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A Moving Mesh Isogeometric Method Based on Harmonic Maps

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Abstract. Although isogeometric analysis has shown great potential in achieving highly accurate numerical solutions of partial differential equations, how to efficiently implement the method is one of the challenges that makes it more competitive in practical simulations. In this paper, an integration of isogeometric analysis and a moving mesh method is proposed, providing a competitive approach to resolve the efficiency issue. Focusing on the Poisson equation, the implementation of the algorithm is presented in detail, including the numerical discretization of the governing equation using isogeometric analysis, and a mesh redistribution technique developed via harmonic maps. It is found that the isogeometric analysis brings attractive features in the realization of moving mesh method, such as providing an accurate expression for moving direction of mesh nodes, and allowing for more choices to construct monitor functions. Through a series of numerical experiments, the efficiency and effectiveness of the proposed method are successfully verified. Moreover, the potential of the method towards the practical applications is also well presented with the simulation of a helium atom in Kohn-Sham density functional theory.

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1. Introduction

In 2005, Hughes *et al.* [26] proposed a novel numerical method for solving partial differential equations, known as isogeometric analysis (IGA), which utilizes the same spline basis functions for both geometry representation and solution space construction. The main motivations behind IGA are to exactly represent the geometry and to bridge the gap between computer-aided design and computer-aided engineering. Furthermore, in this approach, the mesh refinement [15, 26] is simplified as the physical domain is constructed by non-uniform rational B-splines (NURBS). There are three types of refinement strategies in IGA, i.e., the h-, p-, and k-refinements, see [26] for the details. The novel k-refinement can not only increase the smoothness and degree of basis functions but also significantly reduce the number of degrees of freedom (DOFs).

Compared to the classic finite element methods, the IGA framework can easily construct an approximation space of high-order and highly regular basis functions using the k-refinement strategy. However, maintaining a balance between accuracy and computational efficiency remains a persistent challenge. Although higher accuracy can be obtained by increasing the number of DOFs, this always results in challenges towards efficiency as a larger linear system of equations is required to be solved. Recall that in the finite element methods (FEMs), to partially resolve the computational efficiency issue, adaptive FEMs have been developed. Generally speaking, there are three types of adaptive FEMs: the h-adaptive method (locally refine the mesh), the p-adaptive method (locally elevate the degree of basis functions), and the r-adaptive method (moving mesh method), and we refer to [12, 25, 36, 45] and the references therein for the details.

There have been several attempts in the literature to develop adaptive methods within the framework of IGA. Note that the development of h-adaptive techniques for the NURBS-based IGA [26] is limited by the tensor-product structure of B-splines in multi-dimensional space [15]. To overcome this limitation, several suitable extensions for B-splines have been proposed to make local mesh refinement possible. For instance, an adaptive isogeometric method based on hierarchical splines was developed in [13], ensuring the capability of the hierarchical mesh refinement. By allowing the hanging nodes, a series of T-splines was proposed, including the analysis-suitable (dual-compatible) T-splines [10, 40], polynomial splines over (hierarchical) T-meshes [16, 17], and locally refined splines [29]. The p-adaptive method requires the ability to locally elevate the degree of basis functions. However, the NURBS basis functions [38] are constructed with the same degree, limiting the development of p-adaptive method in NURBS-based IGA, and the hybrid-degree weighted T-splines were introduced by Liu *et al.* [35] to implement the p-adaptive method in IGA. Given these considerations, the moving mesh method stands out as a promising approach for improving the computational efficiency of IGA. In fact, the moving mesh method has found widespread applications in various fields, such as computational fluid dynamics [18, 22, 42, 48, 53], phase-field models [44], Burgers' equation [52], fracture mechanics [1, 2, 37], Landau-Lifshitz-Gilbert equation [20], and density functional the-