JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **38**, No. 4, pp. 377-410

doi: 10.4208/jpde.v38.n4.1
December 2025

Uniform Attractors for Non-Autonomous Reaction-Diffusion Equation with Mixed Delay

RAN Yanping[1,*], SUN Xiaoke[1], LIU Yueli[1], LIANG Maolin[1] and HU Zhiqiang[2]

[1] *School of Mathematics and Statistics, Tianshui Normal University, Tianshui 741001, China;*
[2] *School of Mathematics and Statistics, Taishan University, Taian 271000, China.*

Received 30 January 2021; Accepted 23 July 2023

Abstract. This paper is concerned with the asymptotic behavior of solutions of non-autonomous reaction-diffusion equation with delays. The well-posedness theory of equation for the initial data belonging to $C_{L^r(\Omega)}(1<r<\infty)$ and $C_{W^{1,r}(\Omega)}(1<r<N)$ is established respectively. In addition, the existence of uniform attractors in $C_{L^r(\Omega)}$ for the family of processes with translation bounded external force is proved. Moreover, the long time behavior of solution with higher regularity in $C_{W^{1,r}(\Omega)}$ is considered as well.

AMS Subject Classifications: 37B52, 35K40, 35K57, 35D05

Chinese Library Classifications: O175.27

Key Words: Uniform attractors; reaction-diffusion equation with delays; non-autonomous equation.

---

1 Introduction

Let $\Omega\subseteq\mathbb{R}^n$ be a smooth bounded domain. Consider the long-time behavior of the following non-autonomous nonlinear reaction-diffusion equation

$$\begin{cases}\dfrac{\partial u}{\partial t}-\Delta u+\lambda u=f(x,u_t)+g(t,x), & \text{in } (\tau,+\infty]\times\Omega,\\ u|\partial\Omega=0, & t>\tau,\\ u(t,x)=\phi(t-\tau,x), & t\in[\tau-h,\tau],\ x\in\Omega.\end{cases}\tag{1.1}$$

*Corresponding author. *Email addresses:* `ranypmath@163.com` (R. Ran)

Here $\lambda\geq 0$, $g\in L^r_{loc}(\mathbb{R};L^r(\mathbb{R}^n))(r>1)$ and the nonlinear term satisfies the condition:

$$f(x,u_t(t,x))=F(x,u(t-\rho(t),x))+\int_{-h}^{0}G(x,z,u(t+z,x))\mathrm{d}z,$$

in which there exists a positive constant k_2, a positive scalar function $m_1\in L^1([-h,0],\mathbb{R}^+)$ and functions $k_1,m_0\in L^r(\Omega)$ such that $F\in C(\Omega\times\mathbb{R};\mathbb{R})$, $\rho\in C^1(\mathbb{R};[0,h])$ and $G\in C(\Omega\times[-h,0]\times\mathbb{R};\mathbb{R})$ satisfing

$$|F(x,v)|^r\leq|k_1(x)|^r+k_2^r|v|^r,\qquad \forall x\in\Omega,\ v\in\mathbb{R},\ r\geq 2, \tag{1.2}$$

$$|\rho'(t)|\leq\rho^*<1,\qquad \forall t\in\mathbb{R}, \tag{1.3}$$

$$|G(x,s,v)|\leq|m_0(x)|+m_1(s)|v|,\qquad \forall x\in\Omega,\ s\in[-h,0],\ v\in\mathbb{R}, \tag{1.4}$$

$$|F(x,v)-F(x,\nu)|\leq C_1|v-\nu|,\qquad \forall x\in\Omega,\ v,\nu\in\mathbb{R}, \tag{1.5}$$

$$|G(x,s,v)-G(x,s,\nu)|\leq C_2|v-\nu|,\qquad \forall x\in\Omega,\ s\in[-h,0],\ v,\nu\in\mathbb{R}, \tag{1.6}$$

$\overline{m}:=\int_{-h}^{0}m_1(s)\mathrm{d}s$.The non-autonomous term $g(t)=g(x,t)$ is translation bounded in $L^r_b(\mathbb{R};X),r>1$, i.e. $g\in L^r_b(\mathbb{R};X)$,

$$\|g\|^r_{L^r_b(\mathbb{R};X)}=\sup_{t\in\mathbb{R}}\int_t^{t+1}\|g(s)\|^r_X\mathrm{d}s<+\infty, \tag{1.7}$$

where the local $r-$power integral is the Bochner integral.

Let X be $L^r(\Omega)$ or $W^{1,r}(\Omega)$ ($r\geq 2$). $h>0$ is a given positive number, which will represent the delay time later. Let C_X denote the Banach space $C([-h,0];X)$ endowed with the norm

$$\|\phi\|_{C_X}=\sup_{z\in[-h,0]}\|\phi(z)\|_X.$$

Given $\tau\in\mathbb{R},T>\tau$ and a function $u:[\tau-h,T]\to X$, for each $t\in[\tau,T]$, $u_t:[-h,0]\to X$ denote the function defined by $u_t(s)=u(t+s)$ for $s\in[-h,0]$. We are interested in initial condition $\phi\in C_X=C_{L^r(\Omega)}$ or $C_{W^{1,r}(\Omega)}$. In the sequel C denotes an arbitrary positive constant, which may be different from line to line and even in the same line.

Recently, Wang and Kloeden [1] proved the existence of uniform attractors of (1.1) in the space $C_{L^2(\mathbb{R}^n)}$ under the assumption of the linear growth for $F(x,u(t-\rho(t),x))$ and $G(x,z,u(t+z,x))$, and the uniform boundedness in $L^2_b(\mathbb{R};L^2(\mathbb{R}^n))$ of the external term. In the present paper, we will prove the existence of solution and the uniform attractors of (1.1) in C_X on the bounded domain under the conditions (1.2)-(1.7).

Actually, there are lots of results on the existence of solutions or attractors for evolution equations defined on bounded domain without delays. In the autonomous case, the problem

$$\begin{cases}\dfrac{\partial u}{\partial t}-\Delta u=f(x,u), & \text{in } [0,+\infty]\times\Omega,\\ u|\partial\Omega=0, & t>0,\\ u(0,x)=u_0, & x\in\Omega,\end{cases} \tag{1.8}$$