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Properties Around the First Eigenvalue of a Partial Discrete Dirichlet Boundary Value Problem

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Abstract In this article, we study some results around the first eigenvalue $\lambda_{1,m}$ for the partial discrete Dirichlet boundary value problem, like the constant sign of the eigenfunction associated with $\lambda_{1,m}$, the simplicity of $\lambda_{1,m}$ and the sign change of any eigenfunction associated with $\lambda > \lambda_{1,m}$.

Keywords Partial discrete Dirichlet problem, first eigenvalue, simplicity

MSC(2010) 39A14, 39A70, 39A10, 34B15

1. Introduction

Let λ be a positive parameter and $\mathbb{Z}[N, M]$ represent the discrete interval $\{N, N+1, ..., M\}$, where N and M are integers and $N < M$. We consider the following partial discrete problem

$$(P_\lambda)\begin{cases} -\Delta_1(\varphi_p(\Delta_1 x(i-1,j))) - \Delta_2(\varphi_p(\Delta_2 x(i,j-1))) = \lambda m(i,j)\varphi_p(x(i,j)), \\ \qquad\qquad\qquad\qquad\qquad\qquad (i,j) \in \mathbb{Z}[1,\alpha]\times\mathbb{Z}[1,\beta], \\ x(0,j) = x(\alpha+1,j) = 0, \qquad j \in \mathbb{Z}[1,\beta], \\ x(i,0) = x(i,\beta+1) = 0, \qquad i \in \mathbb{Z}[1,\alpha], \end{cases}$$

where $\alpha, \beta \geq 2$ are fixed positive integers, Δ_1 and Δ_2 denote the forward difference operators defined by $\Delta_1 x(i,j) = x(i+1,j) - x(i,j)$ and $\Delta_2 x(i,j) = x(i,j+1) - x(i,j)$, $m \in M_+ =: \{m : \mathbb{Z}[1,\alpha]\times\mathbb{Z}[1,\beta] \to \mathbb{R}/\exists(i_0,j_0) \in \mathbb{Z}[1,\alpha]\times\mathbb{Z}[1,\beta] : m(i_0,j_0) > 0\}$, φ_p denotes the p-Laplacian operator defined by $\varphi_p(u) = |u|^{p-2}u$ and $1 < p < \infty$.

Recent mathematical literature has given difference equations a great deal of attention. Research on these equations can be found at the intersection of many mathematics disciplines, including numerical analysis and nonlinear differential equations. Moreover, they are strongly motivated by their applicability to various fields of research, such as artificial or biological neural networks, mechanical engineering, control systems, and more. For studying these equations, using techniques for nonlinear analysis, many authors arrived at a variety of results, such as fixed point

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methods, Brouwer degree, and critical point theory [1–5,9,12,13,17]. Partial difference equations are those difference equations that include two or more variables and are less frequently studied [7,11,14,16]. Partial difference equations have recently been extensively utilized in a variety of fields.

In 2010, Galewski and Orpe [10] demonstrated the existence of solutions to the following problem utilizing critical point theory:

$$(E_\lambda^f)\begin{cases} -\Delta_1^2(x(i-1,j))) - \Delta_2^2(x(i,j-1) = \lambda f((i,j),x(i,j)), \\ \qquad\qquad\qquad\qquad\qquad\qquad (i,j)\in\mathbb{Z}[1,\alpha]\times\mathbb{Z}[1,\beta], \\ x(0,j)=x(\alpha+1,j)=0, \qquad j\in\mathbb{Z}[1,\beta], \\ x(i,0)=x(i,\beta+1)=0, \qquad i\in\mathbb{Z}[1,\alpha], \end{cases}$$

where Δ_1^2 and Δ_2^2 are two second-order forward difference operators given by $\Delta_1^2 x(i,j)=\Delta_1(\Delta_1 x(i,j))$ and $\Delta_2^2 x(i,j)=\Delta_2(\Delta_2 x(i,j))$, respectively.

Heidarkhani and Imbesi [15] in 2015 presented certain conditions to establish multiple solutions for the problem (E_λ^f).

Du and Zhou [8] studied the following problem in 2020:

$$(E_\lambda^{f,p})\begin{cases} -\Delta_1(\varphi_p(\Delta_1 x(i-1,j))) - \Delta_2(\varphi_p(\Delta_2 x(i,j-1))) = \lambda f((i,j),x(i,j)), \\ \qquad\qquad\qquad\qquad\qquad\qquad (i,j)\in\mathbb{Z}[1,\alpha]\times\mathbb{Z}[1,\beta] \\ x(0,j)=x(\alpha+1,j)=0, \qquad j\in\mathbb{Z}[1,\beta], \\ x(i,0)=x(i,\beta+1)=0, \qquad i\in\mathbb{Z}[1,\alpha], \end{cases}$$

where $1<p<\infty$ and $f((i,j),.)\in C(\mathbb{R},\mathbb{R})$, for all $(i,j)\in\mathbb{Z}[1,\alpha]\times\mathbb{Z}[1,\beta]$. By using critical point theory, the authors established the existence of infinitely many solutions for $(E_\lambda^{f,p})$.

