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# Analysis of Dirichlet-Neumann and Neumann-Dirichlet Methods for Time-Periodic Parabolic Optimal Control Problems

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**Abstract.** Dirichlet-Neumann and Neumann-Neumann methods are not only the parallel strategies in the spatial domain, but also can be used as a class of parallel methods in time. In this paper, we propose the Dirichlet-Neumann and Neumann-Dirichlet algorithms for time-periodic parabolic optimal control problems. By the Lagrange multiplier approach, a coupled system is obtained with a special time-periodic condition. For this coupled system, the Dirichlet-Neumann and Neumann-Dirichlet algorithms and their three variants are derived. We present the conver-gence analysis for all proposed algorithms. The numerical performance of the con-vergence factors is shown to illustrate our theoretical analysis. Based on our anal-ysis, there is a class of algorithms with the better convergence compared with the natural Dirichlet-Neumann algorithm. Finally, numerical experiments are provided to illustrate the theoretical results.

**AMS subject classifications**: 65N12, 65N55, 65Y05

**Key words**: Time-periodic parabolic optimal control problems, Dirichlet-Neumann algorithm, Neumann-Dirichlet algorithm, time domain decomposition, convergence analysis.

## 1. Introduction

Optimal control problems governed by the time-periodic parabolic problem are widely used in many different fields, such as Stokes problems [2], eddy current problems [3, 25], power generating kite systems [24], and so on. Discretized parabolic

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optimal control problems lead to very large systems of equations in time and space. Domain decomposition methods have been used to solve this difficulty [8, 30]. In addition, parabolic optimal control problems can be described as a forward-backward optimality system in time, like [6, 38], and for this forward-backward optimality system, the time variable plays a vital role. Therefore, it is very interesting to derive time domain decomposition methods and parallel solvers in time for such systems. Different parallel-in-time preconditioners have been developed and analyzed for this class of optimal control problems, see, e.g., [32, 39, 40] and the references therein.

Recently, various kinds of time parallel strategies have been well developed, and the development of time parallel methods has a history of more than five decades which is completely reviewed in [14]. Parareal algorithm is a classical time parallel method proposed by Lions *et al.* [33] has been analyzed or used in many different areas, like Navier-Stokes equations [12], time-periodic problems [15], and optimal control problems governed by the initial-value parabolic problems [9, 36]. And ParaOpt algorithm is a new variant of parareal for optimal control problems, which has been derived in [17]. Besides, several new algorithms parallel in time have been proposed, such as PFASST [10], MGRiT [11, 13], ParaDiag [19, 34]. Furthermore, these new parallel-in-time methods have been applied to optimal control problems by some scholars, such as PFASST [22, 23], MGRiT [1], ParaExp [28].

Various parallel strategies have been developed both in space and time directions for optimal control problems. For the elliptic optimal control problems, the domain decomposition methods and their convergence analysis were discussed by Benamou in [4, 5]. And Dirichlet-Neumann waveform relaxation (DNWR) and Neumann-Neumann waveform relaxation (NNWR) methods were constructed to solve the parabolic optimal control problems in the context [35]. With regard to time-periodic parabolic control problems, optimized Schwarz waveform relaxation (OSWR) method was studied and analyzed in [7]. The context [29] presented time-domain decomposition method for optimal control problems governed by the wave equation, and this method was extended to optimal control problems for semilinear hyperbolic systems in [26]. Time domain decomposition methods can also be applied to solve initial-value parabolic optimal control problems. For example, Schwarz method as a time parallel algorithm was proposed by Gander and Kwok [16], and optimized Schwarz method was discussed by Kwok [27]. Furthermore, the Dirichlet-Neumann/Neumann-Neumann algorithms as the time domain decomposition algorithms were presented for initial-value parabolic optimal control problems in [20, 21]. Furthermore, space-time decomposition method applied to optimal control problems was studied in [31, 41].

In this work, we propose several time domain decomposition algorithms based on the Dirichlet-Neumann (DN) and Neumann-Dirichlet (ND) methods to solve time-periodic parabolic optimal control problems. Different from [7] which introduced OSWR for time-periodic parabolic optimal control problems, we use the DN and ND methods as time domain decomposition algorithms to solve this problem. Compared with initial-value parabolic optimal control problems [20, 21], the time-periodic condition is more difficult to deal with, because the coupled system obtained by the Lagrange