t$.
3. ${}^sD^\beta[k] = 0$, *where k is constant.*
4. ${}^sD^\beta[HG] = ({}^sD^\beta[H])G + H({}^sD^\beta[G])$.
5. ${}^sD^\beta[H/G] = \frac{({}^sD^\beta[H])G - H({}^sD^\beta[G])}{G^2}$.

Proof. *As a routine verification, the proof of this theorem is omitted.* □

Remark 2.3. For SECH-fractional derivative the superposition principle shows deviation; i.e.

$$ {}^sD^\beta[{}^sD^\beta H(t)] = {}^sD^{2\beta} H(t), \text{ does not hold in general.} $$

Lemma 2.1. *Let $H(t)$, $G(t)$ and $H(G(t))$ be the first order continuously differentiable functions in $\mathbb{R}$. Then the direct order chain rule with respect to SECH-fractional derivative does not hold; i.e.*

$$ {}^sD^\beta H(G(t)) = [{}^sD^\beta_{G(t)} H(G(t))][{}^sD^\beta G(t)], \text{ is not true in general.} $$

But it aligns with indirect chain rule; i.e.

$$ {}^sD^\beta H(G(t)) = [\mathrm{sech}(1-\beta)t][f'(g(t))g'(t)]. $$

Proof. *As a routine verification, the proof of this lemma is omitted.* □

Theorem 2.3. *Let $\beta \in (0,1]$. Then,*

1. ${}^{s}D^{\beta}\left(\frac{\sinh(1-\beta)t}{1-\beta}\right) = 1.$
2. ${}^{s}D^{\beta}\left(\sin\left(\frac{\sinh(1-\beta)t}{1-\beta}\right)\right) = \cos\left(\frac{\sinh(1-\beta)t}{1-\beta}\right).$
3. ${}^{s}D^{\beta}\left(\cos\left(\frac{\sinh(1-\beta)t}{1-\beta}\right)\right) = -\sin\left(\frac{\sinh(1-\beta)t}{1-\beta}\right).$
4. ${}^{s}D^{\beta}\left(e^{\frac{\sinh(1-\beta)t}{1-\beta}}\right) = e^{\frac{\sinh(1-\beta)t}{1-\beta}}.$

Proof. *As a routine verification, the proof of this theorem is omitted.* □

3. Applications

Now we provide some models, which are solved by using SECH-conformable fractional derivative.

3.1. Population growth model with SECH-fractional derivative

The growth of population has been an important question since the ancient ages. Several population growth models were described in [21]. Malthusian population growth model is the simplest growth model, which predicts the growth under ideal condition where the growth is dynamic in nature. The prediction of the population size $z(t)$ plays an important role in sociology, epidemiology, biology, ecology, etc. The variation of population over time is studied in population growth model, which is defined by:

$$\frac{dz}{dt} = cz(t), \tag{3.1}$$

with the initial condition $z(0) = k$. Here c is a constant equal to birth rate minus death rate, k is the initial value of population and z is the population function. The solution of Eqn.(3.1) is given by:

$$z(t) = ke^{ct}. \tag{3.2}$$

A graphical depiction was provided in Fig-(3) between the data provided by United Nation [22] and the first order derivative population growth model Eqn.(3.2).
The SECH-fractional population growth model is represented as

$${}^{s}D^{\beta} z(t) = cz(t), \text{z(0)=k}. \tag{3.3}$$

3.2. Body cooling model with SECH-fractional derivative

Newton's law of cooling states that the heat release from a body is directly proportional to the temperature difference between the body and its surroundings. From experimental perspective, if the environmental medium is fluid or liquid then the

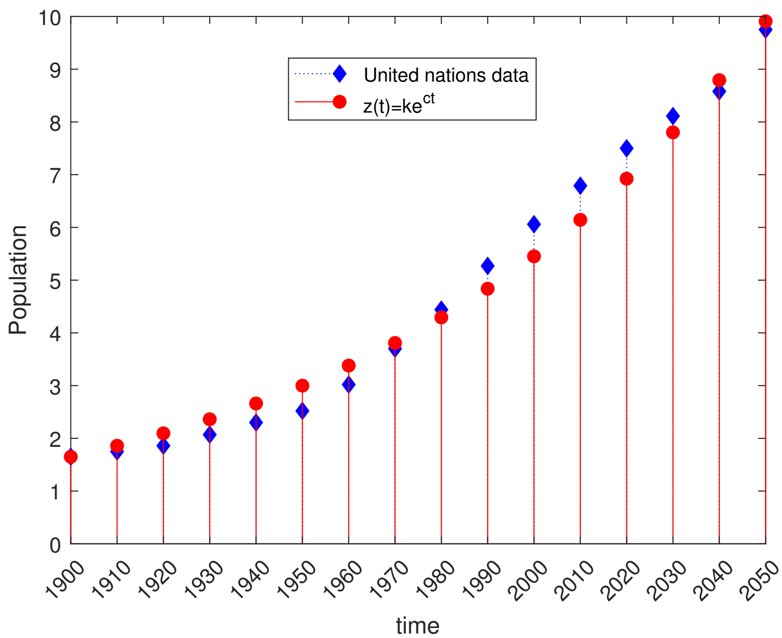

Figure 3. Graphical comparison between United Nations data of growing population [22] from 1900 to 2050 and exponential population growth function data of first order derivative over time. Here the blue stem denotes the population over time in billions and red stem denotes the theoretical values in billions over growing time.