Our study focuses on some results concerning the first eigenvalue $\lambda_{1,m}$ of the problem (P_λ). More precisely, in this article, we investigate the simplicity of $\lambda_{1,m}$, the constant sign of the first eigenfunction associated with $\lambda_{1,m}$, the strict monotonicity characteristic concerning the weight and sign change of any eigenfunction associated with $\lambda>\lambda_{1,m}$.

We consider the $\alpha\beta$-dimensional Banach space

$$H=\Big\{x:\mathbb{Z}[0,\alpha+1]\times\mathbb{Z}[0,\beta+1]\longrightarrow\mathbb{R}: x(0,j)=x(\alpha+1,j)=0,\ j\in\mathbb{Z}[1,\beta] \text{ and } \\ x(i,0)=x(i,\beta+1)=0,\ i\in\mathbb{Z}[1,\alpha]\Big\},$$

endowed with the norm

$$\|x\| = \big(\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha+1}\Delta_1 x(i-1,j)^2 + \sum_{i=1}^{\alpha}\sum_{j=1}^{\beta+1}\Delta_2 x(i,j-1)^2\big)^{\frac{1}{2}}.$$

This article's last section is structured as follows: Section 2 is devoted to mathematical preliminaries, and statements of the main results are covered. Section 3 contains proof of the main results.

2. Preliminaries and statement of main results

Definition 2.1. Let $u, v \in H$. We say that $u \leq v$ (respectively $u \geq v$) if and only if $u(i,j) \leq v(i,j)$ (respectively $u(i,j) \geq v(i,j)$) for all $(i,j) \in \mathbb{Z}[1,\alpha] \times \mathbb{Z}[1,\beta]$.

Definition 2.2. We say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (P_λ), if there exists $w \in H \backslash \{0\}$ such that

$$\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha+1} \varphi_p(\Delta_1 w(i-1,j))\Delta_1 y(i-1,j) + \sum_{i=1}^{\alpha}\sum_{j=1}^{\beta+1} \varphi_p(\Delta_2 w(i,j-1))\Delta_2 y(i,j-1)$$
$$=\lambda \sum_{i=1}^{\alpha}\sum_{j=1}^{\beta} m(i,j)\varphi_p(w(i,j))y(i,j), \tag{2.1}$$

for all $y \in H$.

We define the functionals $\Phi, \Psi : H \longrightarrow \mathbb{R}$ by the formulas

$$\Phi(w) = \frac{1}{p}\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha+1} |\Delta_1 w(i-1,j)|^p + \frac{1}{p}\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta+1} |\Delta_2 w(i,j-1)|^p$$

and

$$\Psi(w) = \frac{1}{p}\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta} m(i,j)|w(i,j)|^p.$$

Clearly, Φ and Ψ are two functionals of class $C^1(H,\mathbb{R})$ and for all $w, y \in H$

$$\begin{aligned} < \Phi'(w), y >= &\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha+1} |\Delta_1 w(i-1,j)|^{p-2}\Delta_1 w(i-1,j)\Delta_1 y(i-1,j) \\ &+ \sum_{i=1}^{\alpha}\sum_{j=1}^{\beta+1} |\Delta_2 w(i,j-1)|^{p-2}\Delta_2 w(i-1,j)\Delta_2 y(i-1,j) \end{aligned}$$

and

$$< \Psi'(w), y >= \sum_{j=1}^{\beta}\sum_{i=1}^{\alpha} m(i,j)|w(i,j)|^{p-2}w(i,j)y(i,j).$$

Proposition 2.1. *Let $v \in H$, the following assertions are equivalent:*

(i) v is an eigenfunction with λ of (P_λ).

(ii) For any $(i,j) \in \mathbb{Z}[1,\alpha] \times \mathbb{Z}[1,\beta]$,

$$-\Delta_1(\varphi_p(\Delta_1 v(i-1,j))) - \Delta_2(\varphi_p(\Delta_2 v(i,j-1))) = \lambda m(i,j)\varphi_p(v(i,j)).$$

Proof. For all $v, y \in H$, we have

$$\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha+1} |\Delta_1 v(i-1,j)|^{p-2}\Delta_1 v(i-1,j)\Delta_1 y(i-1,j)$$

$$
\begin{aligned}
&=\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta}|\Delta_1 v(i-1,j)|^{p-2}\Delta_1 v(i-1,j)y(i,j)\\
&\quad-\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta}|\Delta_1 v(i-1,j)|^{p-2}\Delta_1 v(i-1,j)y(i-1,j)+\sum_{j=1}^{\beta}|\Delta_1 v(\alpha,j)|^{p-1}\Delta_1 y(\alpha,j)\\
&=\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta}|\Delta_1 v(i-1,j)|^{p-2}\Delta_1 v(i-1,j)y(i,j)\\
&\quad-\sum_{i=1}^{\alpha-1}\sum_{j=1}^{\beta}|\Delta_1 v(i,j)|^{p-2}\Delta_1 v(i,j)y(i,j)-\sum_{j=1}^{\beta}|\Delta_1 v(\alpha,j)|^{p-2}\Delta_1 v(\alpha,j)y(\alpha,j)\\
&=-\sum_{j=1}^{\beta}\sum_{i=1}^{\beta}\Delta_1(|\Delta_1 v(i-1,j)|^{p-2}\Delta_1 v(i-1,j))y(i,j).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta+1}|\Delta_2 v(i,j-1)|^{p-2}\Delta_2 v(i,j-1)\Delta_2 y(i,j-1)\\
&=-\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha}\Delta_2(|\Delta_2 v(i,j-1)|^{p-1})y(i,j).
\end{aligned}
$$

