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Parallel and Scalable Two-Grid Nonlinear Preconditioning Technique for Fully Coupled Solution of Thermal Convective Flow Problems

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Abstract. The focus of this paper is on the numerical solution of large sparse nonlinear systems of algebraic equations, which arise from the discretization of the incompressible Navier-Stokes equations for fluid flows. The inexact Newton algorithm is an efficient and popular technique for solving this kind of nonlinear systems. However, the method may not be suitable for large scale calculations because the number of Newton iterations is not scalable with respect to the number of processors, the grid size, and robust with respect to some physical parameters such as the Reynolds number and the Grashof number. In this paper, we investigate some fully coupled parallel two-grid nonlinear preconditioning algorithm, including a grid sequencing method for the Newton iteration and a two-level overlapping Schwarz preconditioner for the linear iteration. We show numerically that it performs well for solving the thermal convective flow problems on parallel computers.

AMS subject classifications: 76D05, 65F08, 49M15, 65M55, 68W10

Key words: Fluid flow, heat transfer, nonlinear preconditioning, grid sequencing, domain decomposition, parallel computing.

1. Introduction

The aim of this work is the construction and analysis of a grid sequencing method and a two-level overlapping Schwarz preconditioner that lead to a fast, scalable, and robust parallel two-grid nonlinear algorithm for solving large, sparse, nonlinear systems of equations arising from the discretization of steady-state incompressible Navier-

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Stokes equations. Many applications in computational science and engineering depend on the Navier-Stokes equations, and hence it is currently an active and important topic in developing efficient and robust algorithms for solving this kind of nonlinear problems. Although years of research have been taken to find fast and reliable methods for solving Navier-Stokes equations, it remains a difficult computing task due to certain characteristics of the equations that are yet to be fully understood mathematically. To resolve the details of the solution, high resolution grids are often required, which implies large condition numbers of the resulting algebraic systems, and also implies the need for a large-scale parallel computer. There are some classes of methods such as projection type methods and multigrid type methods [9, 11] to use for incompressible Navier-Stokes equations. Moreover, in [2, 3, 14, 17, 18, 26, 27], a class of nonlinearly preconditioned inexact Newton algorithms is introduced for some flow problems and showed that the method has good convergence.

In this study, we focus on the class of Newton methods because of their ease of implementation and applicability to general flows and geometry. Since the inexact Newton method has a rapid convergence rate when the initial guess is near the desired solution, there are growing interests in using the inexact Newton method as a popular technique for solving nonlinear systems [6–8, 25]. One drawback of the inexact Newton method is that it is often difficult to obtain a good enough initial guess, especially for multi-components nonlinear systems. The key idea of our proposed two-grid method is to use a grid sequencing method which employs an interpolated coarse grid solution as the initial guess for the fine grid system. This algorithm does not require the splitting of any variable or operator, and this makes it suitable for many nonlinear systems of PDEs. In [24], a class of parallel scalable two-grid is introduced for the nonlinear complementarity problem, and showed numerically that such an approach is totally scalable in the sense that the number of Newton iterations and the number of linear iterations are both nearly independent of the grid size and the number of processors. In addition, the method is not sensitive to the sharp discontinuity often associated with obstacle problems. However, in the paper only one-component system is investigated. In this paper, we extend the work of [24] to the case with a more complicated model of multi-components system and further understand the influent of the grid sequencing method on the accuracy of the numerical simulation. On the other hand, for high fidelity simulations on supercomputers with a large number of processors, the scalability of the algorithm with respect to the number of processors is critically important. To fix the scalability issue, we propose and study a fully coupled two-level multiplicative-type Schwarz preconditioner. With the help of the grid sequencing method and the two-level Schwarz preconditioner, our experiments show that two-grid NKS method provides a good improvement of the overall method in terms of the total computing time, linear and Newton iterations.

The rest of the paper is organized as follows. In Section 2, we discuss two-level Schwarz preconditioned inexact Newton algorithms and introduce a grid sequencing method, which plays the central role in the scalability of the algorithm. In Section 3, we describe two-dimensional steady-state incompressible Navier-Stokes equations and