rate of flow is directly depends on the density and surface area of the medium. We may add different types of catalyst to adjust the rate of flow. In [18] experimental analysis were carried out with different media, where Riemann-Liouville and Caputo derivatives are used. Another fractional representation of Newton's body cooling model was given in [15] with the help of Khalil conformable fractional derivative. Continuing with the fractional operators, [5] showed dynamic properties of different types of fractional operators. If the coefficient of heat transfer between the heat transfer and heat sink is constant, then the equation of body cooling is defined as

$$\frac{dT}{dt} = -c(T(t) - T_a), T(0) = T_0, \tag{3.4}$$

where T_0 is the temperature at 0, T_a is the ambient temperature and c is constant. By using initial condition, we obtain the solution of ordinary differential Eqn.(3.4) as

$$T(t) = T_a + (T_0 - T_a)e^{-ct}. \tag{3.5}$$

As defined in the previous model, we define the equation of body cooling with respect to SECH-fractional derivative as

$$^{s}D^{\beta}T(t) = -c(T(t) - T_a), T(0) = T_0. \tag{3.6}$$

3.3. Heat equation with respect to SECH-fractional derivative

In various scenarios, energy is transferred between two bodies or objects through the medium of heat energy. The heat equation, one of the core fundamental principle in physics and thermal engineering, elucidates how the temperature of a substance evolves or regress with time, influenced by both the applied heat and its thermal characteristics. The main objective of the heat equation is to predict the variation of temperature of a state over time by using the general heat equation, this concept is widely used in meterology, geophysics, thermodynamics, etc. Heat data plays a vital role in predicting weather conditions in meterology, as the other phenomena are either directly or indirectly affected by heat energy. The heat equation with respect to the ordinary derivative has played a crucial role in many fields of science and engineering. Inspired by the ordinary heat equation, the SECH-fractional heat equation is represented as:

$$\frac{{}^s\partial^\beta}{\partial t^\beta}\left(\frac{{}^s\partial^\beta z(x,t)}{\partial t^\beta}\right) = \frac{\partial^2 z(x,t)}{\partial x^2}, \tag{3.7}$$

$$z(0,t) = 0, t > 0, \tag{3.8}$$

$$z(1,t) = 0, t > 0, \tag{3.9}$$

$$\frac{\partial z(x,0)}{\partial t} = 0, \tag{3.10}$$

$$z(x,0) = f(x), 0 < x < 1, \tag{3.11}$$

where $\frac{{}^s\partial^\beta z(x,t)}{\partial t^\beta}$ represents the βth-order SECH-fractional partial derivative. The solution of Eqn.(3.7) can be obtained with the help of SECH-fractional differential equation with constant coefficients. Let us consider a SECH-fractional differential equation as:

$$\frac{{}^s d^\beta}{dy^\beta}\left(\frac{{}^s d^\beta z}{dy^\beta}\right) \pm \mu^2 z = 0. \tag{3.12}$$

In the same manner as an ordinary differential equation, we get an auxiliary equation $m^2 \pm \mu^2 = 0$ for (3.12). For the first case we obtain the roots $m = \pm \iota\mu$ and for the second case we obtain $m = \pm\mu$. With the help of Theorem 2.3, we get two independent solutions of the equation as

$$z_1 = \sin\left(\frac{\mu \sinh(1-\beta)t}{1-\beta}\right), \tag{3.13}$$

$$z_2 = \cos\left(\frac{\mu \sinh(1-\beta)t}{1-\beta}\right). \tag{3.14}$$

Also, for the second case, the two independent solutions are

$$z_1 = e^{\left(\frac{\mu \sinh(1-\beta)t}{1-\beta}\right)}, \tag{3.15}$$

$$z_2 = e^{-\left(\frac{\mu \sinh(1-\beta)t}{1-\beta}\right)}. \tag{3.16}$$

In the similar manner we can find the solutions of the differential equation

$$\frac{{}^s d^\beta}{dy^\beta}\left(\frac{{}^s d^\beta z}{dy^\beta}\right) + a\frac{{}^s d^\beta z}{dy^\beta} + bz = R, \tag{3.17}$$

where a, b are given constants and $R(t)$ is a given function of t. Using the results from Theorem 2.3 we can find the solution of (3.17).

Now we solve fractional conformable fractional differential equation given in Eqns.(3.3),(3.6) and (3.7)-(3.11) by means of the following theorems.

Theorem 3.1. *The solution of population growth model given in Eqn.*(3.3) *with respect to SECH-fractional derivative is exponentially growing, i.e.,*

$$z(t) = ke^{c\frac{\sinh(1-\beta)t}{1-\beta}}.$$

Proof. The SECH-fractional population growth model is represented as,

$$^{s}D^{\beta}z(t) = cz(t), \text{z(0)=k}. \tag{3.18}$$

The important result of ordinary differential equation yields the solution of Eqn.(3.18) as

$$z(t) = ke^{c\frac{\sinh(1-\beta)t}{1-\beta}},$$

where k is the initial value at time '0'. A comprehensive visualization of the solution of (3.18)is in Fig.4. In Fig.5, we provide a visual depiction between the growing value of β(i.e., for $0.01, 0.41$ and 0.81) and the data of united nations from 1900 to 2050. □

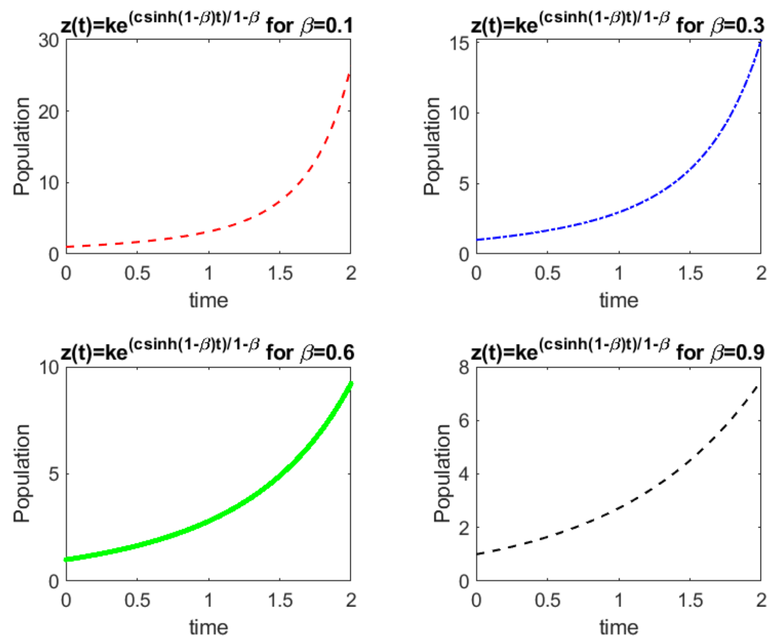

Figure 4. Depiction of solution of population growth model with SECH-fractional derivative for different values β.