We suppose that $v\in H$ is an eigenfunction associated with λ then, from the above equalities,

$$
\begin{aligned}
&\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta}\bigg(-\Delta_1(\Phi_p(\Delta_1 v(i-1,j)))-\Delta_2(\Phi_p(\Delta_2 v(i,j-1)))\\
&-\lambda m(i,j)\Phi_p(v(i,j))\bigg)y(i,j)=0. \qquad (2.2)
\end{aligned}
$$

Since y is arbitrary, then the equality (2.2) is equivalent to

$$
-\Delta_1(\Phi_p(\Delta_1 v(i-1,j)))-\Delta_2(\Phi_p(\Delta_2 v(i,j-1)))=\lambda m(i,j)\Phi_p(v(i,j)),
$$

for all $(i,j)\in\mathbb{Z}[1,\alpha]\times\mathbb{Z}[1,\beta]$.

□

Here is a list of some inequalities that will be used in the future.

Lemma 2.1 (Proposition 1, [8])**.** *For any $v\in H$ and for any $p>1$,*

$$
\begin{aligned}
\max_{(i,j)\in\mathbb{Z}[1,\alpha]\times\mathbb{Z}[1,\beta]}|v(i,j)|\le&\frac{(\alpha+\beta+2)^{\frac{p-1}{p}}}{4}\bigg(\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha+1}\Delta_1 v(i-1,j)^p\\
&+\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta+1}\Delta_2 v(i,j-1)^p\bigg)^{\frac{1}{p}}.
\end{aligned}
$$

Lemma 2.2. *For every $v \in H$ and for any $p > 1$, we have*

$$\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha}|v(i,j)|^p \leq \alpha\beta\frac{(\alpha+\beta+2)^{p-1}}{4^p}\Bigg(\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha+1}\Delta_1 v(i-1,j)^p + \sum_{i=1}^{\alpha}\sum_{j=1}^{\beta+1}\Delta_2 v(i,j-1)^p\Bigg).$$

Now, the main results in this article are the following theorems:

Theorem 2.1.

$$\lambda_{1,m} = \inf\Bigg\{\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha+1}|\Delta_1 w(i-1,j)|^p + \sum_{i=1}^{\alpha}\sum_{j=1}^{\beta+1}|\Delta_2 w(i,j-1)|^p : w \in H \text{ and } \sum_{i=1}^{\alpha}\sum_{j=1}^{\beta} m(i,j)|w(i,j)|^p = 1\Bigg\}, \tag{2.3}$$

is the first eigenvalue of (P_λ).

Remark 2.1.

$$\frac{1}{\lambda_{1,m}} = \sup_{H\setminus\{0\}}\left\{\frac{\sum\limits_{j=1}^{\beta}\sum\limits_{i=1}^{\alpha} m(i,j)|w(i,j)|^p}{\sum\limits_{j=1}^{\beta}\sum\limits_{i=1}^{\alpha+1}|\Delta_1 w(i-1,j)|^p + \sum\limits_{i=1}^{\alpha}\sum\limits_{j=1}^{\beta+1}|\Delta_2 w(i,j-1)|^p}\right\}. \tag{2.4}$$

Theorem 2.2. *If w is an eigenfunction associated with $\lambda_{1,m}$, then w does not change sign.*

Theorem 2.3. *The eigenvalue $\lambda_{1,m}$ is simple .i.e, if v and u are two eigenfunctions associated with $\lambda_{1,m}$, then there exists $r \in \mathbb{R}$ such that $v = ru$.*

Theorem 2.4. *If $\lambda > \lambda_{1,m}$ and w is an eigenfunction associated with λ, then w changes sign on $\mathbb{Z}[1,\alpha] \times \mathbb{Z}[1,\beta]$.*

Theorem 2.5. *$\lambda_{1,m}$ is strictly decreasing with respect to the weight .i.e, if $m, m' \in M_+$ such that $m < m'$, then $\lambda_{1,m'} < \lambda_{1,m}$.*

3. Proof of the main results

By studying the sign of $u(i,j)$, $u(i-1,j)$ and $u(i,j-1)$, we obtain the following lemma.

Lemma 3.1. *For all $u \in H$, we have*

$$\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha+1}|\Delta_1 u^-(i-1,j)|^p + \sum_{i=1}^{\alpha}\sum_{j=1}^{\beta+1}|\Delta_2 u^-(i,j-1)|^p$$
$$\leq -\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha+1}\varphi_p(\Delta_1 u(i-1,j))\Delta_1(u^-(i-1,j))$$

$$-\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta+1}\varphi_p(\Delta_2 u(i,j-1))\Delta_2(u^-(i,j-1)),$$

where $u^- = \max\{-u, 0\}$.

