Numer. Math. Theor. Meth. Appl. Vol. **19**, No. 1, pp. 192-218
doi: 10.4208/nmtma.OA-2025-0080 February 2026

Stability and Optimal Error Estimates Analysis of an LDG Method for the Stochastic Nonlinear KdV Equation

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Received 12 August 2025; Accepted (in revised version) 20 November 2025

Abstract. To address the computational challenges of stochastic nonlinear partial differential equations with high-order derivatives, a local discontinuous Galerkin method is proposed for the stochastic KdV equation. The method is proven to be $\mathcal{L}^2$-stable and to attain optimal error estimates of order $n+1$ measured in the mean-square norm when degree-n polynomials are used. Temporal integration of the spatial semi-discrete stochastic system in the numerical experiments is carried out by using the implicit midpoint method. The simulation results verify the method's accuracy and its consistency with the theoretical analysis.

AMS subject classifications: 65M12, 65M60
Key words: Stochastic nonlinear KdV equation, local discontinuous Galerkin method, $\mathcal{L}^2$-stability, optimal error estimates.

1. Introduction

Consider solving the following stochastic nonlinear KdV equation:

$$\begin{cases} \mathrm{d}\upsilon = \big(\kappa\upsilon_{\chi\chi\chi} + \alpha\upsilon_{\chi\chi} + f(\upsilon,\upsilon_\chi)\big)\mathrm{d}\tau + \sigma(\upsilon,\upsilon_\chi)\mathrm{d}W_\tau, & (\chi,\tau)\in[0,\Gamma]\times[0,T], \\ \upsilon(\chi,0) = \upsilon_0(\chi), & \chi\in[0,\Gamma] \end{cases} \tag{1.1}$$

with the periodic boundary conditions

$$\upsilon(0,\tau) = \upsilon(\Gamma,\tau), \quad \upsilon_\chi(0,\tau) = \upsilon_\chi(\Gamma,\tau), \quad \upsilon_{\chi\chi}(0,\tau) = \upsilon_{\chi\chi}(\Gamma,\tau), \tag{1.2}$$

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where coefficients κ and α, the length Γ and time T are positive constants. W_τ represents the standard one-dimensional Wiener process. The nonlinear KdV equation serves as a fundamental model for various physical phenomena, including shallow water waves, plasma dynamics, and lattice vibrations [9, 11, 24]. To incorporate inherent uncertainties and random perturbations in realistic environments, the stochastic KdV equation (1.1) offers a more accurate and comprehensive framework for modeling, which motivates further theoretical and numerical experiments.

The discontinuous Galerkin (DG) method [19] offers an effective framework for the numerical approximation of high-order partial differential equations (PDEs) because of its inherent flexibility in h- and p-adaptivity, local conservation properties and natural suitability for parallel implementation. The DG method has been generalized to a wide range of PDEs, such as the wave equation [1], the Cahn-Hilliard equation [21] and other equations [2, 5, 18].

Based on these developments, the method has also been applied to stochastic partial differential equations (SPDEs), including the stochastic Helmholtz equations [3], the stochastic KdV equation [16] and stochastic nonlinear conservation laws [12].

The local discontinuous Galerkin (LDG) method as an extension of the classical DG method not only inherits its advantages but also introduces designed numerical fluxes and auxiliary variables, so the LDG method enables localized computations even for equations of high-order spatial derivatives [8]. Relatedly, Li *et al.* [14] proposed an ultra-weak DG method combined with an implicit-explicit time discretization for generalized stochastic KdV equations with multiplicative noise, which provided stability analysis and optimal error estimates. The classical LDG method has been applied to solve the deterministic KdV equation, where its stability and error estimates were analyzed [23]. More recently, Wang *et al.* [20] extended the LDG method for linearized KdV equations by combining implicit-explicit Runge-Kutta time discretization to analyze stability and error estimates. Similarly, Li *et al.* [13] studied the LDG method with downwind-biased numerical fluxes for linearized KdV equations, and they proved uniform stability and optimal error estimates by using generalized Gauss-Radau projections. Subsequently, the LDG method was applied to parabolic SPDEs, demonstrating both stability and optimal error estimates in the $\mathcal{L}^2$ semi-discretization [15]. Moreover, an LDG method addressing the linear stochastic Schrödinger equation driven by multiplicative noise was proposed to make rigorous analysis on its mean-square convergence and stability in the $\mathcal{L}^2$ norm [4]. In particular, Xu and Shu [22] established a unified analytical framework relying on energy-based stability analysis with the aid of auxiliary variables for deriving optimal error estimates of the LDG method and applied it to linear high-order wave equations. With the contribution of their analytical techniques and proof skills, the present work applies error analysis to the stochastic setting. For more studies on the LDG method for other types of PDEs, we refer to [7] and references therein.

