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Numerical Analysis of a Second Order Crank-Nicolson Mixed Finite Element Method for the Swift-Hohenberg Equation

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Abstract. In this paper, we conduct an error analysis for a mixed finite element method combined with Crank-Nicolson time-stepping for the Swift-Hohenberg (SH) equation. The original fourth-order differential equation from the SH equation is reformulated as a coupled system comprising a nonlinear parabolic equation and an elliptic equation. This transformation allows us to propose a mixed finite element method that utilizes only continuous elements for the approximation of the system. We rigorously demonstrate that our proposed scheme is unconditionally uniquely solvable and maintains unconditional nonlinear energy stability. Additionally, we provide a thorough analysis to determine the convergence rate of the method. To substantiate our theoretical findings, we present numerical tests that confirm both the accuracy and energy stability of the scheme.

AMS subject classifications: 35Q99, 65N30, 65M12, 65M70

Key words: Swift-Hohenberg equation, error analysis, mixed finite element, Crank-Nicolson unconditionally energy stable.

1. Introduction

Swift-Hohenberg equation is a nonlinear high-order evolution equation, which was first proposed in 1977 by Swift and Hohenberg [23] when describing Rayleigh Benard convection, that is, to describe the process of the evolution of a specific model in the fluid layer confined to the heat conduction boundary. Because of its complex and in-

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teresting evolution process, it has become one of the important examples of nonlinear dynamic evolution leading to the formation of complex patterns. In addition to the study of complex fluids, it has also been applied in other fields, including the study of neural tissue [9], the simulation of laser problems [12] and the solution of the Kelvin problem on the minimum surface [6].

The SH equation (2.2) can be derived from a variation of Lyapunov functional (2.1), so the continuous equation has nonlinear stability [5, 23]. The equation includes high-order differential terms, nonlinear terms, and some physical parameters. The different selection of parameters makes the SH equation show a banded structure or hexagonal structure [4]. In recent years, various numerical methods for simulating the SH equation (2.2) have also been proposed. In [29], the authors presented a large time-stepping numerical method for the Swift-Hohenberg equation by adding an extra artificial term to preserve the unconditional energy stability. The famous scalar auxiliary variable (SAV) approach [22] has been widely used for the Swift-Hohenberg equation, see e.g., [18, 24, 25]. In [16, 17, 20], authors use various invariant energy quadratization (IEQ) schemes for the temporal discretization that also preserve the unconditional energy stability. High-order time-accurate (BDF2/BDF3) method has been adopted in [15, 19, 27]. For the spatial discretization, the finite difference methods [8, 14, 26], finite element method [19, 20, 24], discontinuous Galerkin (DG) methods [16], local discontinuous Galerkin method (LDG) [7, 30] and Fourier spectral methods [11, 28], have been widely reported.

It is worth noting that the Swift-Hohenberg equation (2.2) exhibits three critical features: high nonlinearity, small parameters ($0 < \epsilon \ll 1$), and high-order derivatives. In practical applications, these characteristics make it challenging to approximate the biharmonic terms using standard methods, potentially leading to more stringent requirements for time step sizes. To address this challenge, an effective approach is to split the biharmonic operator into two Laplacian operators, as demonstrated in [19–21]. In this paper, we propose a new fully discrete finite element scheme for the SH equation using a mixed formulation based on the Crank-Nicolson (CN) scheme, which ensures unconditionally unique solvability. We also establish the unconditional energy stability of this scheme. By employing the operator splitting method to handle the biharmonic term in (2.2), we use the mixed finite element method for spatial discretization and the CN time-stepping scheme for temporal discretization. Our primary focus is on providing a rigorous error analysis for the proposed numerical scheme. Furthermore, the presence of the biharmonic term complicates the application of conforming finite element methods, as it necessitates the use of C^1 elements. The mixed finite element method, however, allows us to utilize standard C^0 elements, which are supported by many computational software packages. Numerical results are provided to validate both the energy stability and accuracy of our numerical scheme. Additionally, we obtain and confirm the convergence properties through numerical experiments.

The paper is organized as follows. In Section 2, we introduce the Swift-Hohenberg equation and derive its mixed weak formulation. Section 3 focuses on the spatial discretization using a mixed finite element method. We establish the unconditional energy