Lemma 3.2. *Let* $u, v \in H$ *and* $(i,j) \in \mathbb{Z}[1,\alpha] \times \mathbb{Z}[1,\beta]$ *such that* $u \leq v$ *and there exists* $(i_0, j_0) \in Z[1,\alpha] \times Z[1,\beta]$ *such that*

$$u(i_0, j_0) = v(i_0, j_0). \tag{3.1}$$

If $u(i,j) = v(i,j)$ *implies that*

$$\begin{aligned}-\Delta_1\big(\varphi_p(\Delta_1(u(i-1,j))) - \varphi_p(\Delta_1(v(i-1,j)))\big) = &-\Delta_2\big(\varphi_p(\Delta_2(v(i,j-1))) \\ &- \varphi_p(\Delta_2(u(i,j-1)))\big),\end{aligned} \tag{3.2}$$

then $u = v$.

Proof. We have,

$$\begin{aligned}-\Delta_1\Big(\varphi_p(\Delta_1 v(i_0-1,j_0)) - \varphi_p(\Delta_1 u(i_0-1,j_0))\Big) =& \big(\varphi_p(\Delta_1 v(i_0-1,j_0)) \\ &- \varphi_p(\Delta_1 u(i_0-1,j_0))\big) \\ &+ \big(\varphi_p(\Delta_1 u(i_0,j_0)) \\ &- \varphi_p(\Delta_1 v(i_0,j_0))\big)\end{aligned}$$

and

$$\begin{aligned}-\Delta_2\Big(\varphi_p(\Delta_2 u(i_0,j_0-1)) - \varphi_p(\Delta_2 v(i_0,j_0-1))\Big) =& \big(\varphi_p(\Delta_2 u(i_0,j_0-1)) \\ &- \varphi_p(\Delta_2 v(i_0,j_0-1))\big) \\ &+ \big(\varphi_p(\Delta_1 v(i_0,j_0)) \\ &- \varphi_p(\Delta_1 u(i_0,j_0))\big).\end{aligned}$$

Since, $x \to \varphi_p(x)$ is increasing in $\mathbb{R}$ for $p > 1$, then

$$-\Delta_1\Big(\varphi_p(\Delta_1 v(i_0-1,j_0)) - \varphi_p(\Delta_1 u(i_0-1,j_0))\Big) \leq 0 \tag{3.3}$$

and

$$-\Delta_2\Big(\varphi_p(\Delta_2 u(i_0,j_0-1)) - \varphi_p(\Delta_2 v(i_0,j_0-1))\Big) \geq 0. \tag{3.4}$$

From (3.1), (3.2), (3.3) and (3.4),

$$-\Delta_1\Big(\varphi_p(\Delta_1 v(i_0-1,j_0)) - \varphi_p(\Delta_1 u(i_0-1,j_0))\Big) = 0 \tag{3.5}$$

and

$$-\Delta_2\Big(\varphi_p(\Delta_2 u(i_0,j_0-1)) - \varphi_p(\Delta_2 v(i_0,j_0-1))\Big) = 0. \tag{3.6}$$

Since, $u \leq v$ and $u(i_0, j_0) = v(i_0, j_0)$, then

$$\begin{aligned}
&\varphi_p(\Delta_1 v(i_0-1, j_0)) - \varphi_p(\Delta_1 u(i_0-1, j_0)) \leq 0,\\
&\varphi_p(\Delta_1 u(i_0, j_0)) - \varphi_p(\Delta_1 v(i_0, j_0)) \leq 0,\\
&\varphi_p(\Delta_2 v(i_0, j_0-1)) - \varphi_p(\Delta_1 u(i_0, j_0-1)) \leq 0
\end{aligned}$$

and

$$\varphi_p(\Delta_2 u(i_0, j_0)) - \varphi_p(\Delta_2 v(i_0, j_0)) \leq 0.$$

Consequently, from (3.5) and (3.6), we get

$$\begin{aligned}
&\varphi_p(\Delta_1 v(i_0-1, j_0)) - \varphi_p(\Delta_1 u(i_0-1, j_0)) = 0,\\
&\varphi_p(\Delta_1 u(i_0, j_0)) - \varphi_p(\Delta_1 v(i_0, j_0)) = 0,\\
&\varphi_p(\Delta_2 v(i_0, j_0-1)) - \varphi_p(\Delta_1 u(i_0, j_0-1)) = 0
\end{aligned}$$

and

$$\varphi_p(\Delta_2 u(i_0, j_0)) - \varphi_p(\Delta_2 v(i_0, j_0)) = 0.$$

By the strictly increasing nature of φ_p and the fact that $v(i_0, j_0) = u(i_0, j_0)$, we obtain

$$\begin{aligned}
v(i_0-1, j_0) &= u(i_0-1, j_0),\\
u(i_0+1, j_0) &= v(i_0+1, j_0),\\
v(i_0, j_0-1) &= u(i_0, j_0-1)
\end{aligned}$$

and

$$u(i_0, j_0+1) = v(i_0, j_0+1).$$

Approach by approach, we prove that for all $(i,j) \in \mathbb{Z}[1,\alpha] \times \mathbb{Z}[1,\beta]$,

$$u(i,j) = v(i,j).$$

□

Lemma 3.3. *If w is an eigenfunction associated with λ and $w \geq 0$, then $w \gneqq 0$.*

