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An Asymptotic Preserving Method for the Weakly Nonlinear Klein-Gordon Equation

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Abstract. In this paper, we propose an asymptotic preserving (AP) method for the weakly nonlinear Klein-Gordon equation (NKGE). Firstly, we apply a multi-scale expansion for the weakly NKGE and obtain the equation for the leading-order term, for which an error estimate has been provided. Secondly, by solving the equation for the leading-order term numerically, we construct an AP method for the weakly NKGE. Finally, numerical results in one spatial dimension are provided to show that:

(i) The method is asymptotic preserving, i.e., the error between the leading-order term and the solution of the weakly NKGE behaves as $\mathcal{O}(\varepsilon)$ as $\varepsilon \to 0$.

(ii) It is uniformly accurate since the numerical solution obtained by the method is independent of the small parameter ε.

(iii) It can make correct predictions about the solution of the original NKGE. Moreover, extension of the method to the two-dimensional weakly NKGE are also provided.

AMS subject classifications: 35C20, 35L70, 81-08

Key words: Weakly nonlinear Klein-Gordon equation, multi-scale expansion, leading-order term, error estimate, asymptotic preserving method.

1. Introduction

Since the Klein-Gordon equation was derived in 1926 by Oskar Klein and Walter Gordon, it is the most fundamental equation to describe the motion of spin-less particle in relativistic quantum mechanics and quantum field theory [11,12,33,46]. Mean-

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while, it plays an important role in mathematical physics, condensed matter physics and plasma physics [13, 23, 27, 52]. The nonlinear Klein-Gordon equation with different types of nonlinearities appears in many scientific applications. For example, the NKGE with power-type nonlinearity can be applied to investigate solid state physics, nonlinear optics and quantum field theory [42, 52, 53]. The NKGE with cubic nonlinearity is widely used in studying the dynamics of relativistic Bose gas, superconductors and other related systems [28, 41].

Up to now, there are extensive numerical and analytical research in the standard nonlinearity strength regime (i.e., $\varepsilon = \mathcal{O}(1)$), but most of those results focus on studying the short-time dynamics of the NKGE. Along with the attention of the life-span of the solution to the NKGE, we find that the analysis and numerical computation of the long-time dynamics of the NKGE in the weak nonlinearity strength regime (i.e., $0 < \varepsilon \ll 1$) is rather important. In analytical aspect, the existence of global classical solutions and the Cauchy problem with weak nonlinearity and the asymptotic behavior of solutions have been considered in the literatures [18, 19, 37, 39, 43, 44, 48]. And the analytical results in the literature have shown that the life-span of a smooth solution is at least up to the time at $\mathcal{O}(\varepsilon^{-2})$, i.e., the final time for the existence of the solution is dependent on the value of the power-type nonlinearity, see [20–22, 29, 44] and references therein.

There are several numerical methods have been applied and analyzed in the literatures to the NKGE with weak nonlinearity [2–5, 9, 30, 36, 47], e.g., the finite difference time domain methods, exponential wave integrator Fourier pseudospectral method, multiscale time integrator Fourier pseudospectral method and some asymptotic preserving schemes, etc. However, the error bounds of those numerical results are normally valid up to the time at $\mathcal{O}(1)$. Recently, Bao *et al.* have proposed some numerical schemes to resolve the weakly NKGE up to the time at $\mathcal{O}(\varepsilon^{-2})$ and established error bounds of those numerical methods in the long time instead of the classical error bounds which are only valid up to the time at $\mathcal{O}(1)$. For example, in [31], the fourth-order compact finite difference method with $h = \mathcal{O}(\varepsilon^{p/4})$ and $\tau = \mathcal{O}(\varepsilon^{p/2})$ has better spatial resolution than the finite difference time domain methods [7], where h and τ are the space and time steps and $p \in \mathbb{N}^+$ is the value of the power-type nonlinearity. Furthermore, with the help of Fourier pseudospectral methods in space [6, 32], Bao *et al.* found that the technique of regularity compensation oscillation (RCO) can improve the uniform error bounds for the second-order semi-discretization in time as $\mathcal{O}(\varepsilon^2\tau^2)$ and for the full-discretization as $\mathcal{O}(h^m + \varepsilon^2\tau^2)$. For more details related to the RCO technique can be seen in [1] and references therein. Now, achieving a fixed accuracy for varying values of ε requires maintaining ε-scalability (or meshing strategy requirements), which becomes prohibitively costly as $\varepsilon \to 0$. The goal of this article is, therefore, to develop new numerical schemes whose accuracy does not deteriorate for vanishing ε.

Following the validity of the nonlinear Schrödinger (NLS) approximation for the NKG systems with a quasilinear quadratic nonlinearity illustrated in [16, 17, 24–26, 35, 38, 40], it is a natural question to ask why the small spatio-temporal modulations of