Theorem 3.2. *The solution of fractional body cooling method given in Eqn.*(3.6) *is*

$$T(t) = T_a + (T_0 - T_a)e^{-\frac{c\sinh(1-\beta)t}{1-\beta}}. \tag{3.19}$$

Proof. The proof is straightforward by using results from ordinary derivative. □

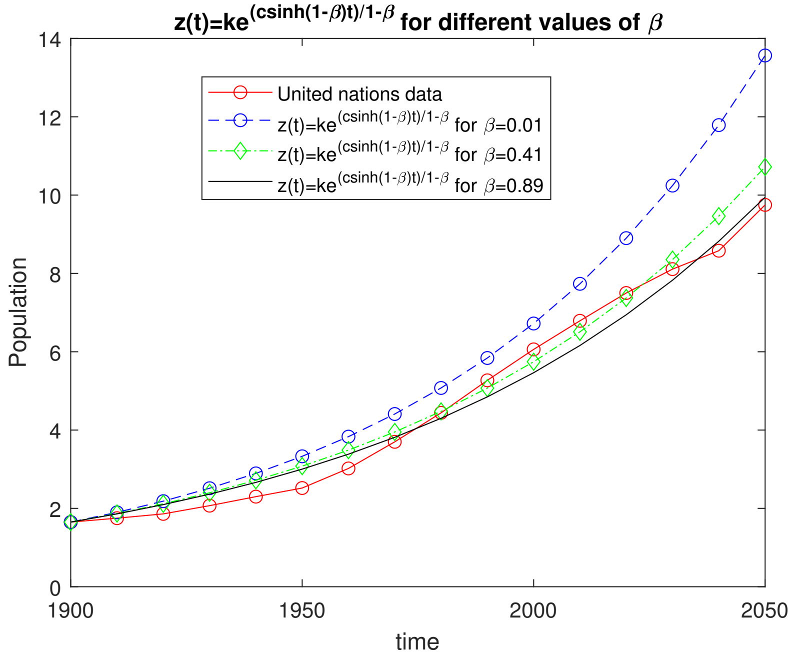

Figure 5. Graphical depiction between United Nations data of growing population [22] from 1900 to 2050 and data of exponential population growth function with respect to SECH-fractional derivative over time. Here the red line(line with circles) denotes the population over time in billions and blue line(dashed line with circles) denotes the values of SECH-fractional derivative for $\beta = 0.01$, green line(dotted line with squares) denotes the values of SECH-fractional derivative for $\beta = 0.41$ and black line denotes the values of SECH-fractional derivative for $\beta = 0.81$ in billions over growing time.

In Fig.6 we, demonstrate the convergence of temperature with the ambient temperature(with $T_a = 26.1$) for different values of β. Riemann-Liouville fractional derivative, Caputo fractional derivative, the first order and SECH-fractional derivative body cooling models are taken into consideration.

CASE-1(Observation for $\beta = 0.09, 0.39$ and $\beta = 0.59$) Caputo and Riemann-Liouville fractional derivative body cooling models need more time to match the body temperature with ambient temperature which is intolerable. The first order derivative body cooling model converges more rapidly than fractional Caputo and Riemann-Liouville model. SECH-fractional derivative model achieves ambient temperature rapidly.

CASE-1(Observation for $\beta = 0.89$) Caputo fractional derivative model shows a higher convergence rate than Riemann-Liouville model but this still very high. For $\beta = 0.89$ SECH-fractional body cooling model reaches ambient temperature at a slower rate, so for lower values of β the rate of convergence is higher.

Due to the weight function in SECH-fractional derivative, the time of convergence of SECH-fractional body cooling model is faster than others. In the physical way the weight function acts like a catalyst which turns around the time taken by the process to minimum. If it is possible we can add a catalyst to the medium which satisfies the characteristics of weight function. For different types of β we can get

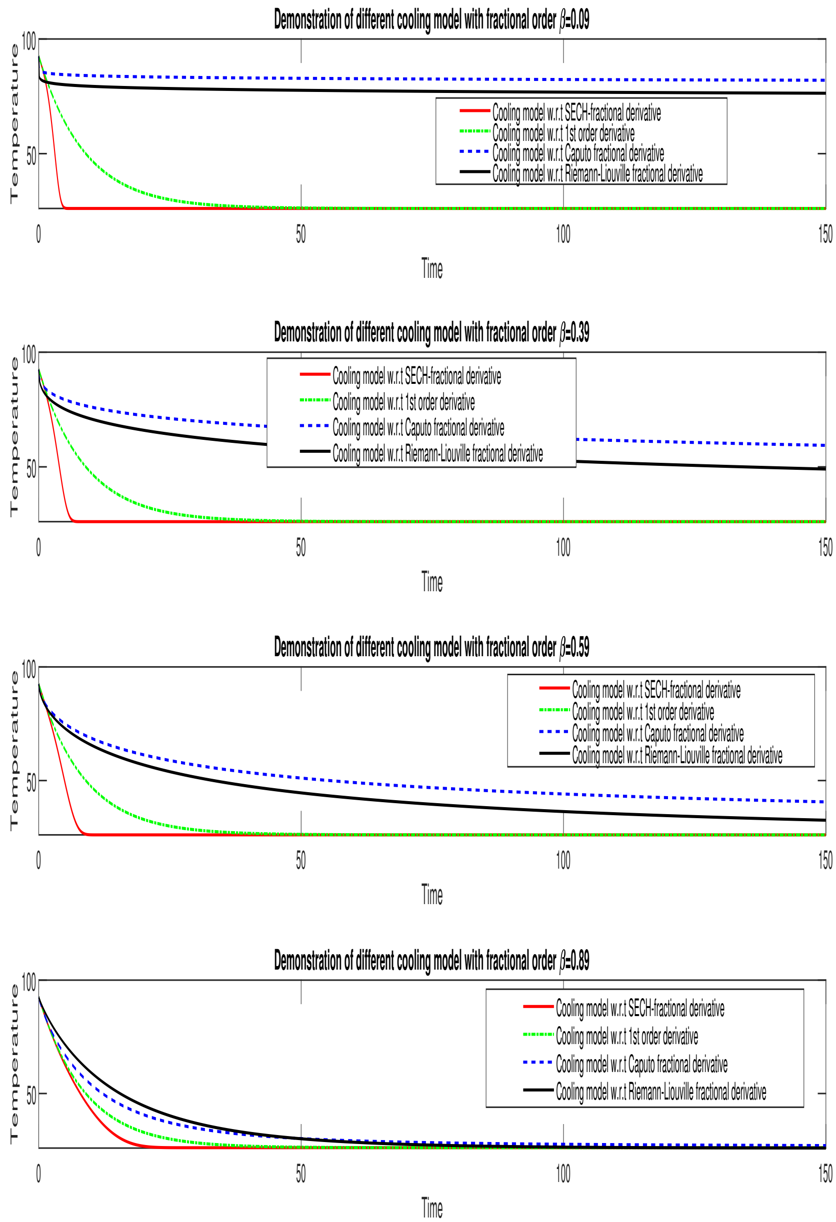

Figure 6. Newton's body cooling model for variable fractional order under different fractional derivatives

different values of catalysts but the catalyst corresponding to small values of β can achieve the ambient temperature in less time.

