

# An Asymptotic Preserving Method for the Weakly Nonlinear Klein-Gordon Equation

Dandan Wang<sup>1,2</sup> and Hanquan Wang<sup>3,\*</sup>

<sup>1</sup> Eastern Institute for Advanced Study, Eastern Institute of Technology,  
Ningbo 315200, P.R. China

<sup>2</sup> College of Mathematics, Sichuan University, Chengdu 610065, P.R. China

<sup>3</sup> School of Statistics and Mathematics and Center for Applied Mathematics,  
Yunnan University of Finance and Economics, Kunming 650221, P.R. China

Received 24 February 2025; Accepted (in revised version) 4 November 2025

---

**Abstract.** In this paper, we propose an asymptotic preserving (AP) method for the weakly nonlinear Klein-Gordon equation (NKGE). Firstly, we apply a multi-scale expansion for the weakly NKGE and obtain the equation for the leading-order term, for which an error estimate has been provided. Secondly, by solving the equation for the leading-order term numerically, we construct an AP method for the weakly NKGE. Finally, numerical results in one spatial dimension are provided to show that:

- (i) The method is asymptotic preserving, i.e., the error between the leading-order term and the solution of the weakly NKGE behaves as  $\mathcal{O}(\varepsilon)$  as  $\varepsilon \rightarrow 0$ .
- (ii) It is uniformly accurate since the numerical solution obtained by the method is independent of the small parameter  $\varepsilon$ .
- (iii) It can make correct predictions about the solution of the original NKGE. Moreover, extension of the method to the two-dimensional weakly NKGE are also provided.

**AMS subject classifications:** 35C20, 35L70, 81-08

**Key words:** Weakly nonlinear Klein-Gordon equation, multi-scale expansion, leading-order term, error estimate, asymptotic preserving method.

---

## 1. Introduction

Since the Klein-Gordon equation was derived in 1926 by Oskar Klein and Walter Gordon, it is the most fundamental equation to describe the motion of spin-less particle in relativistic quantum mechanics and quantum field theory [11, 12, 33, 46]. Mean-

---

\*Corresponding author. *Email addresses:* dandan95\_wang@163.com (D. Wang), wang\_hanquan@hotmail.com (H. Wang)

while, it plays an important role in mathematical physics, condensed matter physics and plasma physics [13, 23, 27, 52]. The nonlinear Klein-Gordon equation with different types of nonlinearities appears in many scientific applications. For example, the NKGE with power-type nonlinearity can be applied to investigate solid state physics, nonlinear optics and quantum field theory [42, 52, 53]. The NKGE with cubic nonlinearity is widely used in studying the dynamics of relativistic Bose gas, superconductors and other related systems [28, 41].

Up to now, there are extensive numerical and analytical research in the standard nonlinearity strength regime (i.e.,  $\varepsilon = \mathcal{O}(1)$ ), but most of those results focus on studying the short-time dynamics of the NKGE. Along with the attention of the life-span of the solution to the NKGE, we find that the analysis and numerical computation of the long-time dynamics of the NKGE in the weak nonlinearity strength regime (i.e.,  $0 < \varepsilon \ll 1$ ) is rather important. In analytical aspect, the existence of global classical solutions and the Cauchy problem with weak nonlinearity and the asymptotic behavior of solutions have been considered in the literatures [18, 19, 37, 39, 43, 44, 48]. And the analytical results in the literature have shown that the life-span of a smooth solution is at least up to the time at  $\mathcal{O}(\varepsilon^{-2})$ , i.e., the final time for the existence of the solution is dependent on the value of the power-type nonlinearity, see [20–22, 29, 44] and references therein.

There are several numerical methods have been applied and analyzed in the literatures to the NKGE with weak nonlinearity [2–5, 9, 30, 36, 47], e.g., the finite difference time domain methods, exponential wave integrator Fourier pseudospectral method, multiscale time integrator Fourier pseudospectral method and some asymptotic preserving schemes, etc. However, the error bounds of those numerical results are normally valid up to the time at  $\mathcal{O}(1)$ . Recently, Bao *et al.* have proposed some numerical schemes to resolve the weakly NKGE up to the time at  $\mathcal{O}(\varepsilon^{-2})$  and established error bounds of those numerical methods in the long time instead of the classical error bounds which are only valid up to the time at  $\mathcal{O}(1)$ . For example, in [31], the fourth-order compact finite difference method with  $h = \mathcal{O}(\varepsilon^{p/4})$  and  $\tau = \mathcal{O}(\varepsilon^{p/2})$  has better spatial resolution than the finite difference time domain methods [7], where  $h$  and  $\tau$  are the space and time steps and  $p \in \mathbb{N}^+$  is the value of the power-type nonlinearity. Furthermore, with the help of Fourier pseudospectral methods in space [6, 32], Bao *et al.* found that the technique of regularity compensation oscillation (RCO) can improve the uniform error bounds for the second-order semi-discretization in time as  $\mathcal{O}(\varepsilon^2\tau^2)$  and for the full-discretization as  $\mathcal{O}(h^m + \varepsilon^2\tau^2)$ . For more details related to the RCO technique can be seen in [1] and references therein. Now, achieving a fixed accuracy for varying values of  $\varepsilon$  requires maintaining  $\varepsilon$ -scalability (or meshing strategy requirements), which becomes prohibitively costly as  $\varepsilon \rightarrow 0$ . The goal of this article is, therefore, to develop new numerical schemes whose accuracy does not deteriorate for vanishing  $\varepsilon$ .

Following the validity of the nonlinear Schrödinger (NLS) approximation for the NKG systems with a quasilinear quadratic nonlinearity illustrated in [16, 17, 24–26, 35, 38, 40], it is a natural question to ask why the small spatio-temporal modulations of

an underlying carrier wave can form the NLS approximation. And we take the cubic nonlinearity of weakly NKGE as example in this paper to introduce how the asymptotic preserve method can be constructed, and obtain one approximation result in terms of  $\varepsilon$  over a time interval  $[0, T/\varepsilon^2]$ . Following [24, 38, 40], the error estimate of AP method for the weakly NKGE can be established with proof in  $H^1(\mathbb{R}) \times L^2(\mathbb{R})$  space.

The paper is organized as follows. In Section 2, based on multi-scale expansion, we obtain a leading-order approximation and construct an AP method to the weakly NKGE in one dimension. Following the existing analytical results, the error estimate on the leading-order approximation can be rigorously showed again. Numerical results are reported in Section 3 to confirm our error estimate. A conclusion is drawn in Section 4. Throughout this paper, we directly adopt the fourth-order time-splitting spectral (TSSP) method to get a reference solution of the weakly NKGE up to the time at  $\mathcal{O}(\varepsilon^{-2})$ . And we also construct a TSSP method to solve NLS equation numerically, the more details can be seen in Appendix A. Finally, we extend the AP method and its rough error estimate to 2D weakly NKGE in Appendix B.

## 2. An AP method to the weakly NKGE

In this section, we consider the initial value problem for weakly NKGE [1, 7, 14, 15, 35, 49–51],

$$\partial_{tt}u(x, t) = \partial_{xx}u(x, t) - u(x, t) + \varepsilon^2 u^3(x, t), \quad x \in (a, b), \quad t \geq 0, \quad (2.1)$$

$$u(x, 0) = \phi(x), \quad x \in [a, b], \quad (2.2)$$

$$\partial_t u(x, 0) = \gamma^\varepsilon(x), \quad x \in [a, b], \quad (2.3)$$

where the small parameter  $\varepsilon$  ( $0 < \varepsilon \ll 1$ ) defines the weakly term. The initial condition  $\phi(x)$  is at  $\mathcal{O}(1)$ . The initial condition  $\gamma^\varepsilon(x)$  depends on  $\varepsilon$ . The unknown function  $u = u(x, t)$  is real-valued function. In particular, if the NKGE (2.1) is considered on a torus  $\mathbb{T}$  (i.e.,  $a = 0, b = 2\pi$ ) and  $u(x, t) \in H^1(\mathbb{T})$ ,  $\partial_t u(x, t) \in L^2(\mathbb{T})$ , the NKGE will become time reversible and its energy is conservative [1, 6, 7, 18, 31, 32, 34].

### 2.1. A multi-scale expansion for the weakly NKGE

On the one hand, Eq. (2.1) describes the motion of a wave packet with a constant group velocity  $c = k/\omega$ . On the other hand, inspired by the work [35] on ordinary differential equations, we aim to expand the solution of the weakly NKGE (2.1) as a perturbation series [25, 26, 35], i.e.,

$$u(x, t) = \sum_{n=0}^{\infty} \varepsilon^n u_n(x_0, x_1, \dots, t_0, t_1, \dots). \quad (2.4)$$

Here the fast spatial variable  $x$ , the slow spatial variables  $x_j$  ( $j \geq 1$ ), the fast temporal variable  $t$ , and the slow temporal scales  $t_j$  ( $j \geq 1$ ) are defined respectively as follows:

$$x_0 = x, \quad x_1 = \varepsilon^1 x, \dots, \quad x_j = \varepsilon^j x, \dots, \quad j = 0, 1, \dots, \infty,$$

$$t_0 = t, \quad t_1 = \varepsilon^1 t, \dots, \quad t_j = \varepsilon^j t, \dots, \quad j = 0, 1, \dots, \infty.$$

Noting that

$$\begin{aligned} \partial_x &= \partial_{x_0} + \varepsilon \partial_{x_1} + \varepsilon^2 \partial_{x_2} + \dots, \\ \partial_t &= \partial_{t_0} + \varepsilon \partial_{t_1} + \varepsilon^2 \partial_{t_2} + \dots. \end{aligned}$$

When the ansatz (2.4) is inserted into Eq. (2.1), the weakly NKGE (2.1) can be rewritten as

$$\begin{aligned} 0 &= \partial_{tt} u(x, t) - \partial_{xx} u(x, t) + u(x, t) - \varepsilon^2 u^3(x, t) \\ &= \left\{ \partial_{t_0 t_0} u_0 + \varepsilon (\partial_{t_0 t_1} u_0 + \partial_{t_1 t_0} u_0 + \partial_{t_0 t_0} u_1) \right. \\ &\quad \left. + \varepsilon^2 (\partial_{t_0 t_2} u_0 + \partial_{t_2 t_0} u_0 + \partial_{t_1 t_1} u_0 + \partial_{t_0 t_1} u_1 + \partial_{t_1 t_0} u_1 + \partial_{t_0 t_0} u_2) + \dots \right\} \\ &\quad - \left\{ \partial_{x_0 x_0} u_0 + \varepsilon (\partial_{x_0 x_1} u_0 + \partial_{x_1 x_0} u_0 + \partial_{x_0 x_0} u_1) \right. \\ &\quad \left. + \varepsilon^2 (\partial_{x_0 x_2} u_0 + \partial_{x_2 x_0} u_0 + \partial_{x_1 x_1} u_0 + \partial_{x_0 x_1} u_1 + \partial_{x_1 x_0} u_1 + \partial_{x_0 x_0} u_2) + \dots \right\} \\ &\quad + \left\{ u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots \right\} - \varepsilon^2 \left\{ u_0^3 + \dots \right\} + \dots. \end{aligned} \quad (2.5)$$

