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A Biquadratic Characteristic Finite Volume Element Method for Time-Fractional Convection-Dominated Equations

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Abstract. We propose a biquadratic characteristic finite volume element (CFVE) method for two classes of time-fractional convection-dominated equations on quadrilateral meshes. For the mixed time-fractional problem (i.e., involving both time-fractional and integer-order derivatives), optimal convergence rate in L^2-norm is established for the CFVE method with temporal discretization based on the $L1$-formula. For the pure time-fractional problem, we decompose the equation by separating an integer-order time derivative term, thereby enabling the application of the method of characteristics. Furthermore, the influence of the decomposition parameter is investigated through numerical experiments, providing practical insights into parameter selection. Extensive numerical experiments validate the efficacy of the proposed CFVE method in simulating both classes of time-fractional convection-dominated equations.

AMS subject classifications: 65N30, 65M60

Key words: Characteristic finite volume element method, time-fractional, convection-dominated, finite volume element method.

1. Introduction

Time-fractional models have emerged as powerful tools for capturing anomalous transport phenomena in complex systems, particularly where memory effects and heterogeneous media interactions dominate. These models excel in simulating sub-diffusive processes encountered in polymer fluid dynamics [26, 32], contaminant transport in porous media [33, 42], and biomedical applications [45]. Their mathematical foundation lies in fractional calculus' ability to describe non-Markovian dynamics through convolution kernels that encode historical dependencies. However, the nonlocal nature of fractional operators and potential convection dominance pose unique computational challenges that demand specialized numerical approaches.

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In recent years, many scholars have begun to use the finite volume element method to solve problems in fields such as fluid mechanics. For fractional partial differential equations, Liu and Ran [19] studied arbitrarily high-order explicit energy-conserving methods for generalized nonlinear fractional Schrödinger wave equations, Yin, Liu, and Li [43] gave a class of shifted high-order numerical methods for the fractional mobile/immobile transport equations and Song *et al.* [31] solved the time fractional cable model using the $L2$ formula. For anisotropic diffusion equations, there have been many new advancements [6, 23, 30, 40, 47]. For other equations, Huang and Xi [7] studied the general self-adjoint elliptic problems, Wang *et al.* [34] studied the energy-preserving finite volume element method for the improved Boussinesq equation and Wang *et al.* [39] proposed the Bogner-Fox-Schmit element finite volume method based on Shishkin mesh for solving fourth-order singularly perturbed elliptic problems. Meanwhile, some scholars have embarked on researching high-order finite volume element methods, constructing new formats and presenting corresponding error proofs [2, 8, 14, 16, 21, 35, 37, 41, 48, 51]. Among them, authors of [8, 16, 35] conducted the construction and analysis of finite volume element schemes with optimal L^2 convergence rates and Yang *et al.* [41] presented the construction and analysis of quadratic finite volume methods on tetrahedral meshes. In addition, the phenomenons of superconvergence are also studied [1, 5, 36, 38].

This work focuses on two fundamental classes of time-fractional convection-dominated equations. The mixed time-fractional (MTF) equation couples integer-order and fractional derivatives

$$\begin{cases} \partial_t u + \boldsymbol{a}\cdot\nabla u - \nabla\cdot(\boldsymbol{b}\nabla u) + \lambda\, {}_0D_t^{1-\alpha}u = f, & (x,t)\in\Omega\times(0,T], \\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

while the pure time-fractional (PTF) equation omits the integer-order time derivative

$$\begin{cases} \boldsymbol{a}\cdot\nabla u - \nabla\cdot(\boldsymbol{b}\nabla u) + \lambda\, {}_0D_t^{1-\alpha}u = f, & (x,t)\in\Omega\times(0,T], \\ u(x,0) = u_0(x), & x\in\Omega. \end{cases} \tag{1.2}$$

Here, $\Omega \subseteq \mathbb{R}^2$ is a convex polygonal domain, $\boldsymbol{a}:\Omega\to\mathbb{R}^2$ is a bounded convection vector, $\boldsymbol{b}:\Omega\to\mathbb{R}^{2\times2}$ is a symmetric positive definite diffusion tensor, $\lambda\geq 0$ is a non-negative constant, and ${}_0D_t^{1-\alpha}(\cdot)$ $(0<\alpha<1)$ denotes the Riemann-Liouville fractional derivative [12, 27]

$${}_0D_t^{1-\alpha}\, g(t) = {}_0^CD_t^{1-\alpha}\, g(t) + \frac{g(0)}{\Gamma(\alpha)}t^{\alpha-1}, \tag{1.3}$$

where the Caputo derivative ${}_0^CD_t^{1-\alpha}g(t)$ and fractional integral ${}_0I_t^{1-\alpha}g(t)$ are defined as

$$\begin{aligned} {}_0^CD_t^{1-\alpha}g(t) &= {}_0I_t^{\alpha}g'(t), \\ {}_0I_t^{1-\alpha}g(t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{g(s)}{(t-s)^{\alpha}}ds. \end{aligned}$$