Theorem 3.3. *Let fractional heat equation be represented as,*

$$\frac{{}^{s}\partial^{\beta}}{\partial t^{\beta}}\left(\frac{{}^{s}\partial^{\beta} z(x,t)}{\partial t^{\beta}}\right) = \frac{\partial^{2} z(x,t)}{\partial x^{2}}, \tag{3.20}$$

$$z(0,t) = 0, t > 0, \tag{3.21}$$

$$z(1,t) = 0, t > 0, \tag{3.22}$$

$$\frac{\partial z(x,0)}{\partial t} = 0, \tag{3.23}$$

$$z(x,0) = f(x), 0 < x < 1, \tag{3.24}$$

where $\frac{{}^{s}\partial^{\beta} z(x,t)}{\partial t^{\beta}}$ *represents the* β*th-order SECH-fractional partial derivative. Then its solution is given by,*

$$z(x,t) = P(x)Q(t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) cos\left(\frac{n\pi \sinh(1-\beta)t}{1-\beta}\right). \tag{3.25}$$

Proof. Now, we find the solution of SECH-fractional heat equation with the help of the above results and by using the concept of separation of variable. Let us assume $z(x,t) = P(x)Q(t)$. By substituting $z = PQ$, we obtain

$$\frac{{}^{s}d^{\beta}}{dt^{\beta}}\left(\frac{{}^{s}d^{\beta} Q(t)P(x)}{dt^{\beta}}\right) = \frac{d^{2} Q(t)P(x)}{dx^{2}}$$

$$\Longrightarrow \frac{\frac{{}^{s}d^{\beta}}{dt^{\beta}}\left(\frac{{}^{s}d^{\beta} Q(t)}{dt^{\beta}}\right)}{Q(t)} = \frac{\frac{d^{2}P(x)}{dx^{2}}}{P(x)} = \sigma, \tag{3.26}$$

where σ is some constant. Then we get

$$\frac{{}^{s}d^{\beta}}{dt^{\beta}}\left(\frac{{}^{s}d^{\beta} Q(t)}{dt^{\beta}}\right) - \sigma Q(t) = 0 \text{ and} \tag{3.27}$$

$$\frac{d^{2}P(x)}{dx^{2}} - \sigma P(t) = 0. \tag{3.28}$$

For different values for σ we get different solutions of Eqns.(3.27) and (3.28). There are three cases as follows,

1. $\sigma = 0$.
2. $\sigma = \mu^{2} > 0, \forall \mu \in \mathbb{R} - \{0\}$.
3. $\sigma = -\mu^{2} < 0, \forall \mu \in \mathbb{R} - \{0\}$.

Case-I For $\sigma = 0$, the ODEs given in (3.27) and (3.28) become

$$\frac{{}^{s}d^{\beta}}{dt^{\beta}}\left(\frac{{}^{s}d^{\beta} Q(t)}{dt^{\beta}}\right) = 0 \text{ and} \tag{3.29}$$

$$\frac{d^{2}P(x)}{dx^{2}} = 0. \tag{3.30}$$

By using Theorem 2.1 and Theorem 2.2, we obtain the solutions of (3.29) and (3.30) as

$$Q(t) = a_1 \frac{\sinh(1-\beta)t}{1-\beta} + a_2,$$
$$P(x) = b_1 x + b_2.$$

By using conditions (3.9),(3.10) and (3.11) we obtain $z(t) = 0$, which is a trivial solution.

Case-II For $\sigma = \mu^2 > 0$, Eqns.(3.27) and (3.28) become

$$\frac{{}^s d^\beta}{dt^\beta}\left(\frac{{}^s d^\beta Q(t)}{dt^\beta}\right) - \mu^2 Q(t) = 0 \text{ and} \tag{3.31}$$

$$\frac{d^2 P(x)}{dx^2} - \mu^2 P(x) = 0, \tag{3.32}$$

which yields the individual solutions as

$$Q_1(t) = e^{\frac{\mu \sinh(1-\beta)t}{1-\beta}} \text{ and } Q_2(t) = e^{\frac{-\mu \sinh(1-\beta)t}{1-\beta}},$$

and

$$P_1(x) = e^{\mu x} \text{ and } P_2(x) = e^{-\mu x}.$$

By substituting the conditions of SECH-fractional heat equation we obtain the trivial solution.

Case-III For $\sigma = -\mu^2 < 0$, the ODEs given in (3.27) and (3.28) transform to

$$\frac{{}^s d^\beta}{dt^\beta}\left(\frac{{}^s d^\beta Q(t)}{dt^\beta}\right) + \mu^2 Q(t) = 0 \text{ and} \tag{3.33}$$

$$\frac{d^2 P(x)}{dx^2} + \mu^2 P(x) = 0. \tag{3.34}$$

Using the results discussed above we get the independent solutions of (3.33) and (3.34) as

$$Q_1(t) = \sin\left(\frac{\mu \sinh(1-\beta)t}{1-\beta}\right) \text{ and } Q_2(t) = \cos\left(\frac{\mu \sinh(1-\beta)t}{1-\beta}\right),$$

and

$$P_1(x) = \sin \mu x \text{ and } p_2(x) = \cos -\mu x.$$

By using conditions (3.9) and (3.10) we get

$$\mu = n\pi \text{ (for non-trivial solutions).}$$

For $\mu = n\pi$ the solutions become

$$Q(t) = b_1 \sin\left(\frac{n\pi \sinh(1-\beta)t}{1-\beta}\right) + b_2 \cos\left(\frac{n\pi \sinh(1-\beta)t}{1-\beta}\right),$$