At the same time, when the ansatz (2.4) is inserted into Eqs. (2.2) and (2.3), the initial conditions can be rewritten as

$$\begin{aligned} \sum_{n=0}^{\infty} \varepsilon^n u_n(x_0, x_1, \dots, t_0, t_1, \dots) \Big|_{t=0} &= \phi(x_0, x_1, \dots), \\ \partial_t \sum_{n=0}^{\infty} \varepsilon^n u_n(x_0, x_1, \dots, t_0, t_1, \dots) \Big|_{t=0} &= \gamma^\varepsilon(x_0, x_1, \dots). \end{aligned}$$

Next we collect like powers of  $\varepsilon$  in Eq. (2.5),

- (1) For leading-order terms, i.e., terms of order  $\mathcal{O}(\varepsilon^0)$ , one obtains a wave equation for  $u_0$

$$\partial_{t_0 t_0} u_0 - \partial_{x_0 x_0} u_0 + u_0 = 0. \quad (2.6)$$

- (2) Proceeding with our multi-scale expansion and obtaining at order  $\mathcal{O}(\varepsilon^1)$  the first-order equation for  $u_1$

$$\partial_{t_0 t_0} u_1 - \partial_{x_0 x_0} u_1 + u_1 = -\partial_{t_0 t_1} u_0 - \partial_{t_1 t_0} u_0 + \partial_{x_0 x_1} u_0 + \partial_{x_1 x_0} u_0. \quad (2.7)$$

- (3) Continuing to consider the terms of order  $\mathcal{O}(\varepsilon^2)$ , i.e.,  $u_2$  satisfies the second-order equation

$$\begin{aligned} &\partial_{t_0 t_0} u_2 - \partial_{x_0 x_0} u_2 + u_2 \\ &= -\partial_{t_0 t_1} u_1 - \partial_{t_1 t_0} u_1 - \partial_{t_1 t_1} u_0 - \partial_{t_0 t_2} u_0 \\ &\quad - \partial_{t_2 t_0} u_0 + \partial_{x_0 x_1} u_1 + \partial_{x_1 x_0} u_1 + \partial_{x_1 x_1} u_0 \\ &\quad + \partial_{x_0 x_2} u_0 + \partial_{x_2 x_0} u_0 + u_0^3. \end{aligned} \quad (2.8)$$

We need to find  $u_n$  ( $n = 0, 1, 2$ ) from the above three equations so as to obtain an approximate solution of the weakly NKGE (2.1).

Firstly, considering the initial value problem for the leading-order equation (2.6),

$$\begin{aligned} \partial_{t_0 t_0} u_0 - \partial_{x_0 x_0} u_0 + u_0 &= 0, \\ u_0(x_0, x_1, \dots, t_0, t_1, \dots) \Big|_{t=0} &= \phi(x_0, x_1, \dots), \\ \partial_t u_0(x_0, x_1, \dots, t_0, t_1, \dots) \Big|_{t=0} &= \gamma^\varepsilon(x_0, x_1, \dots). \end{aligned} \quad (2.9)$$

In the theory of waves, under the most cases that the corresponding initial datum can be viewed as a single wave packet or finite sums of wave packets. For simplicity, we consider the following single wave packet with a right-moving velocity as the solution of Eq. (2.9):

$$u_0(x_0, x_1, \dots, t_0, t_1, \dots) = A_0(x_1, \dots, t_1, \dots) e^{i(kx_0 - \omega t_0)} + c.c., \quad (2.10)$$

where the complex number  $i = \sqrt{-1}$ , *c.c.* denotes ‘complex conjugate’. The wave number  $k$  and the frequency  $\omega$  satisfy the following dispersion relation:

$$\omega = \omega(k) = \sqrt{1 + k^2}.$$

Secondly, substituting (2.10) into the right-hand-side of Eq. (2.7), we solve

$$\begin{aligned} \partial_{t_0 t_0} u_1 - \partial_{x_0 x_0} u_1 + u_1 &= 2(i\omega \partial_{t_1} A_0 + ik \partial_{x_1} A_0) e^{i(kx_0 - \omega t_0)} + c.c., \\ u_1(x_0, x_1, \dots, t_0, t_1, \dots) \Big|_{t=0} &= 0, \\ \partial_t u_1(x_0, x_1, \dots, t_0, t_1, \dots) \Big|_{t=0} &= -\partial_{t_1} u_0(x_0, x_1, \dots, t_0, t_1, \dots) \Big|_{t=0}. \end{aligned}$$

Using the idea from Remark 2.1, we can easily cancel the secular terms  $2(i\omega \partial_{t_1} A_0 + ik \partial_{x_1} A_0) e^{i(kx_0 - \omega t_0)} + c.c.$  by removing them so that  $u_1$  is bounded, i.e.,

$$\begin{aligned} 2(i\omega \partial_{t_1} A_0 + ik \partial_{x_1} A_0) &= 0 \Leftrightarrow \partial_{t_1} A_0 = -\frac{k}{\omega} \partial_{x_1} A_0, \\ 2(i\omega \partial_{t_1} \overline{A_0} + ik \partial_{x_1} \overline{A_0}) &= 0 \Leftrightarrow \partial_{t_1} \overline{A_0} = -\frac{k}{\omega} \partial_{x_1} \overline{A_0}. \end{aligned} \quad (2.11)$$

It is interesting to find that  $u_1$  is the solution of the following wave equation:

$$\partial_{t_0 t_0} u_1 - \partial_{x_0 x_0} u_1 + u_1 = 0.$$

For simplicity, we choose

$$u_1(x_0, x_1, \dots, t_0, t_1, \dots) = 0. \quad (2.12)$$

Thirdly, plugging Eqs. (2.10) and (2.12) into (2.8), we obtain

$$\begin{aligned} &\partial_{t_0 t_0} u_2 - \partial_{x_0 x_0} u_2 + u_2 \\ &= \{ (2i\omega \partial_{t_2} A_0 - \partial_{t_1 t_1} A_0 + 2ik \partial_{x_2} A_0 + \partial_{x_1 x_1} A_0 + 3|A_0|^2 A_0) e^{i(kx_0 - \omega t_0)} \\ &\quad + A_0^3 e^{3i(kx_0 - \omega t_0)} \} + c.c. \end{aligned} \quad (2.13)$$

with the initial conditions

$$\begin{aligned} u_2(x_0, x_1, \dots, t_0, t_1, \dots) \Big|_{t=0} &= 0, \\ \partial_t u_2(x_0, x_1, \dots, t_0, t_1, \dots) \Big|_{t=0} &= -\partial_{t_2} u_0(x_0, x_1, \dots, t_0, t_1, \dots) \Big|_{t=0}. \end{aligned}$$

Similar to the process that we get the analytical solution for  $u_1$ , we must find the secular terms and eliminate them to keep the solution  $u_2$  bounded. Therefore, postulating that

$$\begin{aligned} 2i\omega\partial_{t_2}A_0 - \partial_{t_1t_1}A_0 + 2ik\partial_{x_2}A_0 + \partial_{x_1x_1}A_0 + 3|A_0|^2A_0 &= 0, \\ -2i\omega\partial_{t_2}\overline{A_0} - \partial_{t_1t_1}\overline{A_0} - 2ik\partial_{x_2}\overline{A_0} + \partial_{x_1x_1}\overline{A_0} + 3|A_0|^2\overline{A_0} &= 0. \end{aligned} \quad (2.14)$$

Since  $\omega(3k) \neq 3\omega(k)$  for almost all  $k$  in the mathematical field, the remaining terms  $A_0^3 e^{3i(kx_0 - \omega t_0)} + \overline{A_0^3} e^{-3i(kx_0 - \omega t_0)}$  in Eq. (2.13) are not solutions to the homogeneous equation

$$\partial_{t_0t_0}u_2 - \partial_{x_0x_0}u_2 + u_2 = 0.$$

So Eq. (2.13), the equation for  $u_2$  can be simplified into,

$$\partial_{t_0t_0}u_2 - \partial_{x_0x_0}u_2 + u_2 = A_0^3 e^{3i(kx_0 - \omega t_0)} + c.c..$$

We do not seek the equation for the higher-order term  $u_n$  ( $n \geq 3$ ) in (2.4) since we only consider the leading-order term

$$\tilde{u}(x, t) = A_0(x_1, x_2, \dots, t_1, t_2, \dots) e^{i(kx_0 - \omega t_0)} + c.c. \quad (2.15)$$

for approximating the solution of the weakly NKGE (2.1). The remaining task now is left to find  $A_0$  defined in (2.15).

**Remark 2.1** (Secular Term). If the terms as the solutions to the homogeneous equation contribute to the inhomogeneity, we will think that the terms maybe lead to the solution of the inhomogeneous equation grows unboundedly with the time evolution. And these terms are also called secular terms [10, 45].

## 2.2. Construction of an AP method for the weakly NKGE

We assume that

$$A(x, t) \equiv A_0(x_1, x_2, \dots, t_1, t_2, \dots), \quad (2.16)$$

where  $A(x, t)$  satisfies both Eqs. (2.11) and (2.14). Then we find

$$\varepsilon(2i\omega\partial_{t_1}A + 2ik\partial_{x_1}A) + \varepsilon^2(2i\omega\partial_{t_2}A - \partial_{t_1t_1}A + 2ik\partial_{x_2}A + \partial_{x_1x_1}A + 3|A|^2A) = 0.$$

Rearranging it, the above equation can be rewritten as,

$$\begin{aligned} 2i\omega(\partial_{t_0} + \varepsilon\partial_{t_1} + \varepsilon^2\partial_{t_2})A + 2ik(\partial_{x_0} + \varepsilon\partial_{x_1} + \varepsilon^2\partial_{x_2})A \\ - (\partial_{t_0} + \varepsilon\partial_{t_1} + \varepsilon^2\partial_{t_2})^2A + (\partial_{x_0} + \varepsilon\partial_{x_1} + \varepsilon^2\partial_{x_2})^2A + 3\varepsilon^2|A|^2A = 0. \end{aligned}$$

Noting that  $x_j = \varepsilon^j x$ ,  $t_j = \varepsilon^j t$ ,  $j = 0, 1, \dots, \infty$ , the above equation can be reformulated as

$$i\partial_t A = -\frac{ik}{\omega}\partial_x A + \frac{1}{2\omega}\partial_{tt}A - \frac{1}{2\omega}\partial_{xx}A - \frac{3\varepsilon^2}{2\omega}|A|^2 A. \quad (2.17)$$

Taking the first-order time derivative of both sides of Eq. (2.11), one get

$$\partial_{tt}A = \left(\frac{k}{\omega}\right)^2 \partial_{xx}A.$$

Finally, by replacing  $\partial_{tt}A$  with  $(k/\omega)^2\partial_{xx}A$  in Eq. (2.17), the amplitude of the leading-order term satisfies the following equation:

$$i\partial_t A = -\frac{ik}{\omega}\partial_x A - \frac{1}{2\omega^3}\partial_{xx}A - \frac{3\varepsilon^2}{2\omega}|A|^2 A. \quad (2.18)$$

Although Eq. (2.18) may have a unique solution when a given initial-boundary condition is provided, the weak self-interaction  $(3\varepsilon^2/2\omega)|A|^2 A$  still exists and hinders further approximation and analysis.

