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New Refinements of Hermite-Hadamard Inequalities for Quantum Integrals

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Abstract In this study, we first introduce two functions including quantum integrals. Then, we prove some properties of these mappings such as convexity and monotony. Moreover, by using the newly defined mappings, we prove some refinements of the left hand side of Hermite-Hadamard inequality for left and right quantum integrals.

Keywords Convex functions, Hermite-Hadamard inequality, quantum integrals

MSC(2010) 26A51, 26D10, 26D15

1. Introduction

Quantum calculus is a mathematical framework that is distinct from traditional infinitesimal calculus, as it does not rely on limits or examine calculations with limits. The term "quantum" originates from the Latin word "Quantus," meaning "how much," or "Kvant" in Swedish. There are two main branches of quantum calculus: the q-calculus and the h-calculus, both of which were investigated by P. Cheung and V. Kac in the early last century. While FH Jackson was studying quantum calculus or q-calculus at the same time, this type of calculus had already been solved by Euler and Jacobi. However, quantum calculus, which originated with Euler, has become a significant area of study in mathematics and physics today, providing solutions to previously unsolvable problems, particularly for discontinuous functions. Tariboon and Ntouyas introduced the concepts of quantum calculus on finite intervals and acquired several q-analogues of classical mathematical materials. This has led to numerous new results in the literature regarding quantum analogues of classical mathematical investigations. Additionally, recent research has shown that quantum calculus is a subfield of the more general mathematical area of time scales calculus. Time scales provide a unified framework for investigating dynamic equations on both discrete and continuous domains, making quantum calculus applicable in many fields such as number theory, combinatorics, orthogonal polynomials, simple hypergeometric functions, quantum theory, physics, and relativity theory. For more information, see [3, 5–12, 15] and references cited therein.

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The Hermite-Hadamard inequality was independently proven by C. Hermite and J. Hadamard. It's one of the most recognized inequalities in the theory of convex functional analysis, which is stated as follows:

Let $f:[a,b]\to\mathbb{R}$ be a convex mapping. Then

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{2}. \tag{1.1}$$

If f is concave, both inequalities hold in the reverse direction. Finding many studies in inequality theory, the quantum integral has gone through various searches by researchers to establish the quantum version of the famous Hermite-Hadamard inequality above.

In [2, 4], Alp et al. and Bermudo et al., with the help of the q-derivatives and integrals (defined in Section 2), derive two different versions of q-Hermite-Hadamard inequalities and some estimates. The q-Hermite-Hadamard inequalities are defined as:

Theorem 1.1. *[2,4] For a convex mapping $f:[a,b]\to\mathbb{R}$, the following inequalities hold:*

$$f\left(\frac{qa+b}{[2]_q}\right)\leq\frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx\leq\frac{qf(a)+f(b)}{[2]_q}, \tag{1.2}$$

$$f\left(\frac{a+qb}{[2]_q}\right)\leq\frac{1}{b-a}\int_a^b f(x)\ {}^bd_qx\leq\frac{f(a)+qf(b)}{[2]_q}. \tag{1.3}$$

Remark 1.1. It is very easy to observe that by adding (1.2) and (1.3), we have the following q-Hermite-Hadamard inequality (see, [4]):

$$\begin{aligned} f\left(\frac{a+b}{2}\right)&\leq\frac{1}{2(b-a)}\left[\int_a^b f(x)\ {}_ad_qx+\int_a^b f(x)\ {}^bd_qx\right] \\ &\leq\frac{f(a)+f(b)}{2}. \end{aligned} \tag{1.4}$$

Hereabout, Ali et al. [1] and Sitthiwirattham et al. [13] present the following two different and new versions of q-Hermite-Hadamard type inequalities:

Theorem 1.2. *[1, 13] For a convex mapping $f:[a,b]\to\mathbb{R}$, the following inequalities hold:*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) \\ \leq & \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(x)\ {}^{\frac{a+b}{2}}d_qx+\int_{\frac{a+b}{2}}^b f(x)\ {}_{\frac{a+b}{2}}d_qx\right] \\ \leq & \frac{f(a)+f(b)}{2}, \end{aligned} \tag{1.5}$$

$$f\left(\frac{a+b}{2}\right) \tag{1.6}$$