This work develops an LDG method for the stochastic nonlinear KdV equation with high-order spatial derivatives and periodic boundary conditions. The numerical fluxes are adopted from existing formulations, whereas a new selection of test functions is designed to ensure the stability and convergence of the proposed method. Based on this framework, $\mathcal{L}^2$-stability and optimal error estimates are derived by using martingale theory and Itô's formula. These theoretical results are further validated through numerical experiments in terms of mean-square convergence.

The organization of this paper is as follows. Some preliminary notations, assumptions and concepts are provided in Section 2. In Section 3, the LDG method for solving the stochastic nonlinear KdV equation and orthogonal basis functions are introduced. The $\mathcal{L}^2$-stability of the proposed method is constructed in Section 4. Then Section 5 focuses on the optimal error estimates for LDG method according to the projection operators. Numerical experiments are presented to demonstrate the effectiveness of the proposed method, combined with time discretization in Section 6. Finally, the conclusions are presented in Section 7.

2. Preliminary

In this section, after reviewing notations, assumptions and martingale properties, some projection operator properties and lemmas which are used for proving the error estimation are introduced.

Suppose that the spatial domain is the interval $I = [0, \Gamma]$, which is partitioned into N subintervals $I_\iota = [\chi_{\iota-1/2}, \chi_{\iota+1/2}]$ for $\iota = 1, 2, \ldots, N$, with endpoints $\chi_{1/2} = 0$ and $\chi_{N+1/2} = \Gamma$. The size of each subinterval is defined as $h_\iota = \chi_{\iota+1/2} - \chi_{\iota-1/2}$, and the maximum mesh size is described as $h = \max_{1\le\iota\le N} h_\iota$. For a function υ, the left and right limits at the node $\chi_{\iota+1/2}$ are denoted as $\upsilon^-_{\iota+1/2}$ and $\upsilon^+_{\iota+1/2}$ respectively. The jump operator of υ at the interface $\chi_{\iota+1/2}$ is defined as $[\upsilon]_{\iota+1/2} = \upsilon^+_{\iota+1/2} - \upsilon^-_{\iota+1/2}$.

Then the piecewise polynomial space S_h is defined as

$$S_h = \big\{ \mathrm{v} \,|\, \mathrm{v} \in P^n(I_\iota) \ \text{ for each } \ \iota = 1, 2, \ldots, N \big\},$$

where $P^n(I_\iota)$ denotes the space of polynomials of degree at most n on the subinterval I_ι.

For simplicity, $\|\cdot\|$ and $\|\cdot\|_{H^m}$ denote the norms in $\mathcal{L}^2(I)$ and the Sobolev space $H^m(I) := H^{m,2}(I)$ respectively, where $H^m(I)$ denotes the space of all functions whose weak derivatives up to order m belong to $\mathcal{L}^2(I)$. Define $\Upsilon^2([0,T];\mathcal{L}^2)$ as the space of all adapted processes $\upsilon : [0,T] \to \mathcal{L}^2(I)$ that are strongly continuous in $\mathcal{L}^2(I)$ and satisfy $(\mathbb{E}\sup_{0\le\tau\le T}\|\upsilon(\tau)\|^2)^{1/2} < \infty$.

With the notations and concepts established, recall some fundamental tools from stochastic analysis, including Itô's formula and properties of martingales.

For continuous semimartingales X and Y, Itô's formula yields the following equation:

$$X_\tau Y_\tau = X_0 Y_0 + \int_0^\tau X_s \,\mathrm{d}Y_s + \int_0^\tau Y_s \,\mathrm{d}X_s + \langle X, Y\rangle_\tau, \quad \tau \in [0,T], \tag{2.1}$$

where $\langle X, Y\rangle_\tau = \langle Y, X\rangle_\tau$, and $\langle X, Y\rangle_\tau$ represents the quadratic covariation process. Moreover, for any locally bounded adapted process H, the following equation for stochastic integrals and covariations holds:

$$\left\langle \int_0^{\cdot} H_s \, \mathrm{d}X_s, Y \right\rangle_\tau = \int_0^\tau H_s \, \mathrm{d}\langle X, Y\rangle_s. \tag{2.2}$$