If we define

$$A(x, t) \equiv \tilde{A}(\xi, \sigma), \quad \xi = \varepsilon \left( x - \frac{k}{\omega}t \right), \quad \sigma = \varepsilon^2 t, \quad (2.19)$$

then from Eq. (2.18) we find  $\tilde{A} = \tilde{A}(\xi, \sigma)$  satisfying the following nonlinear Schrödinger equation (NLSE):

$$i\partial_\sigma \tilde{A} = \nu_1 \partial_{\xi\xi} \tilde{A} + \nu_2 |\tilde{A}|^2 \tilde{A}, \quad (2.20)$$

where the coefficients  $\nu_1 = -1/(2\omega^3) < 0$  and  $\nu_2 = -3/(2\omega) < 0$ .

Summing up, from Eqs. (2.15), (2.16) and (2.19), we can construct a leading-order approximation  $\tilde{u}(x, t)$  for  $u(x, t)$  which is the solution of the weakly NKGE (2.1), i.e.,

$$\tilde{u}(x, t) = \tilde{A}(\xi, \sigma)e^{i(kx - \omega t)} + c.c., \quad \tilde{A} \text{ satisfies Eq. (2.20)}. \quad (2.21)$$

According to the above two subsections, we know the reason why the small spatio-temporal modulations of an underlying carrier wave can form the NLS approximation. Next, we show a theorem to illustrate that the leading-order approximation  $\tilde{u}(x, t)$  makes correct predictions about the dynamics of the solutions of Eq. (2.1).

### 2.3. The error estimate for the solution $\tilde{u}(x, t)$

Referring to the analytical techniques in the literature [24, 38, 40], we know the residual  $Res(\tilde{u}) = \mathcal{O}(\varepsilon^2)$  and provide a long-time error estimate in  $Y = H^1(\mathbb{R}) \times L^2(\mathbb{R})$  space.

**Theorem 2.1.** Let  $\tilde{A} = \tilde{A}(\xi, \sigma)$  be a solution of the NLS equation (2.20) and  $\partial_\xi^n \partial_\sigma^l \tilde{A} \in C([0, T_0], L^2)$ ,  $n+l \leq 2$ . Then  $\tilde{u}(x, t) = \tilde{A}(\xi, \sigma)e^{i(kx - \omega t)} + c.c.$  is the approximated solution to the weakly NKGE (2.1). For every  $T_0 \leq T$ , there exists  $\varepsilon_0 \in (0, 1]$  and a constant  $C > 0$  such that for all  $\varepsilon \in (0, \varepsilon_0)$  there holds that if  $u = u(x, t)$  be the solution of the weakly NKGE (2.1) and

$$\|(u(\cdot, 0), \partial_t u(\cdot, 0)) - (\tilde{u}(\cdot, 0), \partial_t \tilde{u}(\cdot, 0))\|_Y \leq d\varepsilon^{1/2},$$

then

$$\|(u(\cdot, t), \partial_t u(\cdot, t)) - (\tilde{u}(\cdot, t), \partial_t \tilde{u}(\cdot, t))\|_Y \leq C\varepsilon^{1/2}, \quad \forall t \in [0, T_0/\varepsilon^2].$$

*Proof.* Similar to the idea in [24, 38, 40], we can show that the error  $u(x, t) - \tilde{u}(x, t)$  remains of order  $\mathcal{O}(\varepsilon^{1/2})$  for time  $t \leq T_0/\varepsilon^2$ . We omit the details here, the more information can be found in the literature and reference therein.  $\square$

**Remark 2.2.** From Theorem 2.1, it follows that  $\tilde{u}(x, t)$  is an approximation to  $u(x, t)$  for  $t \in [T_0/\varepsilon^2]$ . Specially,

$$\sup_{t \in [0, T_0/\varepsilon^2]} \|u(\cdot, t) - \tilde{u}(\cdot, t)\|_{H^1} \leq C\varepsilon^{1/2}.$$

Here, the constant  $C$  is independent of  $\varepsilon$ . Furthermore, the error estimate in the  $L^\infty$  norm behaves as  $\mathcal{O}(\varepsilon)$  (c.f., [40, Remark 3.6]).

**Notations.**  $C([0, T_0], L^2)$  is a space of continuously differentiable function from time  $t = 0$  to  $T_0$  and where the function belongs to the space  $L^2$ . The energy space

$$Y = H^1(\mathbb{R}) \times L^2(\mathbb{R})$$

is equipped with the norm

$$\|(u, v)\|_Y = (\|u\|_{H^1}^2 + \|v\|_{L^2}^2)^{1/2}.$$

Here, the space

$$L^p(\mathbb{R}) = \left\{ u : \mathbb{R} \rightarrow \mathbb{R} \mid \text{Lebesgue measurable and } \int_{\mathbb{R}} |u(x)|^p dx < +\infty \right\}$$

is equipped with the norm

$$\|u\|_{L^p} = \begin{cases} \left( \int_{\mathbb{R}} |u(x)|^p dx \right)^{1/p}, & 1 \leq p < \infty, \\ \text{ess sup}_{x \in \mathbb{R}} |u(x)|, & p = \infty \end{cases}$$

the Sobolev space

$$H^m(\mathbb{R}) = \left\{ u \in L^2(\mathbb{R}) \mid \partial_x^j u \in L^2(\mathbb{R}) \text{ for } 0 \leq j \leq m \right\}$$

is equipped with the norm

$$\|u\|_{H^m} = \max_{j \in \{0,1,\dots,m\}} \|\partial_x^j u(x)\|_{L^2}.$$

In the upcoming section, we numerically show that the leading-order approximation  $\tilde{u}(x, t)$  is asymptotic preserving, and the error between  $\tilde{u}(x, t)$  and  $u(x, t)$ .

### 3. Numerical results

In this section, we firstly verify the leading-order approximation (2.21) is an AP one in 1D and provide some long-time numerical simulation based on this leading-order approximation. Secondly we provide some 2D numerical simulation results based on the leading-order term of a 2D multi-scale expansion which is derived in Appendix B.

In order to obtain the AP results for different values of  $\varepsilon$ , we choose the error estimate in the  $l^\infty$  sense, i.e.,

$$\|u(\cdot, t) - \tilde{u}(\cdot, t)\|_{l^\infty} = \max |u(\cdot, t) - \tilde{u}(\cdot, t)|.$$

#### 3.1. Numerical results for the 1D weakly NKGE

We solve the following initial-boundary value problem for the weakly NKGE (2.1), i.e.:

$$\begin{aligned} \partial_{tt}u(x, t) &= \partial_{xx}u(x, t) - u(x, t) + \varepsilon^2 u^3(x, t), & x \in (a, b), & 0 \leq t \leq 2/\varepsilon^2, \\ u(x, 0) &= \phi(x), \quad \partial_t u(x, 0) = \gamma^\varepsilon(x), & x \in [a, b], & \\ u(a, t) &= u(b, t), \quad \partial_x u(a, t) = \partial_x u(b, t), & 0 \leq t \leq 2/\varepsilon^2, & \end{aligned} \quad (3.1)$$

where  $[a, b] = [-524, 524]$ .

**Case I.** We consider the following 1-soliton solution of NLSE (2.20):

$$\tilde{A}(\xi, \sigma) = \sqrt{-2\eta\nu_2^{-1}} \operatorname{sech}\left(\sqrt{-\eta\nu_1^{-1}}\xi\right) e^{i\eta\sigma}, \quad \eta \in \mathbb{R}^+,$$

and construct the following initial conditions:

$$\begin{aligned} \phi(x) &= 2\operatorname{Re} \left\{ \sqrt{-2\eta\nu_2^{-1}} \operatorname{sech}(\tilde{x}) e^{ikx} \right\}, \\ \gamma^\varepsilon(x) &= -2\operatorname{Re} \left\{ \left[ i\omega \sqrt{-2\eta\nu_2^{-1}} \operatorname{sech}(\tilde{x}) + \varepsilon \frac{-k\eta \sqrt{2(\nu_1\nu_2)^{-1}} \sinh(\tilde{x})}{\omega \cosh^2(\tilde{x})} \right. \right. \\ &\quad \left. \left. + i\varepsilon^2 \frac{\eta \sqrt{-2\eta\nu_2^{-1}} (1 - \sinh^2(\tilde{x}))}{\cosh^3(\tilde{x})} \right. \right. \\ &\quad \left. \left. + i\varepsilon^2 \nu_2 (-2\eta\nu_2^{-1})^{3/2} \operatorname{sech}^3(\tilde{x}) \right] e^{ikx} \right\}. \end{aligned}$$

**Case II.** We choose another initial condition for the NLSE (2.20), i.e.,

$$\tilde{A}(\xi, 0) = \frac{\sqrt{-2\eta\nu_2^{-1}}}{4} \left\{ 1 - \tanh \left( \sqrt{-\eta\nu_1^{-1}}(\xi - 11\eta) \right) \tanh \left( \sqrt{-\eta\nu_1^{-1}}(\xi + 11\eta) \right) \right\}.$$

And we set the initial data  $\phi(x)$  and  $\gamma^\varepsilon$  as

$$\begin{aligned} \phi(x) &= 2\text{Re} \left\{ \tilde{A}_0 e^{ikx} \right\}, \\ \gamma^\varepsilon &= -2\text{Re} \left\{ \left( i\omega \tilde{A}_0 + \frac{k}{\omega} \varepsilon \partial_\xi \tilde{A}_0 + i\varepsilon^2 \nu_1 \partial_{\xi\xi} \tilde{A}_0 + i\varepsilon^2 |\tilde{A}_0|^2 \tilde{A}_0 \right) e^{ikx} \right\}. \end{aligned}$$

Here,  $\text{Re}(\cdot)$  denotes the real part of its complex quantity.  $\tilde{x} = (-\eta\nu_1^{-1})^{1/2} \varepsilon x$ ,  $\nu_1 = -1/(2\omega^3)$ ,  $\nu_2 = -3/(2\omega)$ ,  $\eta = 0.5$ , the wave number  $k = 0.3$  and  $\tilde{A}_0 = \tilde{A}(\xi, 0)$ .

In Appendix A, we provide a fourth-order time-splitting Fourier spectral method for solving Eq. (3.1) and also a fourth-order TSSP method for the NLSE (2.20). We use these two numerical methods and obtain our numerical results:

- (1)  $u(x, t)$ : the numerical solution obtained by a TSSP method for Eq. (3.1) (i.e., Algorithm A.1 in Appendix A).
- (2)  $\tilde{u}(x, t) = A^{INT}(\xi, \sigma) e^{i(kx - \omega t)} + c.c.$  and here  $A^{INT}(\xi, \sigma)$  is the numerical solution of the NLSE (2.20) obtained by the TSSP method (c.f. Remark A.1).

