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A Gradient Recovery Technique for Enhancing the Convergence of Demagnetizing Field Based on PDE Approach

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Abstract. The PDE approach is a popular technique for the demagnetizing field calculation due to the flexibility in handling complex domains. However, it faces a challenge on delivering desired accuracy due to the suboptimal convergence of the function gradient and singularity on the boundary. In this work, a robust gradient recovery technique is applied and analysed for fixing such an issue. An L^2 error estimate of the finite element approximation for the demagnetizing field is derived, which consists of two parts, i.e., of finite element discretization error and boundary approximation error. A gradient recovery method based on the polynomial preserving recovery technique is applied to enhance the accuracy of the finite element approximation, and a superconvergence result is established. An idea of locally refined surface meshes is applied to resolve the singularity in the boundary conditions, thereby reducing boundary approximation errors. Extensive numerical tests are provided to verify our theoretical findings and the efficiency of our proposed method. The results indicate that the proposed method achieves second-order convergence for the approximate demagnetizing field, positioning it as a highly competitive technique to develop optimal algorithms in computational micromagnetics.

AMS subject classifications: 65N12, 65N15, 65N30

Key words: Demagnetizing field, PDE approach, finite element method, gradient recovery technique, polynomial preserving recovery method, superconvergence.

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1. Introduction

Computational micromagnetics has been playing a significant role in several applications, including new data storage, hyper-sound speakers, and neuronal simulations [6, 12, 17, 18]. A fundamental governing equation of micromagnetics is the following Landau-Lifshitz-Gilbert equation:

$$\frac{\partial}{\partial t}\vec{\mathrm{M}} = -\gamma_G \vec{\mathrm{M}} \times \vec{\mathrm{H}}_{eff} + \frac{\alpha}{\vec{\mathrm{M}}_s}\vec{\mathrm{M}} \times \frac{\partial}{\partial t}\vec{\mathrm{M}},$$

where $\vec{\mathrm{M}}$ is the magnetization, $\vec{\mathrm{M}}_s$ is the saturation magnetization, γ_G is the Gilbert gyromagnetic ratio, α is the dimensionless damping coefficient, and $\vec{\mathrm{H}}_{eff}$ is the effective field. The effective field is composed of exchange field, anisotropy field, external field, and the demagnetizing field, where the demagnetizing field ($\vec{\mathrm{H}}_{dem}$) is given by the following equation and its calculation is very challenging [2, 19, 24, 28, 29]:

$$\vec{\mathrm{H}}_{dem}(\vec{x}) = -\nabla\phi(\vec{x}), \quad \phi(\vec{x}) = \frac{1}{4\pi}\int_{\Omega} \vec{\mathrm{M}}(\vec{x}') \cdot \nabla_{\vec{x}'}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) d\vec{x}', \tag{1.1}$$

where Ω is a bounded domain with Lipschitz and piecewise smooth boundary $\partial\Omega$.

It is known that the direct calculation of the demagnetizing field using (1.1) will cause $\mathcal{O}(N^2)$ computational complexity, where N denotes the total amount of the mesh nodes. In order to reduce the computational complexity, a variety of approaches have been developed for the demagnetizing field calculation, including the fast Fourier transform (FFT) method [2, 19], the fast multipole method (FMM) [3, 4], and the PDE approach [15, 28, 29]. The PDE approach is popular for its flexibility in handling complex domains and advantage in computational complexity. In this approach, the potential $\phi(\vec{x})$ is split into two parts, i.e., $\phi = \phi_1 + \phi_2$, where ϕ_1 and ϕ_2 are successively calculated by the following two Poisson equations:

$$\begin{cases} \nabla^2\phi_1(\vec{x}) = \nabla\cdot\vec{\mathrm{M}}(\vec{x}) & \text{in } \Omega, \\ \phi_1 = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

$$\begin{cases} \nabla^2\phi_2(\vec{x}) = 0 & \text{in } \Omega, \\ \phi_2(\vec{x}) = g(\vec{x}) := \displaystyle\int_{\partial\Omega} N(\vec{x}-\vec{y})\left(\vec{\mathrm{M}}\cdot\vec{n} + \frac{\partial\phi_1}{\partial\vec{n}}\right) d\vec{y} & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $N(\vec{x}-\vec{y}) = -1/(4\pi|\vec{x}-\vec{y}|)$ is the Newtonian potential. Once ϕ is obtained, the demagnetizing field $\vec{H}_{dem}$ is calculated by taking the negative gradient of it, i.e., $\vec{H}_{dem} = -\nabla\phi$, and then the following demagnetizing field energy which is important in practice can be obtained:

$$\vec{E}_{dem} = -\frac{\mu_0}{2}\int_{\Omega} \vec{\mathrm{H}}_{dem}\cdot\vec{\mathrm{M}} d\vec{x} = \frac{\mu_0}{2}\int_{\Omega} |\nabla\phi(\vec{x})|^2 d\vec{x},$$

where μ_0 is the magnetic permeability of the vacuum.