JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **38**, No. 4, pp. 411-431

doi: 10.4208/jpde.v38.n4.2
December 2025

Multiple Periodic Solutions of Allen-Cahn System Involving Fractional Laplacian

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Received 9 October 2021; Accepted 28 November 2024

Abstract. We consider periodic solutions of the following nonlinear system associated with the fractional Laplacian

$$(-\partial_{xx})^s \boldsymbol{u}(x) + \nabla F(\boldsymbol{u}(x)) = 0 \quad \text{in } \mathbb{R},$$

where $F:\mathbb{R}^2 \to \mathbb{R}$ is a smooth double-well potential. For the case that F is even in its two variables we obtain the existence of more and more periodic solutions with large period, by using Clark's theorem. For the case that F is only even in its the second variable and the origin is a saddle critical point of F, we give two periodic solutions by using Morse theory.

AMS Subject Classifications: 35B10, 35J50, 35A01

Chinese Library Classifications: O175.25

Key Words: Multiple periodic solutions; fractional Laplacian; Allen-Cahn system; Clark's theorem; Morse theory.

1 Introduction

In this paper, we study multiplicity of periodic solutions to the Allen-Cahn system involving the fractional Laplacian

$$(-\partial_{xx})^s \boldsymbol{u}(x) + \nabla F(\boldsymbol{u}(x)) = 0, \qquad \boldsymbol{u}(x) = \boldsymbol{u}(x+T) \ \text{ in } \mathbb{R}, \tag{1.1}$$

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where $\boldsymbol{u}(x)=(u(x),v(x))$. Here, $(-\partial_{xx})^s$, $0<s<1$, denotes the usual fractional Laplace operator. The function F is a smooth double-well potential with wells at $\boldsymbol{b_1}$ and $\boldsymbol{b_2}$. Without loss of generality we may assume that $\boldsymbol{b_1}=-\boldsymbol{b_2}=(1,0)$. More precisely, we assume that there exists $R_0>1$ such that F satisfies the following conditions

$$\begin{cases} F(\boldsymbol{b_i})=0<F(\boldsymbol{w}), & \forall\ \boldsymbol{w}\neq\boldsymbol{b}_i,\ i=1,2,\\ \nabla F(\boldsymbol{w})\cdot\boldsymbol{w}>0, & \text{for}\ \ |\boldsymbol{w}|\geq R_0. \end{cases} \tag{1.2}$$

We also assume that

$$F_u(u,0)>0 \quad \text{for all} \quad u\in(-1,0); \qquad F_u(u,0)<0 \quad \text{for all} \quad u\in(0,1). \tag{1.3}$$

For the corresponding scalar problem of (1.1), there are several results about periodic solutions recently. Gui-Zhang and the first author [1] obtain the existence of periodic solution for any large period T. For the least positive period, Gui and the first author [2] give an upper bound. The Hamiltonian identity, Modica-type inequality and more general existence results for periodic solutions are also established in [2]. The authors in [3] generalize these results to the corresponding non-autonomous Allen-Cahn equations. Exact value of least positive period, axial symmetry and the result that the limit of periodic solutions being the layer solution as periods tending to infinity are obtained by Feng-Du [4]. Cui and Wang [5] establish multiple solutions of the scalar problem of (1.1) by using Clark's theorem and Morse theory, which are classical tools in the study of periodic solutions for local operators (e.g., [6–8]). For the standard Laplacian case, the existence of nontrivial doubly periodic solutions to (1.1) in two-dimensional space have been found by Shi in [9], and singly periodic solutions have been constructed by Kowalczyk-Liu-Wei in [10].

The general periodic problems for fractional equations have been extensively studied. First, existence and multiplicity of periodic solutions to the so-called pesudo-relativistic Schrödinger equations have been established in [11]- [14]. After the Emden-Fowler change of variable, Dela Torre-del Pino-González-Wei [15] reformulate a Delaunay-type singular fractional Yamabe problem into a variational periodic problem. Barrios-García-Melián-Quaas [16] obtain some existence results of periodic solutions to some fractional Laplace equations by using variational methods and the bifurcation theory. In [17], the authors establish interior and boundary Harnack's inequalities for nonnegative solutions to $(-\Delta)^s u=0$ with periodic boundary conditions, and they obtain regularity properties of the fractional Laplacian with periodic boundary conditions and the point-wise integro-differential formula for the operator.

Gui and the first author [18] prove the existence of periodic solutions of system (1.1) with large period T by using variational methods. Moreover, they draw a conclusion that the second component of periodic solution which minimize its corresponding energy functional is identical to zero if the origin is a saddle point of F, whereas the second