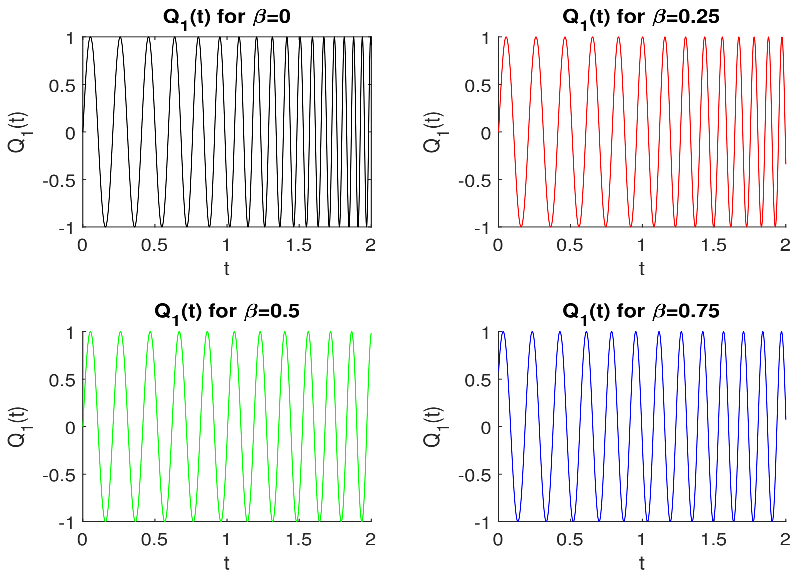

Figure 7. Graphical depiction of $Q_1(t)$ for different values of β.

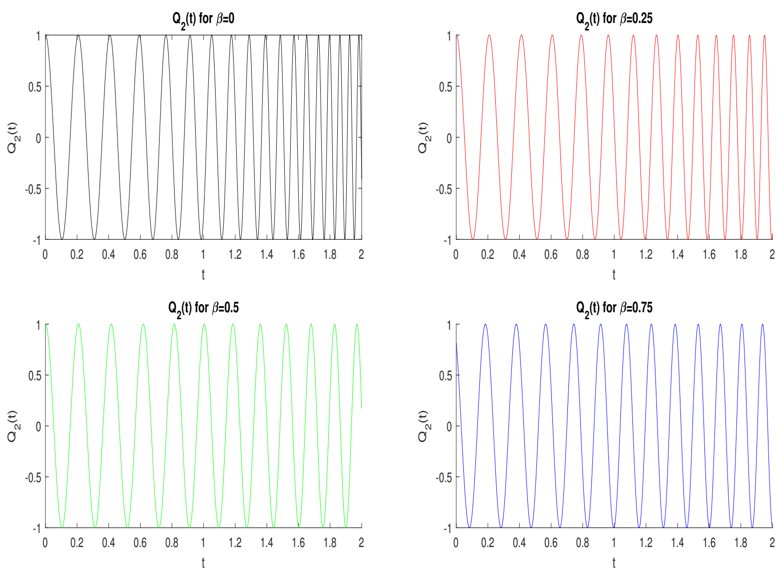

Figure 8. Graphical depiction of $Q_2(t)$ for different values of β.

and

$$P(x) = c_1 \sin(n\pi x).$$

From(3.11), we derive

$$Q(t) = b_2 \cos\left(\frac{n\pi \sinh(1-\beta)t}{1-\beta}\right).$$

Now, using $P(x)$ and $Q(t)$ we get

$$z(x,t) = P(x)Q(t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \cos\left(\frac{n\pi \sinh(1-\beta)t}{1-\beta}\right). \tag{3.35}$$

Using (3.35), and applying Fourier sin series of $f(x)$ we obtain

$$a_n = 2\int_0^1 f(x) sin(x) dx.$$

□

3.4. SIR model through conformable SECH-fractional derivative

From the social, financial and clinical perspectives mathematical modelling plays a crucial role in controlling the damage in all sense. The characteristics of one disease is different from another, so we need different mathematical models. One of the best example is SI-model, which simply gives the dynamics between susceptible and infected individuals. Similarly SIR-model gives the dynamics of a disease where recovered individuals gain complete immunity. There are a large number of diseases which can occur more than once(i.e., the patient does not get complete immunity from the disease and can be susceptible again); those diseases can not be modelled with SI [1] or SIR [1] models. SIRS(Susceptible-Infected-Removed-Susceptile) model is useful to model those kinds of diseases. The disease with a latent period can be modelled with SEIRS(Susceptible-Exposed-Infected-Removed-Susceptible) model, where the E-compartment contains the individuals who get exposed to disease but are still not infectious. SEIRS model is very commonly used to model influenza, COVID-19, Ebola and HIV/AIDS, etc. In [16, 17], global stability and dynamical properties of SEIR model were studied with non-linear incidence rates. The transformation of different models from SEIRS model was given in [17]. As with the ordinary derivative models, some researchers studied the epidemic models with fractional derivative(Riemann-Liouville and Caputo). In [1], Agarwal et al. gave numerous fractional epidemic models(Kermack-McKendric SIR MODEL, SIS, IR, SIRS, SEIR, SEIRS, SIVR, SIRD, MSIR and MSEIR) with the demonstration of transmission dynamics. Also they modelled an SIR-fractional epidemic model with non-linear incidence rates. They found the reproduction number and stability of each model. Nowadays, conformable fractional derivative plays an important role like non-conformal fractional derivative. Harir et al. [8] studied the solution of conformable SECH-fractional order SIR epidemic model with conformable fractional differential transformation method and variational iteration method. In this section we analyse the solutions of conformable fractional order SIR model with bilinear incidence rate. Our main objective is to find the solution of conformable SECH fractional SIR model through Euler and Runge-Kutta method and provide a visual depiction of change in dynamics with varying order.

