

The Existence and Exact Multiplicity of One-Sign Solutions for Semilinear Elliptic Problems in \mathbb{R}^N

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Abstract. In this work, we study the existence of one-sign solutions for the following problem:

$$\begin{cases} -\Delta u = \lambda a(x)f(u), & \text{in } \mathbb{R}^N, \\ u(x) \rightarrow 0, & \text{as } |x| \rightarrow +\infty, \end{cases}$$

where $N \geq 3$, λ is a real parameter and $a \in C_{loc}^\alpha(\mathbb{R}^N, \mathbb{R})$ for some $\alpha \in (0, 1)$ is a weighted function, $f: \mathbb{R} \rightarrow \mathbb{R}$ is a Hölder continuous function with exponent α such that $f(s)s > 0$ for any $s \neq 0$. We determine the intervals of λ for the existence, exact multiplicity of one-sign solutions for this problem. We use bifurcation techniques and the approximation of connected components to prove our main results.

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1 Introduction

Consider the following semilinear elliptic problem

$$\begin{cases} -\Delta u = \lambda a(x)f(u), & \text{in } \mathbb{R}^N, \\ u(x) \rightarrow 0, & \text{as } |x| \rightarrow +\infty, \end{cases} \quad (1.1)$$

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where λ is a real parameter, $N \geq 3$, and $a \in C_{loc}^\alpha(\mathbb{R}^N, \mathbb{R})$ for some $\alpha \in (0, 1)$ is a weighted function which can be sign-changing and $f \in C^\alpha(\mathbb{R}, \mathbb{R})$, and $f(s)s > 0$ for any $s \neq 0$. Edelson and Rumbos [1] and Edelson and Furi [2] have studied the existence of positive solution and the existence of global branches of minimal solutions of the problem (1.1) by the Schauder-Tychonoff fixed point theorem and Dancer global bifurcation theorems [3], respectively. By using Rabinowitz global bifurcation method [4], Rumbos and Edelson [5] have shown the existence of positive minimal solution of the problem (1.1). In 2017, Dai [6] have established a global bifurcation result for the problem (1.1). By [6], set

$$M(\Omega) := \{a \in C_{loc}^\alpha(\Omega, \mathbb{R}) : \{x \in \Omega : a(x) > 0\} \neq \emptyset\}.$$

For any $u \in C_c^\infty(\Omega)$ with $\Omega \subseteq \mathbb{R}^N$, we define

$$\|u\|_1 = \left(\int_{\Omega} |\nabla u|^2 dx \right)^{\frac{1}{2}}.$$

Denote by $\mathcal{D}^{1,2}(\Omega)$ the completion of $C_c^\infty(\Omega)$ with respect to the norm $\|u\|_1$. Denote by $S(\mathbb{R}^N)$ the set of all measurable real functions defined on \mathbb{R}^N . Two functions in $S(\mathbb{R}^N)$ are considered as the same element of $S(\mathbb{R}^N)$ when they are equal almost everywhere. Let $L^2(\mathbb{R}^N; |a|) =: \{u \in S(\mathbb{R}^N) : \int_{\mathbb{R}^N} |a|u^2 dx < +\infty\}$. For $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, $u \neq 0$, define the Rayleigh quotient

$$R(u) = \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\int_{\mathbb{R}^N} au^2 dx}.$$

Assume a satisfies the following condition:

(A1) Let $a \in M(\mathbb{R}^N)$. Assume that $p, P \in C(\mathbb{R}^N, \mathbb{R})$ are positive, radially symmetric and satisfies

$$0 < p(|x|) \leq a(x) \leq P(|x|), \quad \forall x \in \mathbb{R}^N$$

and

$$\int_{\mathbb{R}^N} |x|^{2-N} P(|x|) dx < +\infty.$$

Furthermore, if P satisfies the following more strong condition (with $r = |x|$)

$$\int_{\mathbb{R}^N} r^{N-1} P(r) dr < +\infty. \tag{1.2}$$

Dai [6] established the following spectrum structure:

Lemma 1.1 ([6, Theorem 1.1]). *Let (A1) hold. Then there exists an orthonormal basis $\{\varphi_k\}_1^{+\infty}$ of $L^2(\mathbb{R}^N; |a|)$ and a sequence of positive real numbers $\{\lambda_k\}_1^{+\infty}$ with $\lambda_k \rightarrow +\infty$ as $k \rightarrow +\infty$, such that*

$$0 < \lambda_1 < \lambda_2 \leq \dots \leq \lambda_k \leq \dots,$$