In particular, if Y is continuous and has bounded total variation, then

$$\langle X, Y\rangle_\tau = 0. \tag{2.3}$$

Lemma 2.1 ([10]). *Let H be an adapted process such that $\mathbb{E}((\int_0^T H_s^2 \, \mathrm{d}s)^{1/2}) < \infty$, then the process $\int_0^\tau H_s \, \mathrm{d}W_s$ for τ in $[0,T]$ constitutes a martingale.*

Next, assumptions on the nonlinear function and the properties of projection operators used in the subsequent error analysis are presented.

Assume that nonlinear functions f and σ in Eq. (1.1) are Lipschitz continuous and satisfy the linear growth condition. The Lipschitz continuity and the linear growth condition are described as follows:

- **Lipschitz continuity:** A nonlinear function $\psi(u,v)$ is Lipschitz continuous; that is, there exists a positive constant B_1 such that, for any $u_1, u_2, v_1, v_2 \in \mathbb{R}$,

$$|\psi(u_1,v_1) - \psi(u_2,v_2)|^2 \leq B_1 \left(|u_1-u_2|^2 + |v_1-v_2|^2\right). \tag{2.4}$$

- **Linear growth:** A function $\psi(u,v)$ satisfies a linear growth condition; that is, there exists a positive constant B_2 such that, for any $u, v \in \mathbb{R}$,

$$|\psi(u,v)|^2 \leq B_2 \left(1 + |u|^2 + |v|^2\right). \tag{2.5}$$

Standard projection operator $\mathbf{P}$, the local Gauss-Radau projection operators G and R are defined on the polynomial space S_h and are introduced as follows. These projections satisfy the following orthogonality conditions for each subinterval I_ι:

$$\begin{aligned} &\int_{I_\iota} (\mathbf{P}v - v)\,\mathrm{v}\,\mathrm{d}\chi = 0, \quad \forall \mathrm{v} \in P^n(I_\iota), \qquad (2.6)\\ &\int_{I_\iota} (\mathsf{G}v - v)\,\mathrm{v}\,\mathrm{d}\chi = 0, \quad \forall \mathrm{v} \in P^{n-1}(I_\iota) \quad \text{and} \quad \mathsf{G}v(\chi^-_{\iota-\frac{1}{2}}) = v(\chi_{\iota-\frac{1}{2}}),\\ &\int_{I_\iota} (\mathsf{R}v - v)\,\mathrm{v}\,\mathrm{d}\chi = 0, \quad \forall \mathrm{v} \in P^{n-1}(I_\iota) \quad \text{and} \quad \mathsf{R}v(\chi^+_{\iota-\frac{1}{2}}) = v(\chi_{\iota-\frac{1}{2}}). \end{aligned}$$

According to the results in [6], the projection operators satisfy the approximation property

$$\|\mathbf{P}v - v\| + \|\mathsf{G}v - v\| + \|\mathsf{R}v - v\| \leq C\|v\|_{H^{n+1}} h^{n+1}.$$

Based on the projection above, the following two lemmas provide essential tools for the subsequent error analysis.

Lemma 2.2 ([22]). *Let $u, v \in S_h$ be piecewise smooth functions satisfying the condition in Eq.* (1.2). *For $\iota = 1, 2, \ldots, N$, define the bilinear form $\hat{J}_\iota^\pm$ by*

$$\hat{J}_\iota^\pm(u,v) := -\int_{I_\iota} u v_\chi \mathrm{d}\chi + u^\pm_{\iota+\frac{1}{2}} v^-_{\iota+\frac{1}{2}} - u^\pm_{\iota-\frac{1}{2}} v^+_{\iota-\frac{1}{2}}. \tag{2.7}$$