The dynamics of SIR model with non-linear incidence rate is given below;

$$\begin{aligned}
\frac{dS}{dt} &= -\zeta S^p(t) I^q(t), \\
\frac{dI}{dt} &= \zeta S^p(t) I^q(t) - \gamma I,
\end{aligned}$$

$$\frac{dR}{dt} = \gamma I(t), \text{where } S(0) > 0, I(0) > 0, R(0) \geq 0.$$

Our main objective is to study the bilinear incidence rate of SIR model, where the values of p and q are taken as 1. So the above system becomes

$$\frac{dS}{dt} = -\zeta S(t)I(t),$$
$$\frac{dI}{dt} = \zeta S(t)I(t) - \gamma I,$$
$$\frac{dR}{dt} = \gamma I(t).$$

All the ordinary derivatives of the above system are replaced with corresponding conformable SECH-fractional derivative of order β, where $\beta \in (0, 1]$. Then the above system becomes

$${}^{s}D^{\beta}(S(t)) = -\zeta S(t)I(t),$$
$${}^{s}D^{\beta}(I(t)) = \zeta S(t)I(t) - \gamma I,$$
$${}^{s}D^{\beta}(R(t)) = \gamma I(t).$$

By using (2.2), the above system has been transformed to

$$\frac{dS}{dt} = \frac{-\zeta S(t)I(t)}{\operatorname{sech}(1-\beta)t}, = f(t, S, I), \tag{3.36}$$
$$\frac{dI}{dt} = \frac{\zeta S(t)I(t) - \gamma I}{\operatorname{sech}(1-\beta)t} = g(t, S, I), \tag{3.37}$$
$$\frac{dR}{dt} = \frac{\gamma I(t)}{\operatorname{sech}(1-\beta)t}. \tag{3.38}$$

Before delving into the solution of the above system, let us discuss the behaviour of derivatives analytically. The sech function is increasing for $t \in (-\infty, 0]$ and decreasing for $t \in (0, \infty)$, and the range of the function is $(0, 1]$. The function attains its maximum value for the point 0. Therefore we can clearly say the derivatives $\frac{dS}{dt}$, $\frac{dI}{dt}$and$\frac{dR}{dt}$ are directly proportional to $\frac{1}{\operatorname{sech}((1-\beta)t)}$, so the values of the instantaneous change in S,I and R rapidly increase with a small increase in time. To check the analytic statement we solve the above system by using Euler method(EM) and Runge-Kutta method of order 4(RK-4). The recovered population directly depends on the susceptible and infected population, so we omit the removed compartment(for calculation purpose only).

Euler method Euler method(EM) is a numerical technique for solving an ordinary differential equation with a given initial solution value. Step size for each approximation is considered as 'h'. For different values of $\beta \in (0, 1]$ the EM has been applied. In this paper, we work with the orders $\beta = 0.1, 0.5$ and 0.9. For these three orders the approximations are given in Table.1.

Runge-Kutta method of order 4 Form the perspective of stability and accuracy, Runge-Kutta method(RK) is the most popular method for solving the ordinary differential equations. Our main objective is to see the nature of solution with changing the order of the conformable SECH-fractional derivative. For the differential Eqns.(3.36),(3.37) and (3.38) with initial conditions $S(0), I(0)$ and $R(0)$, and the step size h, the iterations are given in Table.2.

Iteration	t	$S_{0.1}$	$I_{0.1}$	$R_{0.1}$	$S_{0.5}$
1	0	3000	300	200	3000
2000	1.0995	190.6846	3.1013e+03	208.0296	309.3465
4000	2.1995	0.0057	3.2617e+03	238.3237	0.8641
6000	3.2995	1.4847e-14	3.1844e+03	315.5738	8.7807e-05
8000	4.3995	1.5004e-44	2.9867e+03	513.3378	2.3022e-11
10000	5.4995	3.4959e-118	2.5137e+03	986.3230	1.9507e-22
12000	6.5995	5.9986	1.5805e+03	1.9195e+03	5.0650e-41
14000	7.6995	1.0800e-321	453.3228	3.0467e+03	1.0992e-71
16000	8.7995	1.0800e-321	15.6958	3.4843e+03	1.6260e-120
18000	9.8995	1.0800e-321	0.0018	3500	2.3510e-193
20000	10.9995	1.0800e-321	4.1707e-14	3500	7.6023e-290
22000	12.0885	1.0800e-321	4.5805e-43	3500	1.8000e-321
24000	13.1995	1.0800e-321	1.5864e-123	3500	1.8000e-321
26000	14.2995	1.0800e-321	5.4308e-319	3500	1.8000e-321

Iteration	t	$I_{0.5}$	$R_{0.5}$	$S_{0.9}$	$I_{0.9}$	$R_{0.9}$
1	0	300	200	3000	300	200
2000	1.0995	2.9840e+03	206.6213	374.6226	2.9193e+03	206.0639
4000	2.1995	3.2754e+03	223.7474	5.1518	3.2763e+03	218.5530
6000	3.2995	3.2495e+03	250.5013	0.0576	3.2683e+03	231.6429
8000	4.3995	3.2055e+03	294.5304	5.5889e-04	3.2549e+03	245.1486
10000	5.4995	3.1316e+03	368.4057	4.4581e-06	3.2408e+3	259.2227
12000	6.5995	3.0081e+03	491.8699	2.7672e-08	3.2260e+03	274.0265
14000	7.6995	2.8059e+03	694.1403	1.2617e-10	3.2103e+03	289.7276
16000	8.7995	2.4871e+03	1.0129e+03	3.9794e-13	3.1935e+03	306.5009
18000	9.8995	2.0181e+03	1.4819e+03	8.1484	3.1755e+03	324.5303
20000	10.9995	1.4049e+03	2.0951e+03	1.0132e-18	3.1560e+03	344.0098
22000	12.0885	749.8506	2.7501e+03	7.1270e-22	3.1349e+03	365.1451
24000	13.1995	252.5422	3.2475e+03	2.6302e-25	3.1118e+03	388.1547
26000	14.2995	38.2791	3.4617e+03	4.7006e-29	3.0867e+03	413.2704

Table 1. Iterations of SIR model by Euler method

Depiction and Discussion In Figs.12 and 9, dynamics of susceptible population in SIR model are shown for three fractional orders ($\beta = 0.1$, 0.5 and 0.9). For the step size $h = 0.00055$, initial conditions S(0)=3000, I(0)=300, R(0)=200, the rate of transmission form susceptible to infected population is (ζ= 0.00012), the rate of removed individual from infected population (γ=0.00035) the Euler and RK4 approximation were made. For smaller fractional orders the rate of convergence of susceptible population in SIR model to appropriate place(that is the susceptible population approximate to zero) is more faster than the higher one. In fig-12 and fig-10, it is shown that for both the method, the convergence by smaller fractional order is faster. In both the approximation method, removed population is very slowly approximate to total population for higher value of β. When β is close to 1, then the SECH-conformable fractional SIR model is nearly equal to ordinary derivative SIR model.

