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Two Globally Convergent Hybrid CG Methods with Adaptive Restart Strategy for Riemannian Optimization

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Abstract. Hybrid conjugate gradient methods (CGMs) with adaptive restart strategies have been well-researched in Euclidean space. In this paper, we extend two methods of this type to solve optimization problems on Riemannian manifolds. Firstly, we present two Riemannian hybrid CGMs, and their hybrid conjugate parameters are yielded by projection or convex combination of the classical parameters. The first is a Riemannian hybrid CGM that projects between the Dai-Yuan method and another flexible conjugate parameter. The second is a combination of projection and convex combination of the modified Riemannian Liu-Storey method and the modified Riemannian Hestenes-Stiefel method. In the framework of Riemannian CGMs, we apply a uniform adaptive restart strategy to both methods based on the hybrid conjugate parameters we proposed. The search direction of the presented methods satisfies the sufficient descent condition. Under the usual assumption and the weak Wolfe line search, we prove the global convergence of the two proposed methods. Finally, preliminary numerical results are reported and compared with several existing Riemannian CGMs, showing that our methods are effective.

AMS subject classifications: 65K05, 90C30

Key words: Riemannian optimization, hybrid conjugate gradient method, adaptive restart strategy, global convergence.

1. Introduction

For dealing with unconstrained optimization problems in Euclidean space, given by

$$\min f(x), \quad \text{s.t.} \quad x \in \mathbb{R}^n, \tag{1.1}$$

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where $f : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function with its gradient denoted by $\nabla f(x)$, the CGM can be applied to find optimal solutions efficiently. The CGM is well-known and highly efficient in literature. Its outstanding performance is attributed to their simple structure, low storage requirements, and excellent numerical behavior [9,11,12,32]. Usually, the iterative form of CGM for solving problem (1.1) is as follows:

$$x_{k+1} = x_k + \alpha_k \eta_k, \tag{1.2}$$

where $x_k \in \mathbb{R}^n$ is the k-th approximation to the solution, and the positive step size α_k is generated by a suitable line search. η_k is the search direction generated by

$$\eta_k = \begin{cases} -\nabla f(x_k), & k = 1, \\ -\nabla f(x_k) + \beta_k \eta_{k-1}, & k \geq 2, \end{cases} \tag{1.3}$$

where $\beta_k \in \mathbb{R}$ is a conjugate parameter that significantly impacts the numerical performance of the CGMs.

Over the past few decades, many variants of CGMs have emerged to achieve good convergence properties and outstanding numerical performance. Among these, the hybrid CGMs have emerged as particularly attractive, attracting significant research and development efforts. Notable contributions include the hybrid CGM developed by Touati-Ahmed and Storey [31], which combined the Polak-Ribière-Polyak (PRP) and Fletcher-Reeves (FR) methods. Dai and Yuan [6] proposed an effective hybrid CGM that hybridizes the Hestenes-Stiefel (HS) and Dai-Yuan (DY) methods. Zhou *et al.* [35] proposed a hybrid conjugate gradient algorithm with weighted preconditioner. Jian *et al.* [10] presented a new hybrid CGM with descent property. Furthermore, Narayanan and Kaelo [18] proposed another hybrid CGM as a linear combination of the DY method and the HS method.

At the same time, in order to accelerate the algorithm, the adaptive strategy is used in many cases, such as Refs. [4, 16, 26, 33]. Crowder and Wolfe [5] and Powell [19] focused on researching CGMs with adaptive restart strategies and confirmed that the CGMs converge linearly at most without restart strategies in its iterations. To address this, Powell in his other work [20] proposed enhancing the CGM's computational performance through an adaptive restart strategy. Jiang *et al.* [15] established a family of hybrid CGM with adaptive restart strategy to solve large-scale unconstrained optimization problems and image restorations. Moreover, Andrei [3] has highlighted the identification of optimal adaptive restart strategy for CGM as an ongoing challenge. Thus, exploring adaptive restart strategy to improve the CGM presents significant value.

In this paper, we devote ourselves to extending the hybrid CGM in Euclidean space to the setting of the Riemannian manifolds while using an adaptive restart strategy. We consider to solve the following Riemannian optimization problem:

$$\min f(x), \quad \text{s.t.} \quad x \in \mathcal{M}, \tag{1.4}$$

where $f : \mathcal{M} \to \mathbb{R}$ is smooth and $\mathcal{M}$ is a Riemannian manifold endowed with a Riemannian metric $\langle \cdot, \cdot \rangle_x$. Riemannian optimization approach has gained popularity in