Then the following properties hold:

$$\begin{aligned}
&\sum_{\iota=1}^N \big(\hat{J}_\iota^+(u,v) + \hat{J}_\iota^-(v,u)\big) = 0,\\
&\sum_{\iota=1}^N \big(\hat{J}_\iota^+(u,v) + \hat{J}_\iota^+(v,u)\big) = -\sum_{\iota=1}^N [u]_{\iota-\frac{1}{2}}[v]_{\iota-\frac{1}{2}},\\
&\sum_{\iota=1}^N \big(\hat{J}_\iota^-(u,v) + \hat{J}_\iota^-(v,u)\big) = \sum_{\iota=1}^N [u]_{\iota-\frac{1}{2}}[v]_{\iota-\frac{1}{2}},\\
&\sum_{\iota=1}^N \hat{J}_\iota^-(u,u) = \frac{1}{2}\sum_{\iota=1}^N [u]^2_{\iota-\frac{1}{2}},\\
&\sum_{\iota=1}^N \hat{J}_\iota^+(u,u) = -\frac{1}{2}\sum_{\iota=1}^N [u]^2_{\iota-\frac{1}{2}}.
\end{aligned}$$

Lemma 2.3 ([22]). *For all $u \in S_h$, functions $\hat{J}_\iota^+$ and $\hat{J}_\iota^-$ in* (2.7) *about local Gauss-Radau projections* G *and* R *satisfy the following orthogonality conditions:*

$$\hat{J}_\iota^-(v - \mathsf{G}v, u) = 0, \quad \hat{J}_\iota^+(v - \mathsf{R}v, u) = 0.$$

3. LDG method

In this section, a general framework for the LDG method and the definition of numerical fluxes which are used in the spatial discretization are presented.

The LDG method is applied to the stochastic nonlinear KdV equation by introducing auxiliary variables q and p to rewrite Eq. (1.1) as the first-order system

$$\begin{cases} q = v_\chi,\\ p = q_\chi,\\ \mathrm{d}v = \big(\kappa p_\chi + \alpha q_\chi + f(v,q)\big)\mathrm{d}\tau + \sigma(v,q)\mathrm{d}W_\tau, \end{cases} \tag{3.1}$$

subject to the initial condition

$$v(\chi, 0) = v_0(\chi), \quad \chi \in [0, \Gamma].$$

To construct the LDG method, numerical fluxes are required at the cell interfaces to ensure consistency and stability. We define the alternating numerical fluxes as

$$\hat{v}_h = v_h^+, \quad \hat{q}_h = q_h^-, \quad \hat{p}_h = p_h^-, \tag{3.2}$$

where υ_h^+ denotes the right limit of υ_h at the interface, and q_h^-, p_h^- denote the left limits of q_h and p_h, respectively. The alternating fluxes simplify both stability and error analysis, as they naturally eliminate certain interface contributions in the subsequent derivations.

To derive the discrete formulation, system (3.1) is multiplied by test functions z_h, w_h and $\mathrm{v}_h \in S_h$, and is integrated over each subinterval I_ι for $\iota = 1, 2, \ldots, N$. The numerical solutions q_h, p_h, and υ_h which are taken to be piecewise polynomials in the space S_h are designed to approximate the corresponding exact solutions q, p, and υ. Consequently, by combining Eq. (2.7), the following weak formulation is derived:

$$\begin{cases} \displaystyle\int_0^\Gamma q_h z_h \mathrm{d}\chi = \sum_{\iota=1}^N \hat{J}_\iota^+(\upsilon_h, z_h), & \text{(3.3a)} \\ \displaystyle\int_0^\Gamma p_h w_h \mathrm{d}\chi = \sum_{\iota=1}^N \hat{J}_\iota^-(q_h, w_h), & \text{(3.3b)} \\ \displaystyle\int_0^\Gamma \mathrm{v}_h \mathrm{d}\upsilon_h \mathrm{d}\chi = \kappa \sum_{\iota=1}^N \hat{J}_\iota^-(p_h, \mathrm{v}_h)\mathrm{d}\tau + \alpha \sum_{\iota=1}^N \hat{J}_\iota^-(q_h, \mathrm{v}_h)\mathrm{d}\tau & \\ \displaystyle\qquad + \int_0^\Gamma f(\upsilon_h, q_h)\mathrm{v}_h \mathrm{d}\chi \mathrm{d}\tau + \int_0^\Gamma \sigma(\upsilon_h, q_h)\mathrm{v}_h \mathrm{d}\chi \mathrm{d}W_\tau. & \text{(3.3c)} \end{cases}$$

The basis functions $\{\phi_k^\iota\}_{k=0}^n$ are chosen from the polynomial space $P^n(I_\iota)$, and the numerical solutions are represented as

$$\begin{aligned} \upsilon_h &= \sum_{\iota=1}^N \sum_{k=0}^n \upsilon_{k,\iota}(\tau)\phi_k^\iota(\chi), \\ q_h &= \sum_{\iota=1}^N \sum_{k=0}^n q_{k,\iota}(\tau)\phi_k^\iota(\chi), \\ p_h &= \sum_{\iota=1}^N \sum_{k=0}^n p_{k,\iota}(\tau)\phi_k^\iota(\chi). \end{aligned} \tag{3.4}$$

4. Stability property

In this section, after presenting a covariation formula for stochastic integrals, we analyze the stability of the proposed LDG method.

