Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.OA-2025-0010

Vol. **19**, No. 1, pp. 20-38
February 2026

A Space-Time Extrapolation Cascadic Multigrid Method for 2D Linear Parabolic Problems

Huaqing Wang[1,2], Kejia Pan[1], Jin Li[1] and Hongling Hu[3,*]

[1] *School of Mathematics and Statistics, HNP-LAMA, Central South University, Changsha 410083, China*
[2] *School of Mathematics and Big Data, Jining University, Qufu 273155, China*
[3] *MOE-LCSM, School of Mathematics and Statistics, Hunan Normal University, Changsha 410081, China*

Received 27 January 2025; Accepted (in revised version) 1 August 2025

Abstract. We present a new space-time extrapolation cascadic multigrid method with SSOR preconditioned GMRES smoother for linear parabolic equations. This method simultaneously solves all time steps of the system using an all-at-once approach, employing Crank-Nicolson discretization in time and central difference discretization in space. The key components of the new algorithm are the Richardson extrapolation and Lagrange interpolation operators. By utilizing these techniques with numerical solutions of current and previous grids, we generate a good initial guess for the iterative solution on the next finer grid, greatly reducing the number of required iterations and computational time. Finally, we explain how to implement the new multigrid method and show its efficiency through numerical experiments.

AMS subject classifications: 65M10, 78A48

Key words: Parabolic problems, Crank-Nicolson scheme, all-at-once discretization, Richardson extrapolation, multigrid.

1. Introduction

In this paper, we seek an approximate solution for the general second-order parabolic equation under the initial and Dirichlet boundary conditions

$$\begin{aligned} \partial_t u(\mathbf{x},t) &= \nabla\cdot\big(a(\mathbf{x})\nabla u(\mathbf{x},t)\big) - b(\mathbf{x})\cdot\nabla u(\mathbf{x},t) \\ &\quad - c(\mathbf{x})u(\mathbf{x},t) + f(\mathbf{x},t), && (\mathbf{x},t)\in\Omega\times[0,T], \qquad (1.1) \\ u(\mathbf{x},t) &= g(\mathbf{x},t), && (\mathbf{x},t)\in\partial\Omega\times(0,T], \\ u(\mathbf{x},0) &= u_0(\mathbf{x}), && \mathbf{x}\in\Omega, \qquad (1.2) \end{aligned}$$

*Corresponding author. *Email address:* `honglinghu@hunnu.edu.cn` (H.L. Hu)

where Ω is a rectangular domain in R^2, and T is a positive constant. Here the coefficients a, b, c and forcing function f, and the unknown function u are sufficiently smooth functions and have required regularities for the computation and analysis. The functions u_0 and g represent the initial and boundary conditions, respectively, which characterize the exact solution u.

The classical time-stepping method solves parabolic PDEs one-time step after one-time step (i.e., in a fully sequential manner), which would be time-consuming if the number of time steps is large. This challenge has driven the advancement of parallel-in-time (PinT) numerical methods for solving evolutionary PDEs, for example, the parareal algorithm [15, 24, 33], the multigrid-reduction-in-time (MGRIT) method [9, 12, 19, 35], the preconditioned Krylov subspace method [13, 16, 20, 28, 36, 39] and the space-time multigrid (STMG) method [3, 10, 14, 21, 27, 29, 37]. For an in-depth summary of the past 50 years of parallel-in-time research, please refer to [11]. In this paper, we focus on the STMG method.

Multigrid solvers serve as optimal preconditioners for elliptic problems and, with certain considerations, have demonstrated efficiency for space-time discretizations of parabolic problems like the heat equation. By treating time as an additional spatial dimension, STMG extends traditional multigrid paradigms to the temporal domain, enabling simultaneous coarsening and refinement in space and time. Hackbusch [18] first introduced the parabolic multigrid method, which achieved excellent convergence when coarsening in space but struggled with temporal coarsening. Lubich and Ostermann [26] later developed a continuous variant using waveform relaxation as a smoother, which, when discretized, aligned with Hackbusch's original method. Then, Horton and Vandewalle [21] were the first to introduce a general multigrid method that supports coarsening in both space and time, using semi-coarsening strategies to address the strong anisotropy in the time direction. To overcome the time coarsening challenges caused by anisotropy, Gander and Neumuller [14] proposed an STMG method with a block Jacobi smoother for the heat equation, enabling full space-time coarsening to solve the global discrete system at once, with excellent convergence and scalability properties. In recent years, multigrid methods have been used to solve various complex problems, demonstrating remarkable flexibility and efficiency, see [22, 34, 38] for details.

Inspired by the progress mentioned above, we aim to investigate a new space-time extrapolation cascadic multigrid (EXCMG) method for solving all-at-once (AaO) systems derived from full space-time discretization. The EXCMG approach is known for its exceptional efficiency in handling large linear systems, outperforming classical multigrid methods, especially in the context of elliptic problems [4, 6, 30]. In this paper, we first introduce a space-time extrapolation cascadic multigrid (STEXCMG) method to solve parabolic PDEs across all time levels simultaneously. This method utilizes full space-time discretization, resulting in an all-at-once system. Given the popularity of the implicit Crank-Nicolson (CN) scheme [7] for parabolic PDEs due to its unconditional convergence properties, we adopted this method in our study. Unlike existing multigrid methods (see Fig. 1), our method generates a high-quality initial guess for the pre-