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Convergence Analysis of the Legendre Spectral Collocation Methods for Second Order Volterra Integro-Differential Equations

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Abstract. A class of numerical methods is developed for second order Volterra integro-differential equations by using a Legendre spectral approach. We provide a rigorous error analysis for the proposed methods, which shows that the numerical errors decay exponentially in the L^∞-norm and L^2-norm. Numerical examples illustrate the convergence and effectiveness of the numerical methods.

AMS subject classifications: 45L10, 65R20, 65D15

Key words: Second order Volterra integro-differential equations, Gauss quadrature formula, Legendre-collocation methods, convergence analysis.

1. Introduction

Second order Volterra integro-differential equations (VIDEs) arise in the mathematical model of physical and biological phenomena. This fact has led researchers to develop the theoretical and numerical analysis for such equations. For a survey of early results we refer the reader to [12, 19–21, 25]. More recently, polynomial spline collocation methods were investigated in [9, 23]. Bologna [22] found an asymptotic solution for first and second order VIDEs containing an arbitrary kernel. In [24], Sinc-collocation method was developed to approximate the second order VIDEs with boundary conditions.

So far, very few works have touched the spectral approximations to second order VIDEs. Spectral methods have been used in applied mathematics and scientific computing to numerically solve certain partial differential equations (PDEs) [2, 7, 10, 17]. In practice, spectral methods have excellent convergence properties with the so-called "exponential convergence" being the fastest possible. Recently, several authors have developed

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the spectral methods for the solutions of Volterra integral equations (VIEs) of the second kind [27,29,30], pantograph-type delay differential equations [15,16] and singularly perturbed problems [28]. The main purpose of this work is to apply the Legendre spectral-collocation methods for second order VIDEs. We will provide a rigorous error analysis which theoretically justifies the spectral rate of convergence.

For simplicity, denote $y^{(j)}(t) = \left(\partial^j/\partial t^j\right) y(t),\ j = 0,1,2$. In order to discuss the numerical solution of the second order VIDEs we consider the following linear integro-differential equation:

$$y^{(2)}(t) = q(t) + \sum_{j=0}^{1} p_j(t) y^{(j)}(t) + \sum_{j=0}^{1} \int_0^t K_j(t,s) y^{(j)}(s) ds, \qquad t \in \tilde{I} := [0,T], \tag{1.1}$$

with

$$y(0) = y_0, \quad y^{(1)}(0) = y_1, \tag{1.2}$$

where $q : \tilde{I} \to R$, $p_j : \tilde{I} \to R$ and $K_j : D \to R$ $(j = 0,1)$ (with $D := \{(t,s) : 0 \le s \le t \le T\}$) are given functions and are assumed to be sufficiently smooth in the respective domains. The above equation is usually known as basic test equation and is suggested by Brunner and Lambert [14]. It has been widely used for analyzing the solution and stability properties of various methods.

For ease of analysis, we will describe the spectral methods on the standard interval $\hat{I} := [-1,1]$. Hence, we employ the transformation

$$t = \frac{T}{2}(1+x), \qquad x = \frac{2}{T}t - 1.$$

Then the above problem becomes

$$\begin{aligned} u^{(2)}(x) = \left(\frac{T}{2}\right)^2 \sum_{j=0}^{1} \int_0^{\frac{T}{2}(1+x)} K_j\left(\frac{T}{2}(1+x), s\right) y^{(j)}(s) ds \\ + b(x) + \sum_{j=0}^{1} a_j(x) u^{(j)}(x), \qquad x \in \hat{I} := [-1,1], \end{aligned} \tag{1.3}$$

with

$$u(-1) = u_{-1}, \qquad u^{(1)}(-1) = u'_{-1}, \tag{1.4}$$

where

$$\begin{aligned} &u(x) = y\left(\frac{T}{2}(1+x)\right), && b(x) = \left(\frac{T}{2}\right)^2 q\left(\frac{T}{2}(1+x)\right), \\ &a_0(x) = \left(\frac{T}{2}\right)^2 p_0\left(\frac{T}{2}(1+x)\right), && a_1(x) = \frac{T}{2} p_1\left(\frac{T}{2}(1+x)\right), \\ &u_{-1} = y_0, && u'_{-1} = \frac{T}{2} y_1. \end{aligned}$$