Proof. Let w be an eigenfunction associated with λ and $w \geq 0$. According to Proposition 2.1, for all $(i,j) \in \mathbb{Z}[1,\alpha] \times \mathbb{Z}[1,\beta]$,

$$-\Delta_1(\varphi_p(\Delta_1 w(i-1,j))) - \Delta_2(\varphi_p(\Delta_2 w(i,j-1))) = \lambda m(i,j)\varphi_p(w(i,j)). \tag{3.7}$$

On the other hand, if $w(i,j) = 0$, then

$$-\Delta_1(\varphi_p(\Delta_1 w(i-1,j)) \leq 0 \tag{3.8}$$

and

$$-\Delta_2(\varphi_p(\Delta_1 w(i,j-1)) \leq 0. \tag{3.9}$$

According to (3.7), (3.8) and (3.9), we get

$$-\Delta_1(\varphi_p(\Delta_1 w(i-1,j)) - 0) = 0$$

and

$$-\Delta_2(0 - \varphi_p(\Delta_2 w(i,j-1))) = 0.$$

By contradiction, we suppose that there exists $(i_0, j_0) \in Z[1,\alpha] \times Z[1,\beta]$ such that

$$w(i_0, j_0) = 0. \tag{3.10}$$

Thus, by Lemma 3.2, $w = 0$, which is a contradiction.

□

Proof of Theorem 2.1. Let $w_n \in H \setminus \{0\}$ be a minimizing sequence for $\lambda_{1,m}$. We have

$$\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha} m(i,j)|w_n(i,j)|^p = 1$$

and

$$\lambda_{1,m} = \lim_{n\longrightarrow\infty} \sum_{i=1}^{\alpha+1}\sum_{j=1}^{\beta} |\Delta_1 w_n(i-1,j)|^p + \sum_{j=1}^{\beta+1}\sum_{i=1}^{\alpha} |\Delta_2 w_n(i,j-1)|^p.$$

In view of Lemma 2.2, we infer that (w_n) is bounded in H. That information together with the fact that H is a finite dimensional Hilbert space, implies that there exists a subsequence, still denoted by (w_n), and $w_0 \in H$ such that (w_n) converges to w_0 in H.

Consequently,

$$\lambda_{1,m} = \sum_{i=1}^{\alpha+1}\sum_{j=1}^{\beta} |\Delta_1 w_0(i-1,j)|^p + \sum_{j=1}^{\beta+1}\sum_{i=1}^{\alpha} |\Delta_2 w_0(i,j-1)|^p = \Phi(w_0), \tag{3.11}$$

and

$$\Psi(w_0) = \sum_{j=1}^{\beta}\sum_{i=1}^{\alpha} m(i,j)|w_0(i,j)|^p = 1. \tag{3.12}$$

Thus, $\lambda_{1,m} > 0$; otherwise, from (3.11), w_0 is constant, then $w_0 = 0$ and a contradiction is derived from equality (3.12).

Therefore, from (2.4), for all $v \in H$,

$$\frac{\partial}{\partial t}\left(\frac{\sum\limits_{i=1}^{\alpha}\sum\limits_{j=1}^{\beta} m(i,j)|(w_0+tv)(i,j)|^p}{\sum\limits_{i=1}^{\alpha+1}\sum\limits_{j=1}^{\beta} |\Delta_1(w_0+tv)(i-1,j)|^p + \sum\limits_{j=1}^{\beta+1}\sum\limits_{i=1}^{\alpha} |\Delta_2(w_0+tv)(i,j-1)|^p}\right)\Bigg|_{t=0} = 0.$$

Thus,

$$\begin{aligned}
&\Phi(w_0)\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta} m(i,j)\varphi_p(w_0(i,j))v(i,j)\\
=&\Psi(w_0)\Bigg(\sum_{i=1}^{\alpha+1}\sum_{j=1}^{\beta}\varphi_p(\Delta_1 w_0(i-1,j))\Delta_1 v(i-1,j)\\
&+\sum_{j=1}^{\beta+1}\sum_{i=1}^{\alpha}\varphi_p(\Delta_2 w_0(i,j-1))\Delta_2 v(i,j-1)\Bigg).
\end{aligned}$$

The above relation combined with (2.4) shows that $\lambda_{1,m}$ is the first eigenvalue of problem (P_λ).