Lemma 4.1. *For a continuous semimartingale Y and any $\mathrm{v}_h \in S_h$, the numerical solution υ_h satisfies*

$$\int_0^\Gamma \mathrm{v}_h \langle \upsilon_h, Y \rangle_\tau \mathrm{d}\chi = \left\langle \int_0^\cdot \int_0^\Gamma \sigma(\upsilon_h, q_h)\, \mathrm{v}_h \mathrm{d}\chi \mathrm{d}W_s, Y \right\rangle_\tau.$$

Proof. By integrating Eq. (3.3c) over the time interval $[0,\tau]$, we obtain

$$\int_0^\Gamma \mathrm{v}_h(\chi)\upsilon_h(\chi,\tau)\mathrm{d}\chi = H + \int_0^\tau\int_0^\Gamma \sigma(\upsilon_h,q_h)\mathrm{v}_h(\chi)\mathrm{d}\chi\mathrm{d}W_s,$$

where H is denoted as follows and belongs to the space of continuous functions with bounded total variation:

$$\begin{aligned} H := &\int_0^\Gamma \mathrm{v}_h(\chi)\upsilon_h(\chi,0)\mathrm{d}\chi + \kappa\int_0^\tau \sum_{\iota=1}^N \hat{J}_\iota^-\big(p_h,\mathrm{v}_h(\chi)\big)\mathrm{d}s \\ &+ \alpha\int_0^\tau \sum_{\iota=1}^N \hat{J}_\iota^-\big(q_h,\mathrm{v}_h(\chi)\big)\mathrm{d}s + \int_0^\tau\int_0^\Gamma f(\upsilon_h,q_h)\mathrm{v}_h(\chi)\mathrm{d}\chi\mathrm{d}s. \end{aligned}$$

Since Y is a continuous semimartingale and Eq. (2.3) holds, it follows that

$$\begin{aligned} \int_0^\Gamma \mathrm{v}_h\langle\upsilon_h,Y\rangle_\tau\mathrm{d}\chi &= \left\langle\int_0^\Gamma \mathrm{v}_h\upsilon_h\mathrm{d}\chi, Y\right\rangle_\tau \\ &= \left\langle\int_0^\cdot\int_0^\Gamma \sigma(\upsilon_h,q_h)\mathrm{v}_h\mathrm{d}\chi\mathrm{d}W_s, Y\right\rangle_\tau. \end{aligned} \tag{4.1}$$

The proof is complete. □

The following $\mathcal{L}^2$-stability theorem for the proposed LDG method is established based on the results of Lemma 4.1.

Theorem 4.1. *Assume that the initial condition satisfies $\upsilon_0 \in \mathcal{L}^2$, the nonlinearities $f(\upsilon_h,q_h)$ and $\sigma(\upsilon_h,q_h)$ satisfy the linear growth condition* (2.5) *and the coefficient α satisfies the condition $\alpha \ge B_2$, the system* (3.3) *holds*

$$\sup_{0\le\tau\le T}\mathbb{E}\left(\|\upsilon_h(\cdot,\tau)\|^2\right) \le C\left(1+\|\upsilon_h(\cdot,0)\|^2\right).$$

Proof. To facilitate stability analysis and eliminate interface contributions, the test functions $z_h = \kappa p_h$, $w_h = -\alpha\upsilon_h$ and $\mathrm{v}_h = \upsilon_h$ are chosen in the weak formulations (3.3), with the numerical fluxes defined in (3.2), which yields