Fig. 1 shows us  $u(x, t)$  and  $\tilde{u}(x, t)$  with different initial conditions, two approximated solutions to the 1D weakly NKGE (3.1) for fixed  $\varepsilon = 0.1$  at different times

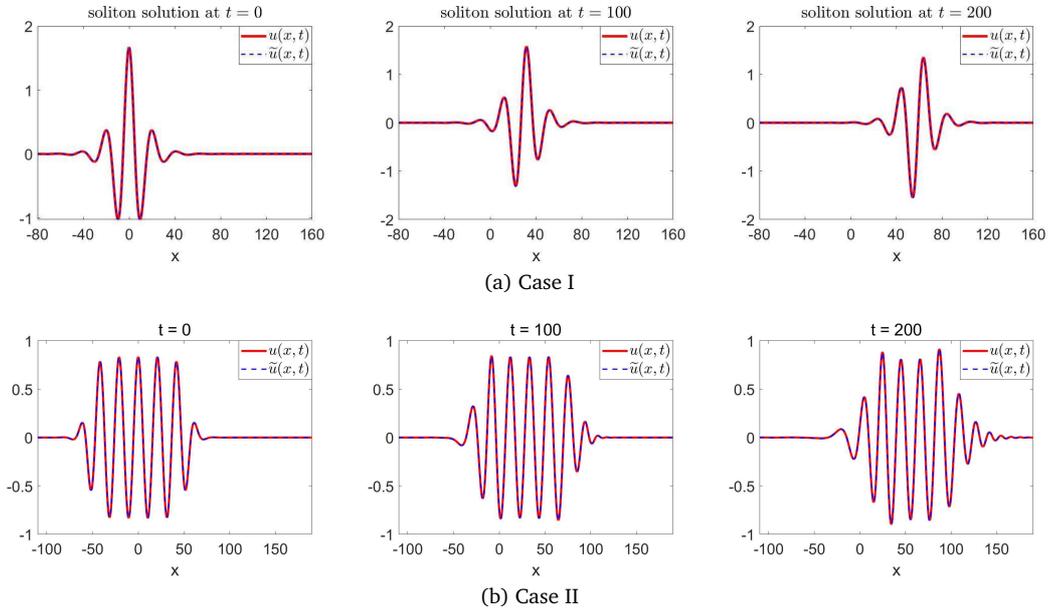


Figure 1:  $\tilde{u}(x, t)$  vs.  $u(x, t)$  at different times ( $t = 0, 100$ , and  $200$  respectively). Here  $\varepsilon = 0.1$  is chosen.

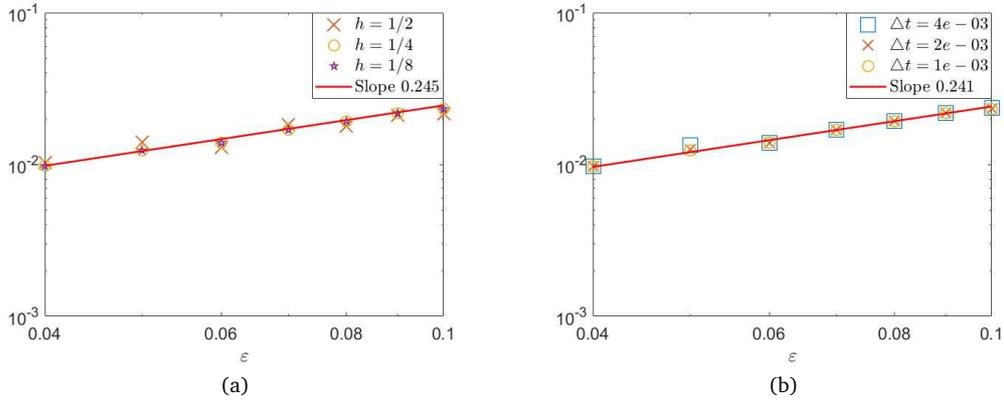


Figure 2: The error  $\|u(x, 2/\varepsilon^2) - \tilde{u}(x, 2/\varepsilon^2)\|_{l^\infty}$  varies against different  $\varepsilon$ . (a) The spatial grid size  $h$  changes but the time step  $\Delta t$  keeps fixed. (b) The temporal size  $\Delta t$  changes but the spatial grid size  $h$  keeps fixed.

$t = 0, 100$ , and  $200$ , respectively. Based on the initial condition Case I, the shape of the envelope remains nearly unchanged. However, the nonsoliton initial value (i.e., Case II) causes the shape of the envelope to change from an almost rectangular form to a completely different outline. Last but not least, from it, compared with the numerical solution of the 1D weakly NKGE (3.1), we find that the leading-order term of the multi-scale expansion provides well a good approximation.

Furthermore, for different spatial mesh size  $h$  and fixed time step  $\Delta t = 1e-03$ , we show that the error  $\|u(x, t) - \tilde{u}(x, t)\|_{l^\infty} = C\varepsilon$  ( $C \in (0.2413, 0.253)$ ) in Fig. 2(a). For different temporal size  $\Delta t$  and fixed spatial step  $h = 1/8$ , we also show that the error  $\|u(x, t) - \tilde{u}(x, t)\|_{l^\infty} = C\varepsilon$  ( $C \in (0.24, 0.243)$ ) in Fig. 2(b). Here  $u(x, t)$  is the numerical solution obtained by a TSSP method for Eq. (3.1) with a very fine spatial and temporal mesh size, which is used as a reference solution. Fig. 2 shows that the leading-order approximation with Case I initial condition is asymptotic preserving one for different  $\varepsilon$ .

From Fig. 2, we have seen that the leading-order approximation gave us a good result. Moreover, the method is asymptotic preserving and its time step and spatial grid size do not depend on the small parameter  $\varepsilon$ . Finally we use it to simulate the long-time behavior of the wave function of the weakly NLKG when  $\varepsilon$  goes near to zero. Fig. 3 gives us the approximation solution  $\tilde{u}(x, t)$  for some small parameters  $\varepsilon = 0.016, 0.008$  and  $0.004$ , respectively. In this simulation, we choose the wave number  $k = 0.05$  and the spatial domain  $[a, b] = [-3142, 3142]$ .

### 3.2. Numerical results for the 2D weakly NKGE

We consider the following initial-boundary problem for the weakly NKGE (B.1), i.e.:

$$\begin{aligned}
 \partial_{tt}u(\mathbf{x}, t) &= \Delta u(\mathbf{x}, t) - u(\mathbf{x}, t) + \varepsilon^2 u^3(\mathbf{x}, t), & \mathbf{x} &= (x, y) \in \Omega, & t &\geq 0, \\
 u(\mathbf{x}, 0) &= \phi(\mathbf{x}), & \partial_t u(\mathbf{x}, 0) &= \gamma^\varepsilon(\mathbf{x}), & \mathbf{x} &= [x, y] \in \bar{\Omega}, \\
 u, \quad \partial_x u &\text{ and } \partial_y u &\text{ are periodic at } \partial\bar{\Omega}, & & t &\geq 0.
 \end{aligned} \tag{3.2}$$

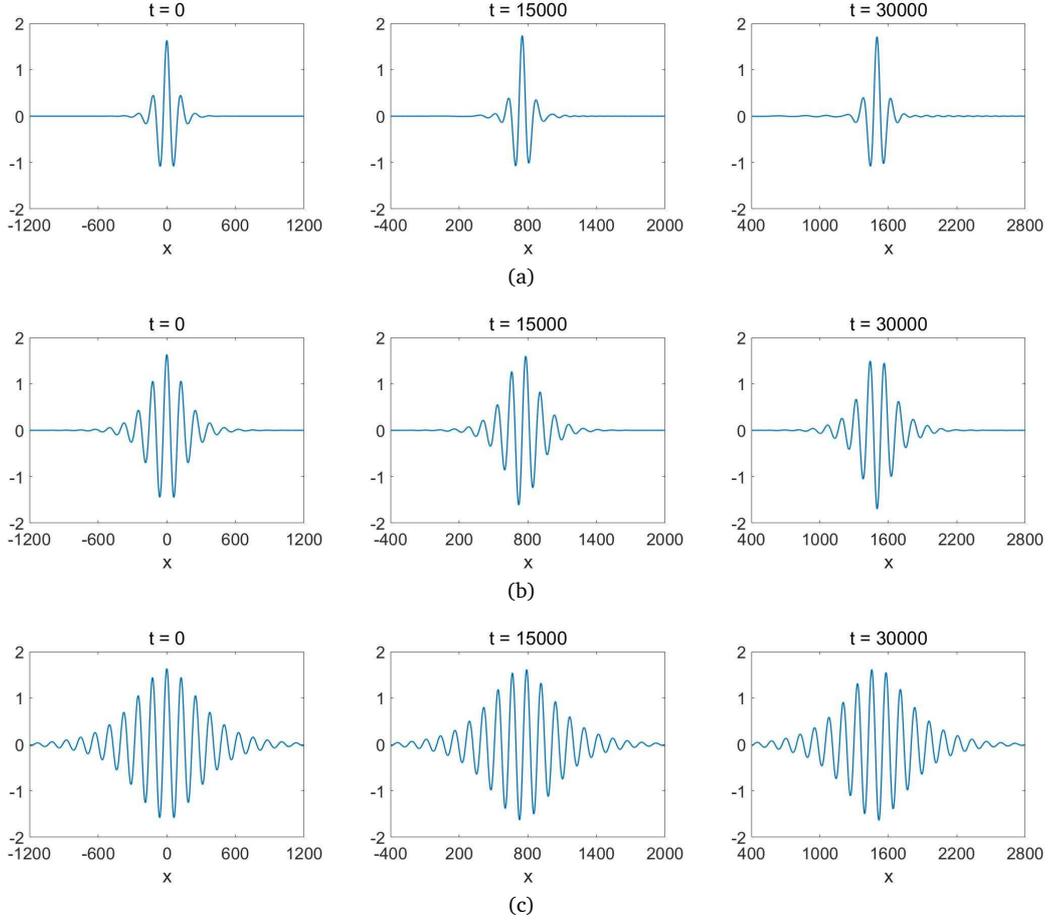


Figure 3: Time evolution of the approximation solution  $\tilde{u}(x, t)$  for different small parameters: (a)  $\varepsilon = 0.016$ ; (b)  $\varepsilon = 0.008$ ; (c)  $\varepsilon = 0.004$ .

