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Stability and Optimal Error Estimates Analysis of an LDG Method for the Stochastic Nonlinear KdV Equation

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Abstract. To address the computational challenges of stochastic nonlinear partial differential equations with high-order derivatives, a local discontinuous Galerkin method is proposed for the stochastic KdV equation. The method is proven to be $\mathcal{L}^2$-stable and to attain optimal error estimates of order $n+1$ measured in the mean-square norm when degree-n polynomials are used. Temporal integration of the spatial semi-discrete stochastic system in the numerical experiments is carried out by using the implicit midpoint method. The simulation results verify the method's accuracy and its consistency with the theoretical analysis.

AMS subject classifications: 65M12, 65M60
Key words: Stochastic nonlinear KdV equation, local discontinuous Galerkin method, $\mathcal{L}^2$-stability, optimal error estimates.

1. Introduction

Consider solving the following stochastic nonlinear KdV equation:

$$\begin{cases} \mathrm{d}\upsilon = \big(\kappa\upsilon_{\chi\chi\chi} + \alpha\upsilon_{\chi\chi} + f(\upsilon,\upsilon_\chi)\big)\mathrm{d}\tau + \sigma(\upsilon,\upsilon_\chi)\mathrm{d}W_\tau, & (\chi,\tau)\in[0,\Gamma]\times[0,T], \\ \upsilon(\chi,0) = \upsilon_0(\chi), & \chi\in[0,\Gamma] \end{cases} \tag{1.1}$$

with the periodic boundary conditions

$$\upsilon(0,\tau) = \upsilon(\Gamma,\tau), \quad \upsilon_\chi(0,\tau) = \upsilon_\chi(\Gamma,\tau), \quad \upsilon_{\chi\chi}(0,\tau) = \upsilon_{\chi\chi}(\Gamma,\tau), \tag{1.2}$$

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where coefficients κ and α, the length Γ and time T are positive constants. W_τ represents the standard one-dimensional Wiener process. The nonlinear KdV equation serves as a fundamental model for various physical phenomena, including shallow water waves, plasma dynamics, and lattice vibrations [9, 11, 24]. To incorporate inherent uncertainties and random perturbations in realistic environments, the stochastic KdV equation (1.1) offers a more accurate and comprehensive framework for modeling, which motivates further theoretical and numerical experiments.

The discontinuous Galerkin (DG) method [19] offers an effective framework for the numerical approximation of high-order partial differential equations (PDEs) because of its inherent flexibility in h- and p-adaptivity, local conservation properties and natural suitability for parallel implementation. The DG method has been generalized to a wide range of PDEs, such as the wave equation [1], the Cahn-Hilliard equation [21] and other equations [2, 5, 18].

Based on these developments, the method has also been applied to stochastic partial differential equations (SPDEs), including the stochastic Helmholtz equations [3], the stochastic KdV equation [16] and stochastic nonlinear conservation laws [12].

The local discontinuous Galerkin (LDG) method as an extension of the classical DG method not only inherits its advantages but also introduces designed numerical fluxes and auxiliary variables, so the LDG method enables localized computations even for equations of high-order spatial derivatives [8]. Relatedly, Li *et al.* [14] proposed an ultra-weak DG method combined with an implicit-explicit time discretization for generalized stochastic KdV equations with multiplicative noise, which provided stability analysis and optimal error estimates. The classical LDG method has been applied to solve the deterministic KdV equation, where its stability and error estimates were analyzed [23]. More recently, Wang *et al.* [20] extended the LDG method for linearized KdV equations by combining implicit-explicit Runge-Kutta time discretization to analyze stability and error estimates. Similarly, Li *et al.* [13] studied the LDG method with downwind-biased numerical fluxes for linearized KdV equations, and they proved uniform stability and optimal error estimates by using generalized Gauss-Radau projections. Subsequently, the LDG method was applied to parabolic SPDEs, demonstrating both stability and optimal error estimates in the $\mathcal{L}^2$ semi-discretization [15]. Moreover, an LDG method addressing the linear stochastic Schrödinger equation driven by multiplicative noise was proposed to make rigorous analysis on its mean-square convergence and stability in the $\mathcal{L}^2$ norm [4]. In particular, Xu and Shu [22] established a unified analytical framework relying on energy-based stability analysis with the aid of auxiliary variables for deriving optimal error estimates of the LDG method and applied it to linear high-order wave equations. With the contribution of their analytical techniques and proof skills, the present work applies error analysis to the stochastic setting. For more studies on the LDG method for other types of PDEs, we refer to [7] and references therein.

This work develops an LDG method for the stochastic nonlinear KdV equation with high-order spatial derivatives and periodic boundary conditions. The numerical fluxes are adopted from existing formulations, whereas a new selection of test functions is