Proof of Theorem 2.2. Let w be an eigenfunction associated with $\lambda_{1,m}$. For all $y \in H$, we have

$$\begin{aligned}&\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha+1}\varphi_p(\Delta_1 w(i-1,j))\Delta_1 y(i-1,j)+\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta+1}\varphi_p(\Delta_2 w(i,j-1))\Delta_2 y(i,j-1)\\ =&\lambda_{1,m}\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta} m(i,j)\varphi_p(w(i,j))y(i,j).\end{aligned} \tag{3.13}$$

If $w^- \neq 0$, we take $y = -w^-$ in (3.13), and we get

$$\begin{aligned}\lambda_{1,m}\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha} m(i,j)|w^-(i,j)|^p =&-\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha+1}\varphi_p(\Delta_1 w(i-1,j))\Delta_1(w^-(i-1,j))\\ &-\sum_{i=1}^{\alpha}\sum_{j=1}^{\alpha+1}\varphi_p(\Delta_2 w(i,j-1))\Delta_2(w^-(i,j-1)).\end{aligned}$$

According to Lemma 3.1,

$$\begin{aligned}&\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha+1}|\Delta_1 w^-(i-1,j)|^p+\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta+1}|\Delta_2 w^-(i,j-1)|^p\\ \leq&\lambda_{1,m}\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha} m(i,j)|w^-(i,j)|^p.\end{aligned} \tag{3.14}$$

By (2.4), we deduce that

$$\frac{1}{\lambda_{1,m}}=\frac{\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta} m(i,j)|w^-(i,j)|^p}{\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha+1}|\Delta_1 w^-(i-1,j)|^p+\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta+1}|\Delta_2 w^-(i,j-1)|^p}.$$

Therefore, w^- is an eigenfunction with eigenvalue $\lambda_{1,m}$ and by Lemma 3.3, $w^- > 0$. If $w^- = 0$, so, $w = w^+ \geq 0$, by using Lemma 3.3, $w = w^+ > 0$.

Proof of Theorem 2.3. Consider two eigenfunctions, u and v, associated with $\lambda_{1,m}$. By Theorem 2.2, we can assume that $v > 0$ and $u > 0$. Let $\alpha = \min_{\mathbb{Z}[1,\alpha]\times\mathbb{Z}[1,\beta]} \frac{v(i,j)}{u(i,j)}$. Then $\alpha u \leq v$ and there exists $(i_0, j_0) \in \mathbb{Z}[1,\alpha] \times \mathbb{Z}[1,\beta]$ such that $\alpha u(i_0, j_0) = v(i_0, j_0)$.
On the other hand, if $v(i,j) = \alpha u(i,j)$, we have

$$\begin{aligned}-\Delta_1(\varphi_p(\Delta_1 v(i-1,j)))-\Delta_2(\varphi_p(\Delta_2 v(i,j-1)))=&\lambda_{1,m}\varphi_p(v(i,j))\\ =&\lambda_{1,m}\varphi_p(\alpha u(i,j))\\ =&-\Delta_1(\varphi_p(\Delta_1\alpha u(i-1,j)))\\ &-\Delta_2(\varphi_p(\Delta_2\alpha u(i,j-1))).\end{aligned}$$

Therefore,

$$-\Delta_1(\varphi_p(\Delta_1(\alpha u(i-1,j)))-\varphi_p(\Delta_1(v(i-1,j))))=-\Delta_2(\varphi_p(\Delta_2(v(i,j-1)))$$

$$- \varphi_p(\Delta_2(\alpha u(i, j-1)))).$$

Thus, by Lemma 3.2, $v = \alpha u$.

The following result has been proved in the one-dimensional case, where u depends on only one variable; see [6].

Lemma 3.4. *If λ is an eigenvalue of (P_λ) and it admits an eigenfunction with a constant sign, then*

$$\lambda = \inf \Bigg\{ \sum_{j=1}^{\beta} \sum_{i=1}^{\alpha+1} |\Delta_1 v(i-1, j)|^p + \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta+1} |\Delta_2 v(i, j-1)|^p + \lambda \sum_{j=1}^{\beta} \sum_{i=1}^{\alpha} m^-(i,j)|v(i,j)|^p : v \in H \text{ and } \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} m^+(i,j)|v(i,j)|^p = 1 \Bigg\},$$

where $m^- = \max\{-m, 0\}$ and $m^+ = \max\{m, 0\}$.

Proof. Let u_0 be an eigenfunction with a constant sign associated with λ. We put

$$w = \frac{u_0}{\left(\sum_{j=1}^{\beta} \sum_{i=1}^{\alpha} m^+(i,j)|u_0(i,j)|^p \right)^{\frac{1}{p}}}$$

and

$$\delta = \inf \Bigg\{ \sum_{j=1}^{\beta} \sum_{i=1}^{\alpha+1} |\Delta_1 v(i-1, j)|^p + \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta+1} |\Delta_2 v(i, j-1)|^p + \lambda \sum_{j=1}^{\beta} \sum_{i=1}^{\alpha} m^-(i,j)|v(i,j)|^p : v \in H \text{ and } \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} m^+(i,j)|v(i,j)|^p = 1 \Bigg\}.$$

We have $w \in H$ and $\sum_{j=1}^{\beta} \sum_{i=1}^{\alpha} m^+(i,j)|w(i,j)|^p = 1$, then

$$\begin{aligned}
\delta \leq & \sum_{j=1}^{\beta} \sum_{i=1}^{\alpha+1} |\Delta_1 w(i-1, j)|^p + \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta+1} |\Delta_2 w(i, j-1)|^p + \lambda \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} m^-(i,j)|w(i,j)|^p \\
= & \frac{1}{\sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} m^+(i,j)|u_0(i,j)|^p} \Bigg(\sum_{j=1}^{\beta} \sum_{i=1}^{\alpha+1} |\Delta_1 u_0(i-1, j)|^p + \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta+1} |\Delta_2 u_0(i, j-1)|^p \\
& + \lambda \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} m^+(i,j)|u_0(i,j)|^p - \lambda \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} m(i,j)|u_0(i,j)|^p \Bigg) \\
= & \lambda.
\end{aligned}$$