In 2D simulation, the spatial domain  $\bar{\Omega} = [-1164, 1164]^2$  when the wave number  $k = 0.135$ , and we construct the following initial datum:

$$\begin{aligned} \phi(x, y) &= 2\text{Re} \left\{ \sqrt{-2\eta\nu_2^{-1}} \text{sech}(\tilde{X}) e^{ik(x+y)} \right\}, \\ \gamma^\varepsilon(x, y) &= -2\text{Re} \left\{ \left[ i\omega \sqrt{-2\eta\nu_2^{-1}} \text{sech}(\tilde{X}) + \varepsilon \frac{-2k\eta \sqrt{2(\nu_1\nu_2)^{-1}} \sinh(\tilde{X})}{\omega \cosh^2(\tilde{X})} \right. \right. \\ &\quad \left. \left. + i\varepsilon^2 \frac{\eta \sqrt{-2\eta\nu_2^{-1}} (1 - \sinh^2(\tilde{X}))}{\cosh^3(\tilde{X})} \right. \right. \\ &\quad \left. \left. + i\varepsilon^2 \nu_2 (-2\eta\nu_2^{-1})^{3/2} \text{sech}^3(\tilde{X}) \right] e^{ik(x+y)} \right\} \end{aligned}$$

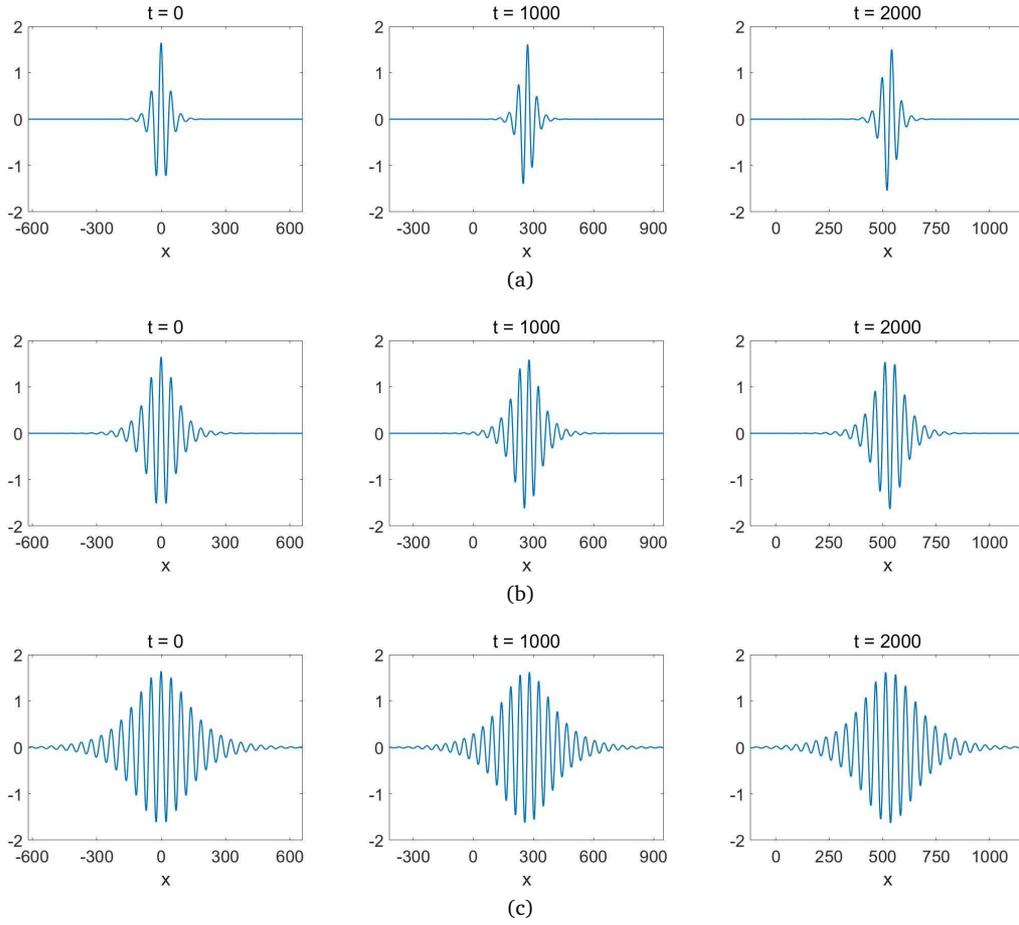


Figure 4: Time evolution of the approximated solution  $\tilde{u}(x, 0, t)$  with different values of  $\varepsilon$ : (a)  $\varepsilon = 0.05$ ; (b)  $\varepsilon = 0.025$ ; (c)  $\varepsilon = 0.0125$ .

for (3.2). Here,

$$\tilde{X} = \sqrt{-\eta\nu_1^{-1}\varepsilon}(x+y), \quad \nu_1 = -\frac{1+k^2}{\omega^3}, \quad \nu_2 = -\frac{3}{2\omega}.$$

The more information can be found in Appendix B.

In Fig. 4, we show the leading-order approximation  $\tilde{u}(x, y=0, t)$  for  $\varepsilon = 0.05, 0.025$  and  $0.0125$ , respectively. From it, we can observe an interesting phenomenon that the wave is right-moving from time  $t = 0$  to  $t = 2000$ . In the large time, the solution is highly oscillatory.

#### 4. Conclusion

Based on a multi-scale expansion, we construct an asymptotic preserving method for the nonlinear Klein-Gordon equation with a weak cubic nonlinearity. We find that

the leading-order term of the multi-scale expansion can be used to predict the dynamics governed by the weakly nonlinear Klein-Gordon equation both in 1D and in 2D, respectively. We have shown that the method is indeed asymptotic preserving one. We have provided the error estimate for the approximated solution in 1D. However, we have not yet given the error estimate for the method in 2D, which will be left as our future work.

### Acknowledgements

The first author thanks the Yunnan University of Finance and Economics for hosting her visit during Year 2024.

The research of H. Wang is supported in part by the Natural Science Foundation of China (Grant Nos. 11871418, 12461081), and by the Yunnan Fundamental Research Projects (Grant No. 202101AS070044). This work is also supported by the research fund from Yunnan University of Finance and Economics (Grant No. 2025H29).

### Appendix A A fourth-order TSSP method to 1D NKGE

As we know, the TSSP method has been widely applied to numerically solve dispersive partial differential equations. And it is more efficient than finite difference methods. We use a fourth-order TSSP method [8, 54] with fine mesh size and small temporal step to obtain the ‘exact’ solution of the weakly NKGE

$$\partial_{tt}u(x, t) = \partial_{xx}u(x, t) - u(x, t) + \varepsilon^2 u^3(x, t), \quad x \in (a, b), \quad 0 \leq t \leq T/\varepsilon^2, \quad (\text{A.1})$$

$$u(x, 0) = \phi(x), \quad \partial_t u(x, 0) = \gamma^\varepsilon(x), \quad x \in [a, b], \quad (\text{A.2})$$

$$u(a, t) = u(b, t), \quad \partial_x u(a, t) = \partial_x u(b, t), \quad 0 \leq t \leq T/\varepsilon^2. \quad (\text{A.3})$$

Defining operator  $\langle \Delta \rangle = (1 - \partial_{xx})^{1/2}$  and utilizing the following ansatz [1, 6, 30]:

$$w_1(x, t) = u - i\langle \Delta \rangle^{-1} \partial_t u,$$

$$w_2(x, t) = \bar{u} - i\langle \Delta \rangle^{-1} \partial_t \bar{u},$$

we transform Eq. (A.1) into a first-order differential equations in time, i.e.,

$$i\partial_t w_1 = -\langle \Delta \rangle w_1 + \frac{\varepsilon^2}{8} \langle \Delta \rangle^{-1} (w_1 + \bar{w}_2)^3 \equiv \mathcal{A}w_1 + \mathcal{B}(w_1, w_2), \quad (\text{A.4})$$

$$i\partial_t w_2 = -\langle \Delta \rangle w_2 + \frac{\varepsilon^2}{8} \langle \Delta \rangle^{-1} (\bar{w}_1 + w_2)^3 \equiv \mathcal{A}w_2 + \mathcal{B}(w_1, w_2).$$

Here, notation  $\mathcal{A}$  denotes the linear operator, and  $\mathcal{B}$  is the nonlinear operator. When ansatz  $w_1(x, t)$  and  $w_2(x, t)$  are inserted into Eq. (A.1), the initial conditions (A.2) and (A.3) can be rewritten as,

$$w_1(x, 0) = \phi(x) - i\langle \Delta \rangle^{-1} \gamma^\varepsilon(x), \quad x \in [a, b],$$

$$w_2(x, 0) = \overline{\phi(x)} - i\langle \Delta \rangle^{-1} \overline{\gamma^\varepsilon(x)}, \quad x \in [a, b].$$

We choose the spatial mesh size  $h = (b - a)/N$  for  $N$  an even positive integer and the time step  $\Delta t = T/M$  for  $M$  a known constant, and we let the grid points and the time step be

$$x^l = a + lh, \quad t^n = n\Delta t, \quad l = 0, 1, \dots, N, \quad n = 0, 1, \dots, M.$$

The approximation solution vector  $w_1^n \approx \{w_1(x^l, t^n)\}_{l=0}^{N-1}$ ,  $w_2^n \approx \{w_2(x^l, t^n)\}_{l=0}^{N-1}$ .

In order to get more efficient numerical results, we develop a fourth-order TSSP method to Eq. (A.4). Setting  $z = \{z(x^l)\}_{l=0}^{N-1}$ , we define the operator  $\langle \Delta \rangle^{\pm 1}$  in Fourier space, i.e.,

$$\left\{ \langle \Delta \rangle^{\pm 1} z(x^l) \right\}_{l=0}^{N-1} \approx \mathcal{F}^{-1} \left\{ (1 + \mu_k^2)^{\pm \frac{1}{2}} \mathcal{F}(z)_k \right\}_{k=-N/2}^{N/2-1} \quad \text{with} \quad \mu_k = \frac{2\pi k}{b-a}.$$

We summarize the fourth-order TSSP method for the 1D weakly NKGE in the following algorithm.

