

# A Fast Fully Discrete Mixed Finite Element Scheme for Fractional Viscoelastic Models of Wave Propagation

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**Abstract.** Due to the nonlocal feature of fractional differential operators, the numerical solution to fractional partial differential equations usually requires expensive memory and computation costs. This paper develops a fast scheme for fractional viscoelastic models of wave propagation. We first apply the Laplace transform to convert the time-fractional constitutive equation into an integro-differential form that involves the Mittag-Leffler function as a convolution kernel. Then we construct an efficient sum-of-exponentials (SOE) approximation for the Mittag-Leffler function. We use mixed finite elements for the spatial discretization and the Newmark scheme for the temporal discretization of the second time-derivative of the displacement variable in the kinematical equation and finally obtain the fast algorithm. Compared with the traditional L1 scheme for time fractional derivative, our fast scheme reduces the memory complexity from  $\mathcal{O}(N_s N)$  to  $\mathcal{O}(N_s N_{\text{exp}})$  and the computation complexity from  $\mathcal{O}(N_s N^2)$  to  $\mathcal{O}(N_s N_{\text{exp}} N)$ , where  $N$  denotes the total number of temporal grid points,  $N_{\text{exp}}$  the number of exponentials in SOE, and  $N_s$  the complexity of memory and computation related to the spatial discretization. Numerical experiments confirm the theoretical results.

**AMS subject classifications:** 65N30, 65M60, 65M12

**Key words:** Fractional viscoelastic model, wave propagation, Mittag-Leffler function, sum-of-exponentials approximation, Caputo derivative, fast scheme.

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## 1. Introduction

Assume that  $\Omega \subset \mathbb{R}^d$  ( $d = 2, 3$ ) is a bounded open domain with boundary  $\partial\Omega$ ,  $T > 0$  is the time length, and  $\alpha \in (0, 1)$  is a constant. Consider the following fractional viscoelastic model of wave propagation which was introduced by Caputo and

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Mainardi [12]:

$$\begin{cases} \rho \mathbf{u}_{tt} - \mathbf{div} \sigma = \mathbf{f}, & (x, t) \in \Omega \times [0, T], & (1.1a) \\ \sigma + \tau_\sigma^\alpha \frac{\partial^\alpha \sigma}{\partial t^\alpha} = \mathbb{D} \left( \varepsilon(\mathbf{u}) + \tau_\varepsilon^\alpha \frac{\partial^\alpha \varepsilon(\mathbf{u})}{\partial t^\alpha} \right), & (x, t) \in \Omega \times [0, T], & (1.1b) \\ \mathbf{u} = 0, & (x, t) \in \partial\Omega \times [0, T], & (1.1c) \\ \mathbf{u}(x, 0) = \mathbf{u}_0, \quad \mathbf{u}_t(x, 0) = \mathbf{v}_0, \sigma(x, 0) = \sigma_0, & x \in \Omega. & (1.1d) \end{cases}$$

Here  $\mathbf{u} = (u_1, \dots, u_d)^T$  is the displacement field,  $\sigma = (\sigma_{ij})_{d \times d}$  the symmetric stress tensor,  $\mathbf{div} \sigma = (\sum_{i=1}^d \partial_i \sigma_{i1}, \dots, \sum_{i=1}^d \partial_i \sigma_{id})^T$ ,  $\varepsilon(\mathbf{u}) = (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)/2$  the strain tensor,  $\tau_\sigma$  the relaxation time,  $\tau_\varepsilon$  the retardation time,  $\rho(x)$  the mass density, and  $\mathbb{D}$  the fourth order symmetric tensor.  $\mathbf{f} = (f_1, \dots, f_d)$  is the body force and  $\mathbf{u}_0(x)$ ,  $\mathbf{v}_0(x)$ ,  $\sigma_0(x)$  are initial data. For any function  $\mathbf{v}(x, t)$ , denote  $\mathbf{v}_t := \partial \mathbf{v} / \partial t$  and  $\mathbf{v}_{tt} := \partial^2 \mathbf{v} / \partial t^2$ , and for  $0 < \alpha < 1$ , let  $\partial^\alpha \mathbf{v} / \partial t^\alpha$  be the  $\alpha$ -order Caputo fractional derivative of  $\mathbf{v}(x, t)$  defined by

$$\frac{\partial^\alpha \mathbf{v}}{\partial t^\alpha}(x, t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t \frac{\mathbf{v}_t(x, s)}{(t - s)^\alpha} ds.$$

We note that the following three classical viscoelastic models correspond to different choices of the relaxation/retardation time in the constitutive equation (1.1b) with  $\alpha = 1$ : the Kelvin-Voigt model ( $\tau_\sigma = 0$ ,  $\tau_\varepsilon > 0$ ); the Maxwell model ( $\tau_\sigma > 0$ ,  $\tau_\varepsilon = 0$ ) and the Zener model ( $\tau_\sigma > 0$ ,  $\tau_\varepsilon > 0$ ).

Many materials display elastic and viscous kinematic behaviours simultaneously. Such a feature, called viscoelasticity, is commonly characterized by using springs, which obey the Hooke's law, and viscous dashpots, which obey the Newton's law. Different combinations of the springs and dashpots lead to various viscoelastic models, e.g. the Zener model, the Kelvin-Voigt model and the Maxwell model. We refer the reader to [10, 15, 16, 19, 24, 26, 41, 42] for several monographs on the development and application of the viscoelasticity theory.

In recent decades, fractional order differential operators, as extension of integer order ones, have been widely used in many scientific and engineering fields such as physics, chemistry, materials science, biology, finance and other sciences, due to their ability to accurately describe states or development processes with memory and hereditary characteristics. As far as the viscoelastic materials with complex rheological properties are concerned, more and more studies indicate that, comparing with the integer order models, time fractional viscoelastic models can more precisely characterize the creep and relaxation dynamic behaviours and capture the effects of "fading" memory [5, 6, 9, 11, 22, 23, 36, 40].

There are some works in the literature on the numerical analysis of time fractional viscoelastic models. Enelund and Josefson [17] rewrote the constitutive equation of fractional Zener model (Riemann Liouville type) as an integro-differential equation with a weakly singular convolution kernel by Laplace transform and carried out finite element simulation. Based on the integro-differential form of constitutive equation from [17], Adolfsson *et al.* [1] proposed a piecewise constant discontinuous Galerkin