Conclusion: In this study we carried out theoretical experiment on population growth model, Newton's body cooling model, heat equation model and Susceptible-Infected-Removed(SIR) model with respect to conformable SECH-fractional derivative. Also we derived some basic properties of SECH-fractional derivative. According to data from United Nation, the population of the world matches with exponential growth with respect to SECH-fractional derivative. It acts as a catalyst in Newton's body cooling model, which speeds up the process and takes less time to

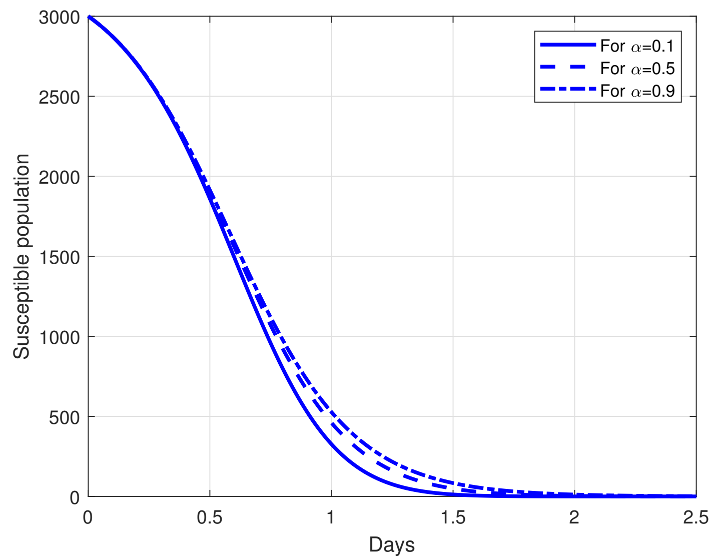

Figure 9. Dynamics of susceptible individuals in SIR model through EM

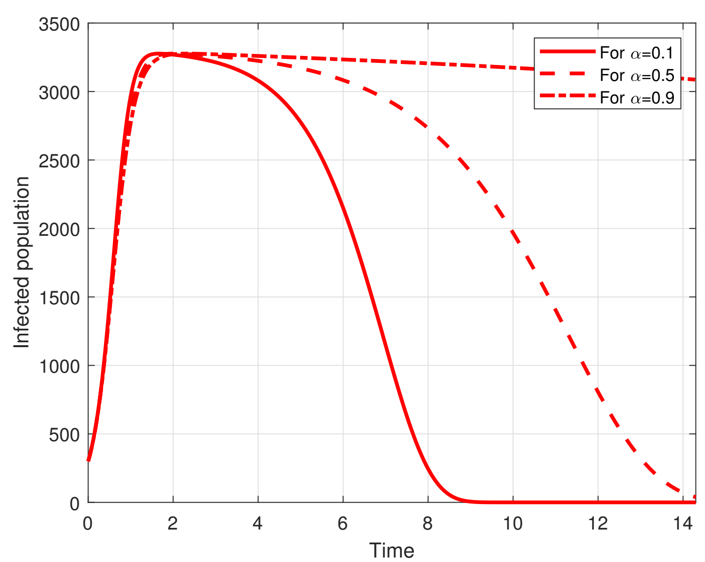

Figure 10. Dynamics of infected individuals in SIR model through EM

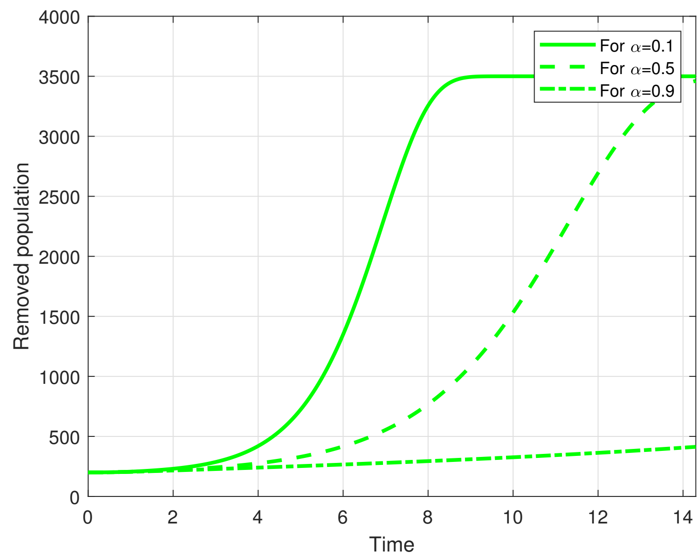

Figure 11. Dynamics of removed individuals in SIR model through EM

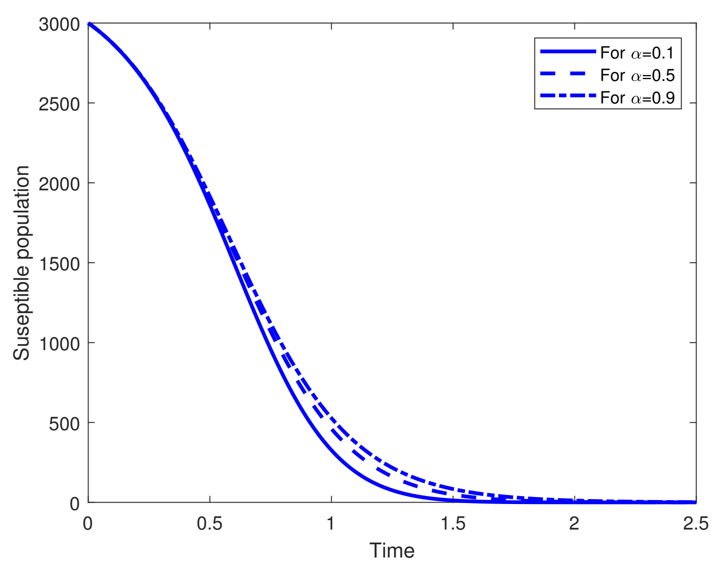

Figure 12. Dynamics of susceptible individuals in SIR model through RK4

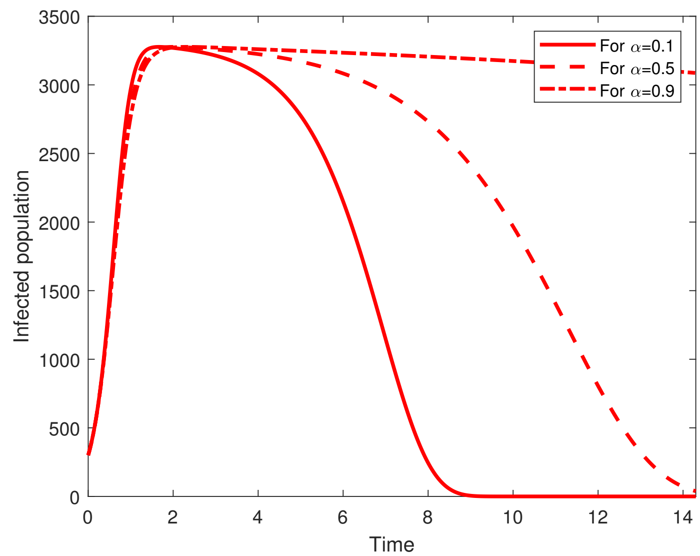

Figure 13. Dynamics of infected individuals in SIR model through RK4

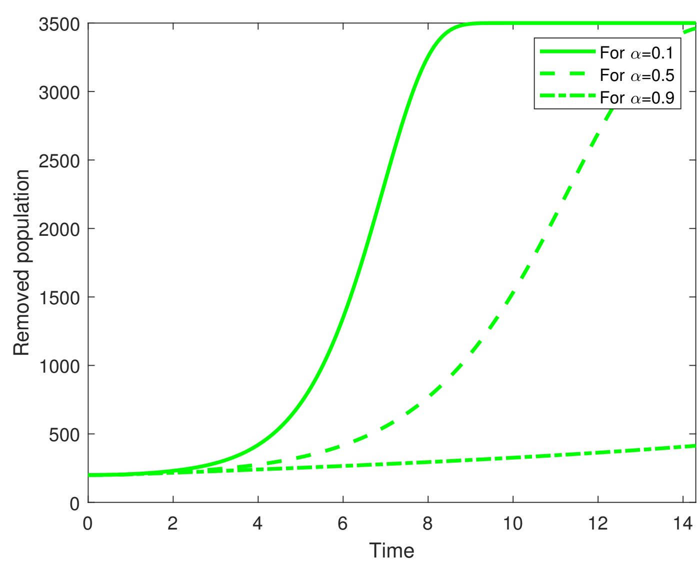

Figure 14. Dynamics of removed individuals in SIR model through RK4

Iteration	t	$S_{0.1}$	$I_{0.1}$	$R_{0.1}$	$S_{0.5}$
1	0	3000	300	200	3000
2000	1.0995	190.8718	3.1011e+03	208.0347	309.3727
4000	2.1995	0.0059	3.2617e+03	238.3286	0.8718
6000	3.2995	1.8511e-14	3.1844e+03	315.5781	9.0519e-05
8000	4.3995	6.6179e-44	2.9867e+03	513.3385	2.5165e-11
10000	5.4995	2.4879e-114	2.5137e+03	986.3034	2.5163e-22
12000	6.5995	2.9179e-253	1.5806e+03	1.9194e+03	1.0400e-40
14000	7.6995	1.7600e-321	453.5447	3.0465e+03	7.9310e-71
16000	8.7995	1.7600e-321	15.7542	3.4842e+03	2.7468e-118
18000	9.8995	1.7600e-321	0.0019	3500	4.000e-188