---

**Algorithm A.1** A Fourth-Order TSSP Method to the 1D Weakly NKGE (A.1)

---

1: Setting

$$\phi = \{\phi(x^l)\}_{l=0}^{N-1}, \quad \gamma^\varepsilon = \{\gamma^\varepsilon(x^l)\}_{l=0}^{N-1}, \quad \mu = \left\{ \sqrt{1 + \mu_k^2} \right\}_{k=-N/2}^{N/2-1}.$$

Based on initial values  $w_1^0 = \{w_1(x^l, 0)\}_{l=0}^{N-1}$  and  $w_2^0 = \{w_2(x^l, 0)\}_{l=0}^{N-1}$ , we set

$$\mathcal{F}(w_1^0) = \mathcal{F}(\phi) - i\mu^{-1}\mathcal{F}(\gamma^\varepsilon), \quad \mathcal{F}(w_2^0) = \mathcal{F}(\bar{\phi}) - i\mu^{-1}\mathcal{F}(\bar{\gamma}^\varepsilon).$$

2: Coefficients

$$\alpha_1 = \frac{1}{2(2 - 2^{1/3})}, \quad \alpha_2 = \frac{1}{2 - 2^{1/3}}, \quad \alpha_3 = \frac{1 - 2^{1/3}}{2(2 - 2^{1/3})}, \quad \alpha_4 = -\frac{2^{1/3}}{2 - 2^{1/3}}.$$

for  $n = 0, 1, \dots, M - 1$  do

$$\begin{aligned} w_1^{(1)} &= \mathcal{F}^{-1} \left\{ \mathcal{F}(w_1^n) e^{i\mu\Delta t\alpha_1} \right\}, \\ w_2^{(1)} &= \mathcal{F}^{-1} \left\{ \mathcal{F}(w_2^n) e^{i\mu\Delta t\alpha_1} \right\}. \end{aligned}$$

Setting  $u^* = (w_1^{(1)} + \overline{w_2^{(1)}})/2$  and  $\bar{u}^* = (\overline{w_1^{(1)}} + w_2^{(1)})/2$ ,

$$\begin{aligned} w_1^{(2)} &= \mathcal{F}^{-1} \left\{ \mathcal{F}(w_1^{(1)}) - i\Delta t\alpha_2\varepsilon^2\mu^{-1}\mathcal{F}((u^*)^3) \right\}, \\ w_2^{(2)} &= \mathcal{F}^{-1} \left\{ \mathcal{F}(w_2^{(1)}) - i\Delta t\alpha_2\varepsilon^2\mu^{-1}\mathcal{F}((\bar{u}^*)^3) \right\}, \\ w_1^{(3)} &= \mathcal{F}^{-1} \left\{ \mathcal{F}(w_1^{(2)}) e^{i\mu\Delta t\alpha_3} \right\}, \\ w_2^{(3)} &= \mathcal{F}^{-1} \left\{ \mathcal{F}(w_2^{(2)}) e^{i\mu\Delta t\alpha_3} \right\}. \end{aligned}$$

Setting  $u^{**} = (w_1^{(3)} + \overline{w_2^{(3)}})/2$  and  $\overline{u^{**}} = (\overline{w_1^{(3)}} + w_2^{(3)})/2$ ,

$$\begin{aligned} w_1^{(4)} &= \mathcal{F}^{-1} \left\{ \mathcal{F}(w_1^{(3)}) - i\Delta t \alpha_4 \varepsilon^2 \mu^{-1} \mathcal{F}((u^{**})^3) \right\}, \\ w_2^{(4)} &= \mathcal{F}^{-1} \left\{ \mathcal{F}(w_2^{(3)}) - i\Delta t \alpha_4 \varepsilon^2 \mu^{-1} \mathcal{F}(\overline{(u^{**})^3}) \right\}, \\ w_1^{(5)} &= \mathcal{F}^{-1} \left\{ \mathcal{F}(w_1^{(4)}) e^{i\mu \Delta t \alpha_3} \right\}, \\ w_2^{(5)} &= \mathcal{F}^{-1} \left\{ \mathcal{F}(w_2^{(4)}) e^{i\mu \Delta t \alpha_3} \right\}. \end{aligned}$$

Setting  $u^{***} = (w_1^{(5)} + \overline{w_2^{(5)}})/2$  and  $\overline{u^{***}} = (\overline{w_1^{(5)}} + w_2^{(5)})/2$ ,

$$\begin{aligned} w_1^{(6)} &= \mathcal{F}^{-1} \left\{ \mathcal{F}(w_1^{(5)}) - i\Delta t \alpha_2 \varepsilon^2 \mu^{-1} \mathcal{F}((u^{***})^3) \right\}, \\ w_2^{(6)} &= \mathcal{F}^{-1} \left\{ \mathcal{F}(w_2^{(5)}) - i\Delta t \alpha_2 \varepsilon^2 \mu^{-1} \mathcal{F}(\overline{(u^{***})^3}) \right\}, \\ w_1^{n+1} &= \mathcal{F}^{-1} \left\{ \mathcal{F}(w_1^{(6)}) e^{i\mu \Delta t \alpha_1} \right\}, \\ w_2^{n+1} &= \mathcal{F}^{-1} \left\{ \mathcal{F}(w_2^{(6)}) e^{i\mu \Delta t \alpha_1} \right\}. \end{aligned}$$

**end for**

$$3: u^M = \frac{1}{2}(w_1^M + \overline{w_2^M}) \approx \{u(x^l, t^M)\}_{l=0}^{N-1}.$$


---

**Remark A.1** (A Fourth-Order TSSP Method to 1D NLS). The complex-valued function  $\tilde{A} = \tilde{A}(\xi, \sigma)$  is governed by the following NLSE:

$$\begin{aligned} i\partial_\sigma \tilde{A} &= \nu_1 \partial_{\xi\xi} \tilde{A} + \nu_2 |\tilde{A}|^2 \tilde{A} \equiv (\mathcal{A} + \mathcal{B})\tilde{A}, \quad \xi \in (-L, L), \quad 0 \leq \sigma \leq T, \\ \tilde{A}(x, t=0) &= \tilde{A}^0, \quad \xi \in [-L, L]. \end{aligned} \tag{A.5}$$

For the given constant  $M$  and  $N$ , we take time step

$$\Delta\sigma = T/M$$

into account to yield the time discretization

$$\sigma^n = n\Delta\sigma, \quad n = 0, 1, \dots, M.$$

The spatial discretization can be got as

$$\xi^l = -L + lh_\xi \quad \text{with step size} \quad h_\xi = 2L/N, \quad l = 0, 1, \dots, N.$$

(1) The linear part is solved in Fourier space, and we get

$$A_{\Delta\sigma}^L = \mathcal{F}^{-1} \left\{ \mathcal{F}(\tilde{A}^n) e^{i\theta^2 \nu_1 \Delta\sigma} \right\} \quad \text{with} \quad \theta = \left\{ \frac{2\pi k}{2L} \right\}_{k=-N/2}^{N/2-1}, \quad \sigma \in [\sigma^n, \sigma^{n+1}].$$

(2) The nonlinear part can be solved exactly as

$$A_{\Delta\sigma}^{NL} = e^{-i\nu_2\Delta\sigma|A^n|^2}\tilde{A}^n \quad \text{with} \quad \sigma \in [\sigma^n, \sigma^{n+1}].$$

Finally, the fourth-order TSSP method to 1D weakly NLSE (A.5) can be summarized as

$$\tilde{A}(\xi, \sigma) \approx A_{\alpha_1\Delta\sigma}^L \circ A_{\alpha_2\Delta\sigma}^{NL} \circ A_{\alpha_3\Delta\sigma}^L \circ A_{\alpha_4\Delta\sigma}^{NL} \circ A_{\alpha_3\Delta\sigma}^L \circ A_{\alpha_2\Delta\sigma}^{NL} \circ A_{\alpha_1\Delta\sigma}^L.$$

Here, the coefficients  $\alpha_1, \alpha_2, \alpha_3$  and  $\alpha_4$  are the same as in Algorithm A.1.

## Appendix B An AP scheme to the weakly NKGE in 2D

We consider the weakly NKGE in 2D

$$\partial_{tt}u(\mathbf{x}, t) = \Delta u(\mathbf{x}, t) - u(\mathbf{x}, t) + \varepsilon^2 u^3(\mathbf{x}, t), \quad \mathbf{x} \in \Omega, \quad t \geq 0, \quad (\text{B.1})$$

$$u(\mathbf{x}, 0) = \phi(\mathbf{x}), \quad \mathbf{x} \in \bar{\Omega}, \quad (\text{B.2})$$

$$\partial_t u(\mathbf{x}, 0) = \gamma^\varepsilon(\mathbf{x}), \quad \mathbf{x} \in \bar{\Omega}. \quad (\text{B.3})$$

Here,  $0 < \varepsilon \ll 1$  is the small parameter. The given initial conditions  $\phi(\mathbf{x}) \in \mathbb{R}$  is at  $\mathcal{O}(1)$  and  $\gamma^\varepsilon(\mathbf{x}) \in \mathbb{R}$  depends on  $\varepsilon$ . The unknown wave  $u = u(\mathbf{x}, t) = u(x, y, t) \in \mathbb{R}^2 \times \mathbb{R}$  is a real-valued function.

Similar to the 1D weakly NKGE (2.1), we expand the solution of (B.1) as a perturbation series, i.e.,

$$u(\mathbf{x}, t) = \sum_{n=0}^{\infty} \varepsilon^n u_n(x_0, x_1, \dots, y_0, y_1, \dots, t_0, t_1, \dots). \quad (\text{B.4})$$

Here, the slow spatial variables  $x_j, y_j$  ( $j \geq 1$ ) and slow temporal scales  $t_j$  ( $j \geq 1$ ) can be defined as

$$\begin{aligned} x_0 &= x, & x_1 &= \varepsilon^1 x, \dots, & x_j &= \varepsilon^j x, \dots, & j &= 0, 1, \dots, \infty, \\ y_0 &= y, & y_1 &= \varepsilon^1 y, \dots, & y_j &= \varepsilon^j y, \dots, & j &= 0, 1, \dots, \infty, \\ t_0 &= t, & t_1 &= \varepsilon^1 t, \dots, & t_j &= \varepsilon^j t, \dots, & j &= 0, 1, \dots, \infty. \end{aligned}$$

So

$$\begin{aligned} \partial_x &= \partial_{x_0} + \varepsilon \partial_{x_1} + \varepsilon^2 \partial_{x_2} + \dots, \\ \partial_y &= \partial_{y_0} + \varepsilon \partial_{y_1} + \varepsilon^2 \partial_{y_2} + \dots, \\ \partial_t &= \partial_{t_0} + \varepsilon \partial_{t_1} + \varepsilon^2 \partial_{t_2} + \dots. \end{aligned}$$

When the ansatz (B.4) is inserted into Eq. (B.1), we get the following partial differential equations with respect to different orders of  $\varepsilon$ :

1.  $\varepsilon^1$ -order:

$$\partial_{t_0 t_0} u_0 - \partial_{x_0 x_0} u_0 - \partial_{y_0 y_0} u_0 + u_0 = 0,$$

2.  $\varepsilon^2$ -order:

$$\begin{aligned} & \partial_{t_0 t_0} u_1 - \partial_{x_0 x_0} u_1 - \partial_{y_0 y_0} u_1 + u_1 \\ &= -\partial_{t_0 t_1} u_0 - \partial_{t_1 t_0} u_0 + \partial_{x_0 x_1} u_0 + \partial_{x_1 x_0} u_0 \\ & \quad + \partial_{y_0 y_1} u_0 + \partial_{y_1 y_0} u_0, \end{aligned}$$

3.  $\varepsilon^3$ -order:

$$\begin{aligned} & \partial_{t_0 t_0} u_2 - \partial_{x_0 x_0} u_2 - \partial_{y_0 y_0} u_2 + u_2 \\ &= -\partial_{t_0 t_1} u_1 - \partial_{t_1 t_0} u_1 - \partial_{t_1 t_1} u_0 - \partial_{t_0 t_2} u_0 \\ & \quad - \partial_{t_2 t_0} u_0 + \partial_{x_0 x_1} u_1 + \partial_{x_1 x_0} u_1 + \partial_{x_1 x_1} u_0 \\ & \quad + \partial_{x_0 x_2} u_0 + \partial_{x_2 x_0} u_0 + \partial_{y_0 y_1} u_1 + \partial_{y_1 y_0} u_1 \\ & \quad + \partial_{y_1 y_1} u_0 + \partial_{y_0 y_2} u_0 + \partial_{y_2 y_0} u_0 + u_0^3, \\ & \quad \dots \end{aligned}$$

Meanwhile, the initial conditions (B.2) and (B.3) can be rewritten as

$$\begin{aligned} & \sum_{n=0}^{\infty} \varepsilon^n u_n(x_0, x_1, \dots, y_0, y_1, \dots, t_0, t_1, \dots) \Big|_{t=0} = \phi(x_0, x_1, \dots, y_0, y_1, \dots), \\ & \partial_t \sum_{n=0}^{\infty} \varepsilon^n u_n(x_0, x_1, \dots, y_0, y_1, \dots, t_0, t_1, \dots) \Big|_{t=0} = \gamma^\varepsilon(x_0, x_1, \dots, y_0, y_1, \dots). \end{aligned}$$

Next we solve the leading-, first- and second-order equations with different initial conditions, respectively, and obtain  $u_0, u_1$  to approximate the solution of the weakly NKGE (B.1).

