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Three Solutions for a Perturbed Integral Equation with Homogeneous Dirichlet Condition

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Abstract We consider the integral equation $-\mathcal{L}_K^p u = \lambda f(x,u) + \mu g(x,u)$, with homogeneous Dirichlet condition on a bounded Lipschitz domain of $\mathbb{R}^N$ where $\lambda, \mu \in \mathbb{R}$, $p \geq 2$, $s \in]0,1[$, $N > ps$, $f, g : \mathbb{R}^N \longrightarrow \mathbb{R}$ are Carathéodory functions with subcritical growth and $-\mathcal{L}^p$ denotes a class of operators that includes $(-\Delta)_p^s$, the fractional p-Laplacian. Here μg represents a small perturbation of λf. Applying an abstract critical point theorem due to Ricceri, a variational setting developed by Xiang et al. and a Minti-Browder's theorem, we prove the existence of three weak solutions.

Keywords Integral elliptic equation, variational methods, critical points

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1. Introduction

Let's consider a theoretical model for the dynamics of a population living in a habitat $\Omega \subseteq \mathbb{R}^N$ which is an open Lipschitz bounded domain,

$$\partial_t u = \Lambda u + \sigma(x,u), \quad t \geq 0, x \in \Omega. \tag{1.1}$$

Here $u(t,x)$ denotes the population density at time t and position $x \in \Omega$, and the function σ represents the population supply due to births and deaths. The operator Λ, which could be integral or integro-differential, models a diffusion process which is affected by non-local population information coming e.g. from cognitive processes and that, therefore, is very far from considering the individuals as non-living particles interacting in a random way, as it was assumed in the pioneer work [22] in the 1950s.

For the study of biological populations, [24], are of interest the stationary counterparts of equations like (1.1),

$$-\Lambda u = \sigma(x,u), \quad x \in \Omega. \tag{1.2}$$

These time-independent problems have attracted the attention of mathematicians as they appear quite naturally not only in the study of population dynamics but also in continuum mechanics, phase transition phenomena, financial mathematics, game theory, etc.; see e.g. [4, 8, 13, 16, 23]. Particular attention has attracted (1.2),

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for situations where the non-local non-linear diffusion operator Λ equals or contains an integral operator given by

$$\mathcal{L}_K^p u(x) = -2 \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} |u(x) - u(y)|^{p-2}(u(x) - u(y))K(x - y)\, dy, \tag{1.3}$$

where $p > 1$. The function $K : \mathbb{R}^N \setminus \{0\} \longrightarrow \mathbb{R}$ could be interpreted as a *perceptual kernel* (or *detection function*), related to what an individual perceives, and $\mathcal{L}_K^p u(\cdot)$ could be interpreted as a kind of *resource perception function*, which is able to capture information of how the individuals perceived the resources in its habitat, [24]. In this way, the population dynamics modeled with (1.1) would be directly affected by the capacity of perception of the individuals.

We shall assume that the kernel K is positive, even and such that

$$mK \in \mathrm{L}^1(\mathbb{R}^N), \tag{1.4}$$

$$K(x) \geq \theta |x|^{-(N+sp)}, \quad x \in \mathbb{R}^N \setminus \{0\}, \tag{1.5}$$

where $m(x) = \min\{|x|^p, 1\}$ and $\theta > 0$. Observe that, in particular, $\mathcal{L}_K^p$ becomes the fractional p-Laplace operator when $K(x) = |x|^{-(N+ps)}$:

$$-(-\Delta)_p^s u(x) = 2 \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}} dy. \tag{1.6}$$

Remark 1.1. To deal with equation (1.2) where $\Lambda = (-\Delta)_p^s$, $0 < s < 1$, a first feasible space to look for weak solutions is the fractional Sobolev space $\mathrm{W}^{s,p}(\Omega)$, [10]. In this context the corresponding embeddings are related to the so-called fractional critical exponent

$$p_s^* = Np/(N - sp).$$

In [25] it was studied a Kirchhoff-type problem with homogeneous Dirichlet condition, that is, equation (1.2) with

$$u(x) = 0, \qquad x \in \mathbb{R}^N \setminus \Omega, \tag{1.7}$$

where

$$\Lambda u = M\Big(\int_{\mathbb{R}^{2N}} |u(x) - u(y)|^p K(x - y)\, dx\, dy\Big)\, \mathcal{L}_K^p u,$$

M is a continuous function and σ is a Carathéodory function verifying Ambrosetti-Rabinowitz condition. When the population supply σ presents a sublinear growth, a direct variational method is applied to obtain solutions, while a mountain-pass solution was found for the case of superlinear population supply.

In [5] and [20] it was studied (1.2) with boundary condition (1.7) for $\Lambda = \mathcal{L}_K^2$. By applying [19, Th.3], three solutions were found in [5] when the population supply has the form

$$\sigma(x, \tau) = \epsilon f(x, \tau) - \lambda g(x, \tau) + \nu h(x, \tau),$$

where $\epsilon, \lambda, \nu \in \mathbb{R}$ and the functions $f, g, h \in \mathrm{C}(\Omega \times \mathbb{R})$ present subcritical growth. On the other hand, in [20] it's found a mountain-pass solution when, for some $a_1, a_2, r > 0$, $\mu > 2$ and $q \in]2, 2_s^*[$, the function σ verifies

$$|\sigma(x, \tau)| \leq a_1 + a_2 |\tau|^{q-1}, \quad \text{for all } \tau \in \mathbb{R} \text{ and a.e. } x \in \Omega;$$