20000	10.9995	1.7600e-321	5.0292e-14	3500	1.9950e-279
22000	12.0995	1.7600e-321	1.8131e-42	3500	1.8400e-321
24000	13.1995	1.7600e-321	5.1465e-119	3500	1.8400e-321
26000	14.2995	1.7600e-321	9.6876e-320	3500	1.8400e-321

Iteration	t	$I_{0.5}$	$R_{0.5}$	$S_{0.9}$	$I_{0.9}$	$R_{0.9}$
1	0	300	200	3000	300	200
2000	1.0995	2.9840e+03	206.6261	374.5541	2.9194e+03	206.0685
4000	2.1995	3.2754e+03	223.7522	5.1745	3.2763e+03	218.5577
6000	3.2995	3.2495e+03	250.5059	0.0582	3.2683e+03	231.6475
8000	4.3995	3.2055e+03	294.5348	5.6724e-04	3.2548e+03	245.1532
10000	5.4995	3.1316e+03	368.4096	4.5512e-06	3.2408e+03	259.2273
12000	6.5995	3.0081e+03	491.8724	3.8433e-08	3.2260e+03	274.0311
14000	7.6995	2.8059e+03	694.1391	1.3059e-10	3.2103e+03	289.7321
16000	8.7995	2.4871e+03	1.0129e+03	4.1530e-13	3.1935e+03	306.5054
18000	9.8995	2.0181e+03	1.4819e+03	8.5859e-16	3.1755e+03	324.5447
20000	10.9995	1.4049e+03	2.0951e+03	1.0796e-18	3.1560e+03	344.0141
22000	12.0995	749.9630	2.7500e+03	7.6951e-22	3.1349e+03	365.1494
24000	13.1995	252.6567	3.2473e+03	2.8846e-25	3.1118e+03	388.1589
26000	14.2995	38.3314	3.4617e+03	5.2522e-29	3.0867e+03	413.2745

Table 2. Iterations of SIR model by RK4 method

achieve ambient/environment temperature. For small value of β the convergence is faster i.e the catalyst acts more strongly. The solution of heat equation is depicted and the behaviour of solution according to different values of β is visualized. In SECH-fractional SIR model the rate of convergence(how much time required to dilute to the original situation) of susceptible, infected and removed population directly depends on the order of the derivative. For higher order of β we get slower convergence.

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