(1) Solving the initial value problem for the leading-order equation  $u_0$  satisfies

$$\begin{aligned} & \partial_{t_0 t_0} u_0 - \partial_{x_0 x_0} u_0 - \partial_{y_0 y_0} u_0 + u_0 = 0, \\ & u_0 \Big|_{t=0} = \phi(x_0, x_1, \dots, y_0, y_1, \dots), \\ & \partial_t u_0 \Big|_{t=0} = \gamma^\varepsilon(x_0, x_1, \dots, y_0, y_1, \dots). \end{aligned}$$

Here,  $u_0 = u_0(x_0, x_1, \dots, y_0, y_1, \dots, t_0, t_1, \dots)$ . For the sake of simplicity, we choose one single linear model with right-moving disturbance as its solution, i.e.,

$$u_0 = A_0(x_1, \dots, y_1, \dots, t_1, \dots) e^{i(k_x x_0 + k_y y_0 - \omega t_0)} + c.c.,$$

where  $k_x$  and  $k_y$  represent the wave numbers in the  $x$ - and  $y$ -direction, respectively. And the frequency  $\omega$  satisfies the following dispersion relation:

$$\omega = \omega(k_x, k_y) = \sqrt{1 + k_x^2 + k_y^2}.$$

(2) Solving the initial value problem for the first-order equation  $u_1$  satisfies

$$\begin{aligned} & \partial_{t_0 t_0} u_1 - \partial_{x_0 x_0} u_1 - \partial_{y_0 y_0} u_1 + u_1 \\ &= 2(i\omega \partial_{t_1} A_0 + ik_x \partial_{x_1} A_0 + ik_y \partial_{y_1} A_0) e^{i(k_x x_0 + k_y y_0 - \omega t_0)} + c.c., \\ & u_1|_{t=0} = 0, \\ & \partial_t u_1|_{t=0} = -\partial_{t_1} u_0|_{t=0}. \end{aligned}$$

Here,  $u_1(x_0, x_1, \dots, y_0, y_1, \dots, t_0, t_1, \dots)$ . Using idea from Remark 2.1, we demand that  $A_0$  satisfies

$$\begin{aligned} \partial_{t_1} A_0 &= -\frac{k_x}{\omega} \partial_{x_1} A_0 - \frac{k_y}{\omega} \partial_{y_1} A_0, \\ \partial_{t_1} \overline{A_0} &= -\frac{k_x}{\omega} \partial_{x_1} \overline{A_0} - \frac{k_y}{\omega} \partial_{y_1} \overline{A_0}, \end{aligned} \tag{B.5}$$

so as to make  $u_1$  bounded. Then the first-order equation can be simplified into the following 2D wave equation:

$$\partial_{t_0 t_0} u_1 - \partial_{x_0 x_0} u_1 - \partial_{y_0 y_0} u_1 + u_1 = 0.$$

And we choose

$$u_1 = 0$$

for simplicity.

(3) Solving the initial value problem for the second-order equation  $u_2$  satisfies

$$\begin{aligned} & \partial_{t_0 t_0} u_2 - \partial_{x_0 x_0} u_2 - \partial_{y_0 y_0} u_2 + u_2 \\ &= \left\{ (2i\omega \partial_{t_2} A_0 - \partial_{t_1 t_1} A_0 + 2ik_x \partial_{x_2} A_0 \right. \\ & \quad + \partial_{x_1 x_1} A_0 + 2ik_y \partial_{y_2} A_0 + \partial_{y_1 y_1} A_0 \\ & \quad + 3|A_0|^2 A_0) e^{i(k_x x_0 + k_y y_0 - \omega t_0)} \\ & \quad \left. + A_0^3 e^{3i(k_x x_0 + k_y y_0 - \omega t_0)} \right\} + c.c., \\ & u_2|_{t=0} = 0, \\ & \partial_t u_2|_{t=0} = -\partial_{t_2} u_0|_{t=0}. \end{aligned}$$

Let  $u_2 = u_2(x_0, x_1, \dots, y_0, y_1, \dots, t_0, t_1, \dots)$  bounded in a similar way (c.f. Remark 2.1), we need to remove the secular terms and postulate that,

$$\begin{aligned} & 2i\omega \partial_{t_2} A_0 - \partial_{t_1 t_1} A_0 + 2ik_x \partial_{x_2} A_0 + \partial_{x_1 x_1} A_0 \\ & \quad + 2ik_y \partial_{y_2} A_0 + \partial_{y_1 y_1} A_0 + 3|A_0|^2 A_0 = 0, \\ & -2i\omega \partial_{t_2} \overline{A_0} - \partial_{t_1 t_1} \overline{A_0} - 2ik_x \partial_{x_2} \overline{A_0} + \partial_{x_1 x_1} \overline{A_0} \\ & \quad - 2ik_y \partial_{y_2} \overline{A_0} + \partial_{y_1 y_1} \overline{A_0} + 3|A_0|^2 \overline{A_0} = 0. \end{aligned} \tag{B.6}$$

Finally, the  $\varepsilon^2$ -order equation can be simplified to

$$\partial_{t_0 t_0} u_2 - \partial_{x_0 x_0} u_2 + u_2 = A_0^3 e^{3i(k_x x_0 - \omega t_0)} + c.c.$$

From the analytical solutions of  $u_1$  and  $u_2$ , we get the leading-order approximation term of the solution  $u(\mathbf{x}, t)$ , i.e.,

$$\tilde{u}(\mathbf{x}, t) = A_0(x_1, \dots, y_1, \dots, t_1, \dots) e^{i(k_x x_0 + k_y y_0 - \omega t_0)} + c.c. \quad (\text{B.7})$$

We utilize the following assumption to define amplitude  $A_0$  in Eq. (B.7),

$$A(\mathbf{x}, t) \equiv A_0(x_1, \dots, y_1, \dots, t_0, t_1, \dots). \quad (\text{B.8})$$

Here,  $A(\mathbf{x}, t)$  is governed by the combination of Eqs. (B.5) and (B.6). Then we find that

$$i\partial_t A = -\frac{i}{\omega} \nabla \cdot \mathbf{k} A + \frac{1}{2\omega} \partial_{tt} A - \frac{1}{2\omega} \Delta A - \frac{3\varepsilon^2}{2\omega} |A|^2 A, \quad (\text{B.9})$$

where  $\nabla = (\partial_x, \partial_y)^T$ ,  $\mathbf{k} = (k_x, k_y)^T$ . By replacing  $\partial_{tt} A$  with  $(k_x^2/\omega)\partial_{xx} A + (k_y^2/\omega)\partial_{yy} A$  in (B.9) and using notation  $\tilde{\mathbf{k}} = (1 + k_x^2, 1 + k_y^2)^T$ , Eq. (B.9) can be reformulated as

$$i\partial_t A = -\frac{i}{\omega} \nabla \cdot \mathbf{k} A - \frac{1}{2\omega^3} \tilde{\mathbf{k}} \cdot \Delta A - \frac{3\varepsilon^2}{2\omega} |A|^2 A.$$

Based on the definition

$$A(\mathbf{x}, t) \equiv \tilde{A}(\boldsymbol{\xi}, \sigma), \quad \boldsymbol{\xi} = (\xi_x, \xi_y)^T = \varepsilon \left( x - \frac{k_x}{\omega} t, y - \frac{k_y}{\omega} t \right)^T, \quad \sigma = \varepsilon^2 t, \quad (\text{B.10})$$

and the same wave number in different spatial directions (i.e.,  $k = k_x = k_y$ ), we find that the small spatio-temporal modulation  $\tilde{A} = \tilde{A}(\boldsymbol{\xi}, \sigma)$  satisfies the following NLSE:

$$i\partial_\sigma \tilde{A} = \frac{1}{2} \nu_1 \tilde{\Delta} \tilde{A} + \nu_2 |\tilde{A}|^2 \tilde{A}. \quad (\text{B.11})$$

Here,  $\tilde{\Delta} = (\partial_{\xi_x \xi_x}, \partial_{\xi_y \xi_y})^T$  and coefficients  $\nu_1 = -(1 + k^2)/\omega^3 < 0$ ,  $\nu_2 = -3/(2\omega) < 0$ .

In summary, from Eqs. (B.7), (B.8) and (B.10), the leading-order approximation  $\tilde{u}(\mathbf{x}, t)$  for the solution of 2D weakly NKGE (B.1) can be constructed as

$$\tilde{u}(\mathbf{x}, t) = \tilde{A}(\boldsymbol{\xi}, \sigma) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} + c.c., \quad \tilde{A} \text{ satisfies Eq. (B.11).}$$

And the amplitude  $\tilde{A}(\boldsymbol{\xi}, \sigma)$  also exist one soliton solution, i.e.,

**Remark B.1** (Soliton Solution).

$$\tilde{A}(\boldsymbol{\xi}, \sigma) = \tilde{A}(\xi_x, \xi_y, \sigma) = \sqrt{-2\eta\nu_2^{-1}} \operatorname{sech} \left( \sqrt{-\eta\nu_1^{-1}} (\xi_x + \xi_y) \right) e^{i\eta\sigma}, \quad \eta \in \mathbb{R}^+.$$

In our numerical experiments, we construct the initial conditions based on the 2D soliton solution for Eq. (B.11) and apply the fourth-order TSSP method to solve 2D NLSE numerically. Following [26], we give a remark to state the error bound of the asymptotic preserving method in two dimension, i.e.,

**Remark B.2.** With the appropriate assumption of the wave vector  $\mathbf{k}$  and a space  $\mathcal{W}$ , let  $\tilde{A} \in C([0, T_0], \mathcal{W})$  be the solution of the NLSE (B.11). Then there exist  $\varepsilon_0 > 0$ ,  $T_1 \in (0, T_0]$ , and a constant  $C > 0$  such that for all  $\varepsilon \in (0, \varepsilon_0)$  we have solutions of Eq. (B.1) satisfying

$$\sup_{t \in [0, T_1/\varepsilon^2]} \|u(\cdot, t) - \tilde{u}(\cdot, t)\|_{\mathcal{W}} \leq C\varepsilon.$$

Of course, this is just a crude conclusion. It will be our future work to prove the above result and to provide the corresponding numerical results.