Let $v \in H \setminus \{0\}$ such that

$$\sum_{j=1}^{\beta} \sum_{i=1}^{\alpha} m^+(i,j)|v(i,j)|^p = 1 \tag{3.15}$$

and

$$\delta = \sum_{j=1}^{\beta}\sum_{i=1}^{\alpha+1}|\Delta_1 v(i-1,j)|^p + \sum_{i=1}^{\alpha}\sum_{j=1}^{\beta+1}|\Delta_2 v(i,j-1)|^p + \lambda\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha} m^-(i,j)|v(i,j)|^p. \tag{3.16}$$

Since,

$$\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha+1}|\Delta_1|v(i-1,j)||^p + \sum_{i=1}^{\alpha}\sum_{j=1}^{\beta+1}|\Delta_2|v(i,j-1)||^p \leq \sum_{j=1}^{\beta}\sum_{i=1}^{\alpha+1}|\Delta_1 v(i-1,j)|^p + \sum_{i=1}^{\alpha}\sum_{j=1}^{\beta+1}|\Delta_2 v(i,j-1)|^p,$$

then, we can suppose that $v > 0$.

Since v achieves the infimum, then in the same way as in the proof of Theorem 2.1, for all $y \in H$, we obtain

$$\begin{aligned}\delta\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta} m^+(i,j)\varphi_p(v(i,j))y(i,j) =& \sum_{j=1}^{\beta}\sum_{i=1}^{\alpha+1}\varphi_p(\Delta_1 v(i-1,j))\Delta_1 y(i-1,j)\\ &+ \sum_{i=1}^{\alpha}\sum_{j=1}^{\beta+1}\varphi_p(\Delta_2 v(i,j-1))\Delta_2 y(i,j-1)\\ &+ \lambda\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta} m^-(i,j)\varphi_p(v(i,j))y(i,j).\end{aligned} \tag{3.17}$$

Thus, according to (3.17), Proposition 2.1 and the fact that $v > 0$, we have

$$\begin{aligned}\delta m^+(i,j)|v(i,j)|^{p-1} =& -\Delta_1(\varphi_p(\Delta_1 v(i-1,j))) - \Delta_2(\varphi_p(\Delta_2 v(i,j-1)))\\ &+ \lambda m^-(i,j)|v(i,j)|^{p-1}.\end{aligned} \tag{3.18}$$

If $u_0 > 0$, we put $c = \max_{(i,j)\in\mathbb{Z}[1,\alpha]\times\mathbb{Z}[1,\beta]} \frac{v(i,j)}{u_0(i,j)}$, then $cu_0 \geq v$ and there exists $(i_0, j_0) \in \mathbb{Z}[1,\alpha] \times \mathbb{Z}[1,\beta]$, such that $c = \frac{v(i_0,j_0)}{u_0(i_0,j_0)}$. From (3.18) and the fact that $\delta \leq \lambda$, if $v(i,j) = cu_0(i,j)$, then

$$\begin{aligned}&-\Delta_1(\varphi_p(\Delta_1 v(i-1,j))) - \Delta_2(\varphi_p(\Delta_2 v(i,j-1))) + \lambda m^-(i,j)|v(i,j)|^{p-1}\\ \leq&\lambda m^+(i,j)|cu_0(i,j)|^{p-1}\\ =&-\Delta_1(\varphi_p(\Delta_1 cu_0(i-1,j))) - \Delta_2(\varphi_p(\Delta_2 cu_0(i,j-1))) + \lambda m^-(i,j)|cu_0(i,j)|^{p-1}.\end{aligned}$$

Therefore,

$$\begin{aligned}&-\Delta_1\Big(\varphi_p(\Delta_1 v(i-1,j)) - \varphi_p(\Delta_1 cu_0(i-1,j))\Big)\\ \leq& -\Delta_2\Big(\varphi_p(\Delta_2 cu_0(i,j-1)) - \varphi_p(\Delta_2 v(i,j-1))\Big).\end{aligned} \tag{3.19}$$

Since, $x \to \varphi_p(x)$ is increasing in $\mathbb{R}$ for $p > 1$, $v(i,j) = cu_0(i,j)$ and $v \leq cu_0$, then

$$-\Delta_1\Big(\varphi_p(\Delta_1 v(i-1,j)) - \varphi_p(\Delta_1 cu_0(i-1,j))\Big) \geq 0 \tag{3.20}$$

and

$$-\Delta_2\Big(\varphi_p(\Delta_2 cu_0(i,j-1)) - \varphi_p(\Delta_2 v(i,j-1))\Big) \leq 0. \tag{3.21}$$

From (3.19), (3.20) and (3.21),

$$-\Delta_1\Big(\varphi_p(\Delta_1 v(i-1,j)) - \varphi_p(\Delta_1 cu_0(i-1,j))\Big) = 0$$

and

$$-\Delta_2\Big(\varphi_p(\Delta_2 cu(i,j-1)) - \varphi_p(\Delta_2 v(i,j-1))\Big) = 0.$$

So, by Lemma 3.2, $v = cu$.