## References

- [1] W. BAO, Y. CAI, AND Y. FENG, *Improved uniform error bounds on time-splitting methods for long-time dynamics of the nonlinear Klein-Gordon equation with weak nonlinearity*, SIAM J. Numer. Anal. 60 (2022), 1962–1984.
- [2] W. BAO, Y. CAI, AND X. ZHAO, *A uniformly accurate multiscale time integrator pseudospectral method for the Klein-Gordon equation in the nonrelativistic limit regime*, SIAM J. Numer. Anal. 52 (2014), 2488–2511.
- [3] W. BAO AND X. DONG, *Analysis and comparison of numerical methods for the Klein-Gordon equation in the nonrelativistic limit regime*, Numer. Math. 120 (2012), 189–229.
- [4] W. BAO, X. DONG, AND X. ZHAO, *An exponential wave integrator pseudospectral method for the Klein-Gordon-Zakharov system*, SIAM J. Sci. Comput. 35 (2013), A2903–A2927.
- [5] W. BAO, X. DONG, AND X. ZHAO, *Uniformly accurate multiscale time integrators for highly oscillatory second order differential equations*, J. Math. Study 47 (2014), 111–150.
- [6] W. BAO, Y. FENG, AND C. SU, *Uniform error bounds of a time-splitting spectral method for the long-time dynamics of the nonlinear Klein-Gordon equation with weak nonlinearity*, Math. Comp. 91 (2022), 811–842.
- [7] W. BAO, Y. FENG, AND W. YI, *Long time error analysis of finite difference time domain methods for the nonlinear Klein-Gordon equation with weak nonlinearity*, Commun. Comput. Phys 26 (2019), 1307–1334.
- [8] W. BAO AND J. SHEN, *A fourth-order time-splitting Laguerre-Hermite pseudospectral method for Bose-Einstein condensates*, SIAM J. Sci. Comput. 26 (2005), 2010–2028.
- [9] W. BAO AND L. YANG, *Efficient and accurate numerical methods for the Klein-Gordon-Schrödinger equations*, J. Comput. Phys. 225 (2007), 1863–1893.
- [10] C. M. BENDER AND S. A. ORSZAG, *Advanced Mathematical Methods for Scientists and Engineers: Asymptotic Methods and Perturbation Theory*, Springer, 1978.
- [11] J. D. BJORKEN AND S. DRELL, *Relativistic Quantum Fields*, McGraw-Hill, 1965.
- [12] N. N. BOGOLIUBOV AND D. V. SHIRKOV, *Introduction to the Theory of Quantized Fields*, Wiley-Interscience, 1959.
- [13] P. J. CAUDREY, J. C. EILBECK, AND J. D. GIBBON, *The sine-Gordon equation as a model classical field theory*, Nuov Cim B 25 (1975), 497–511.
- [14] S. C. CHIKWENDU AND C. V. EASWARAN, *Multiple-scale solution of initial-boundary value problems for weakly nonlinear wave equations on the semi-infinite line*, SIAM J. Appl. Math. 52 (1992), 946–958.
- [15] S. C. CHIKWENDU AND J. KEVORKIAN, *A perturbation method for hyperbolic equations with small nonlinearities*, SIAM J. Appl. Math. 22 (1972), 235–258.

- [16] M. CHIRILUS-BRUCKNER, W.-P. DÜLL, AND G. SCHNEIDER, *NLS approximation of time oscillatory long waves for equations with quasilinear quadratic terms*, Math. Nachr. 288 (2015), 158–166.
- [17] C. CHONG AND G. SCHNEIDER, *Numerical evidence for the validity of the NLS approximation in systems with a quasilinear quadratic nonlinearity*, ZAMM-Z. Angew. Math. Mech. 93 (2013), 688–696.
- [18] D. COHEN, E. HAIRER, AND C. LUBICH, *Conservation of energy, momentum and actions in numerical discretizations of non-linear wave equations*, Numer. Math. 110 (2008), 113–143.
- [19] D. COHEN, E. HAIRER, AND C. LUBICH, *Long-time analysis of nonlinearly perturbed wave equations via modulated Fourier expansions*, Arch. Ration. Mech. Anal. 187 (2008), 341–368.
- [20] J.-M. DELORT, *Temps d'existence pour l'équation de Klein-Gordon semi-linéaire à données petites périodiques*, Amer. J. Math. 120 (1998), 663–689.
- [21] J.-M. DELORT, *On long time existence for small solutions of semi-linear Klein-Gordon equations on the torus*, J. Anal. Math. 107 (2009), 161–194.
- [22] J.-M. DELORT AND J. SZEFTTEL, *Long time existence for small data nonlinear Klein-Gordon equations on tori and spheres*, Int. Math. Res. Not. 2004 (2004), 1897–1966.
- [23] R. K. DODD, J. C. EILBECK, J. D. GIBBON, AND H. C. MORRIS, *Solitons and Nonlinear Wave Equations*, Academic, 1982.
- [24] W. DÖRFLER, A. LECHLEITER, M. PLUM, G. SCHNEIDER, AND C. WIENERS, *The role of the nonlinear Schrödinger equation in nonlinear optics*, in: Photonic Crystals: Mathematical Analysis and Numerical Approximation, Springer, 2011.
- [25] W.-P. DÜLL, *Justification of the nonlinear Schrödinger approximation for a quasilinear Klein-Gordon equation*, Comm. Math. Phys. 355 (2017), 1189–1207.
- [26] W.-P. DÜLL, A. HERMANN, G. SCHNEIDER, AND D. ZIMMERMANN, *Justification of the 2D NLS equation for a fourth order nonlinear wave equation-quadratic resonances do not matter much in case of analytic initial conditions*, J. Math. Anal. Appl. 436 (2016), 847–867.
- [27] S. M. EL-SAYED, *The decomposition method for studying the Klein-Gordon equation*, Chaos Solitons & Fractals 18 (2003), 1025–1030.
- [28] M. FACCIOLI AND L. SALASNICH, *Spontaneous symmetry breaking and Higgs mode: Comparing Gross-Pitaevskii and nonlinear Klein-Gordon equations*, Symmetry 10 (2018), 80.
- [29] D. FANG AND Q. ZHANG, *Long-time existence for semi-linear Klein-Gordon equations on tori*, J. Differential Equations 249 (2010), 151–179.
- [30] E. FAOU AND K. SCHRATZ, *Asymptotic preserving schemes for the Klein-Gordon equation in the nonrelativistic limit regime*, Numer. Math. 126 (2014), 441–469.
- [31] Y. FENG, *Long time error analysis of the fourth-order compact finite difference methods for the nonlinear Klein-Gordon equation with weak nonlinearity*, Numer. Methods Partial Differential Equations 37 (2021), 897–914.
- [32] Y. FENG AND W. YI, *Uniform error bounds of an exponential wave integrator for the long-time dynamics of the nonlinear Klein-Gordon equation*, Multiscale Model. Simul. 19 (2021), 1212–1235.
- [33] H. FESHBACH AND F. VILLARS, *Elementary relativistic wave mechanics of spin 0 and spin 1/2 particles*, Rev. Modern Phys. 30 (1958), 24–45.
- [34] E. HAIRER AND C. LUBICH, *Spectral semi-discretisations of weakly non-linear wave equations over long times*, Found. Comput. Math. 8 (2008), 319–334.
- [35] P. JAKOBSEN, *Introduction to the method of multiple scales*, arXiv:1312.3651.
- [36] S. JIN, *Efficient asymptotic-preserving (AP) schemes for some multiscale kinetic equations*,

- SIAM J. Sci. Comput. 21 (1999), 441–454.
- [37] M. KEEL AND T. TAO, *Small data blow-up for semilinear Klein-Gordon equation*, Amer. J. Math. 121 (1999), 629–669.
- [38] P. KIRRMANN, G. SCHNEIDER, AND A. MIELKE, *The validity of modulation equations for extended systems with cubic nonlinearities*, Proc. Roy. Soc. Edinburgh Sect. A 122 (1992), 85–91.
- [39] S. KLAINERMAN, *Global existence of small amplitude solutions to nonlinear Klein-Gordon equations in four space-time dimensions*, Comm. Pure Appl. Math. 38 (1985), 631–641.
- [40] P. KRÄMER, *The Method of Multiple Scales for Nonlinear Klein-Gordon and Schrödinger Equations*, Diploma Thesis, 2013.
- [41] V. V. KONOTOP, A. SÀNCHEZ, AND L. VÀZQUEZ, *Kink dynamics in the weakly stochastic  $\varphi^4$  model*, Phys. Rev. B 44 (1991), 2554–2566.
- [42] K. LI AND Q. ZHANG, *Existence and nonexistence of global solutions for the equation of dislocation of crystals*, J. Differential Equations 146 (1998), 5–21.
- [43] T. LI AND Y. ZHOU, *Nonlinear Klein-Gordon Equations*, Springer, 2017.
- [44] H. LINDBLAD, *On the lifespan of solutions of nonlinear wave equations with small initial data*, Comm. Pure Appl. Math. 43 (1990), 445–472.
- [45] J. A. MURDOCK, *Perturbations: Theory and Methods*, Wiley, 1991.
- [46] J. J. SAKURAI, *Advanced Quantum Mechanics*, Addison Wesley, 1967.
- [47] W. STRAUSS AND L. VAZQUEZ, *Numerical solution of a nonlinear Klein-Gordon equation*, J. Comput. Phys. 28 (1978), 271–278.
- [48] G. TODOROVA AND B. YORDANOV, *Critical exponent for a nonlinear Klein-Gordon equation with damping*, J. Differential Equations 174 (2001), 464–489.
- [49] A. H. P. VAN DER BURGH, *On the Asymptotic Validity of Perturbation Methods for Hyperbolic Differential Equations*, in: *Asymptotic Analysis: From Theory to Application*, Springer, 1979.
- [50] W. T. VAN HORSSSEN, *An asymptotic theory for a class of initial-boundary value problems for weakly nonlinear wave equations with an application to a model of the galloping oscillations of overhead transmission lines*, SIAM J. Appl. Math. 48 (1988), 1227–1243.
- [51] W. T. VAN HORSSSEN AND A. H. P. VAN DER BURGH, *On initial boundary value problems for weakly semilinear telegraph equations. Asymptotic theory and application*, SIAM J. Appl. Math. 48 (1988), 719–736.
- [52] A.-M. WAZWAZ, *The tanh and the sine-cosine methods for compact and noncompact solutions of the nonlinear Klein-Gordon equation*, Appl. Math. Comput. 167 (2005), 1179–1195.
- [53] A.-M. WAZWAZ, *New travelling wave solutions to the Boussinesq and the Klein-Gordon equations*, Commun. Nonlinear Sci. Numer. Simul. 13 (2008), 889–901.
- [54] H. YOSHIDA, *Construction of higher order symplectic integrators*, Phys. Lett. A 150 (1990), 262–268.