Thus, using the inequality (3.18), we obtain

$$\delta m^+(i,j)|cu_0(i,j)|^{p-1} = \lambda m^+(i,j)|cu_0(i,j)|^{p-1}.$$

Hence, $\lambda = \delta$.

If $u_0 < 0$, we put $c = \max\limits_{(i,j)\in\mathbb{Z}[1,\alpha]\times\mathbb{Z}[1,\beta]} \frac{v(i,j)}{-u_0(i,j)}$, then $-cu_0 \geq v$ and there exists $(i_0,j_0) \in \mathbb{Z}[1,\alpha]\times\mathbb{Z}[1,\beta]$, such that $c = \frac{v(i_0,j_0)}{-u(i_0,j_0)}$. Using the same techniques mentioned above, we prove that $\lambda = \delta$. □

Proof of Theorem 2.4. Let $\lambda > \lambda_{1,m}$ be an eigenvalue of (P_λ) and w a positive eigenfunction with $\lambda_1(m)$ such that

$$\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha} m^+(i,j)|w(i,j)|^p = 1.$$

Since $m = m^+ - m^-$, then

$$\begin{aligned}
&\sum_{i=1}^{\alpha+1}\sum_{j=1}^{\beta}|\Delta_1 w(i-1,j)|^p + \sum_{j=1}^{\beta+1}\sum_{i=1}^{\alpha}|\Delta_2 w(i,j-1)|^p + \lambda_{1,m}\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha} m^-(i,j)|w(i,j)|^p\\
=&\lambda_{1,m}\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha} m^+(i,j)|w(i,j)|^p\\
=&\lambda_{1,m}.
\end{aligned}$$

So,

$$\begin{aligned}
\lambda_{1,m}\big(1 - \sum_{j=1}^{\beta}\sum_{i=1}^{\alpha} m^-(i,j)|w(i,j)|^p\big) =& \sum_{j=1}^{\beta}\sum_{i=1}^{\alpha+1}|\Delta_1 w(i-1,j)|^p\\
&+\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta+1}|\Delta_2 w(i,j-1)|^p\\
>&0.
\end{aligned} \tag{3.22}$$

By contradiction, assume that λ admits an eigenfunction that keeps a constant sign. According to Lemma 3.4,

$$\lambda = \inf\Bigg\{\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha+1}|\Delta_1 u(i-1,j)|^p + \sum_{i=1}^{\alpha}\sum_{j=1}^{\beta+1}|\Delta_2 u(i,j-1)|^p$$

$$+\lambda\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha} m^{-}(i,j)|u(i,j)|^{p} : u\in H \text{ and } \sum_{j=1}^{\beta}\sum_{i=1}^{\alpha} m^{+}(i,j)|u(i,j)|^{p}=1\Big\}.$$

Then

$$\lambda \leq \sum_{j=1}^{\beta}\sum_{i=1}^{\alpha+1}|\Delta_1 w(i-1,j)|^p+\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta+1}|\Delta_2 w(i,j-1)|^p+\lambda\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta} m^{-}(i,j)|w(i,j)|^p. \tag{3.23}$$

Combining (3.22) with (3.23), we obtain

$$\lambda\big(1-\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha} m^{-}(i,j)|w(i,j)|^p\big) \leq \lambda_{1,m}\big(1-\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha} m^{-}(i,j)|w(i,j)|^p\big).$$

Hence, from (3.22), we obtain $\lambda \leq \lambda_{1,m}$, which is a contradiction.

Proof of Theorem 2.5. Let $m, m' \in M_+$, such that $m < m'$ and w_1 an eigenfunction associated with $\lambda_{1,m}$. We have

$$\begin{aligned}
\frac{1}{\lambda_{1,m}} &= \sup_{H\setminus\{0\}}\left\{\frac{\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta} m(i,j)|u(i,j)|^p}{\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha+1}|\Delta_1 u(i-1,j)|^p+\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta+1}|\Delta_2 u(i,j-1)|^p}\right\}\\
&= \frac{\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta} m(i,j)|w_1(i,j)|^p}{\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha+1}|\Delta_1 w_1(i-1,j)|^p+\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta+1}|\Delta_2 w_1(i,j-1)|^p}\\
&\leq \frac{\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta} m'(i,j)|w_1(i,j)|^p}{\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha+1}|\Delta_1 w_1(i-1,j)|^p+\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta+1}|\Delta_2 w_1(i,j-1)|^p}\\
&\leq \sup_{H\setminus\{0\}}\left\{\frac{\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta} m'(i,j)|u(i,j)|^p}{\sum_{j=1}^{\beta}\sum_{i=1}^{\alpha+1}|\Delta_1 u(i-1,j)|^p+\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta+1}|\Delta_2 u(i,j-1)|^p}\right\}\\
&\leq \frac{1}{\lambda_{1,m'}}.
\end{aligned}$$

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