$$\left\{\begin{aligned} &\kappa\int_0^\Gamma q_h p_h\mathrm{d}\chi = \kappa\sum_{\iota=1}^N \hat{J}_\iota^+(\upsilon_h,p_h), && \text{(4.2a)} \\ &-\alpha\int_0^\Gamma p_h\upsilon_h\mathrm{d}\chi = -\alpha\sum_{\iota=1}^N \hat{J}_\iota^-(q_h,\upsilon_h), && \text{(4.2b)} \\ &\int_0^\Gamma \upsilon_h\mathrm{d}\upsilon_h\mathrm{d}\chi = \kappa\sum_{\iota=1}^N \hat{J}_\iota^-(p_h,\upsilon_h)\mathrm{d}\tau + \alpha\sum_{\iota=1}^N \hat{J}_\iota^-(q_h,\upsilon_h)\mathrm{d}\tau \\ &\qquad + \int_0^\Gamma f(\upsilon_h,q_h)\upsilon_h\mathrm{d}\chi\mathrm{d}\tau + \int_0^\Gamma \sigma(\upsilon_h,q_h)\upsilon_h\mathrm{d}\chi\mathrm{d}W_\tau. && \text{(4.2c)} \end{aligned}\right.$$

After multiplying Eqs. (4.2a) and (4.2b) by $\mathrm{d}\tau$ and summing them, and using the properties of Lemma 2.2, the following equation is obtained:

$$\begin{aligned}\int_0^\Gamma \upsilon_h \mathrm{d}\upsilon_h \mathrm{d}\chi = & -\kappa\int_0^\Gamma q_h p_h \mathrm{d}\chi\mathrm{d}\tau + \alpha\int_0^\Gamma p_h\upsilon_h\mathrm{d}\chi\mathrm{d}\tau \\ & + \int_0^\Gamma f(\upsilon_h, q_h)\upsilon_h\mathrm{d}\chi\mathrm{d}\tau + \int_0^\Gamma \sigma(\upsilon_h,q_h)\upsilon_h\mathrm{d}\chi\mathrm{d}W_\tau. \end{aligned} \tag{4.3}$$

According to the Itô's formula (2.1), the integral equation of υ_h is expressed as

$$\int_0^\Gamma \upsilon_h^2(\chi,\tau)\mathrm{d}\chi = \int_0^\Gamma \upsilon_h^2(\chi,0)\mathrm{d}\chi + 2\int_0^\Gamma\int_0^\tau \upsilon_h\mathrm{d}\upsilon_h\mathrm{d}\chi + \int_0^\Gamma \langle\upsilon_h,\upsilon_h\rangle_\tau\mathrm{d}\chi. \tag{4.4}$$

After integrating (4.3) over time from $[0,\tau]$ and substituting the result into (4.4), the expectation is taken to yield

$$\begin{aligned}\mathbb{E}\left(\|\upsilon_h(\cdot,\tau)\|^2\right) = & \|\upsilon_h(\cdot,0)\|^2 - 2\kappa\mathbb{E}\left(\int_0^\tau\int_0^\Gamma p_h q_h\mathrm{d}\chi\mathrm{d}s\right) \\ & + 2\alpha\mathbb{E}\left(\int_0^\tau\int_0^\Gamma p_h\upsilon_h\mathrm{d}\chi\mathrm{d}s\right) + 2\mathbb{E}\left(\int_0^\tau\int_0^\Gamma f(\upsilon_h,q_h)\upsilon_h\mathrm{d}\chi\mathrm{d}s\right) \\ & + 2\mathbb{E}\left(\int_0^\tau\int_0^\Gamma \sigma(\upsilon_h,q_h)\upsilon_h\mathrm{d}\chi\mathrm{d}W_s\right) + \mathbb{E}\left(\int_0^\Gamma\langle\upsilon_h,\upsilon_h\rangle_\tau\mathrm{d}\chi\right). \end{aligned}\tag{4.5}$$

According to the periodic boundary conditions, the second term in the right-hand of (4.5) transforms into

$$\mathbb{E}\left(\int_0^\tau\int_0^\Gamma q_h q_{h\chi}\mathrm{d}\chi\mathrm{d}s\right) = 0.$$

Using integration by parts, the third term on the right-hand side of (4.5) becomes

$$\mathbb{E}\left(\int_0^\tau\int_0^\Gamma p_h\upsilon_h\mathrm{d}\chi\mathrm{d}s\right) = -\mathbb{E}\left(\int_0^\tau\int_0^\Gamma q_h^2\mathrm{d}\chi\mathrm{d}s\right).$$

Therefore, Eq. (4.5) follows that

$$\begin{aligned}\mathbb{E}\left(\|\upsilon_h(\cdot,\tau)\|^2\right) = & \|\upsilon_h(\cdot,0)\|^2 - 2\alpha\mathbb{E}\left(\int_0^\tau\int_0^\Gamma q_h^2\mathrm{d}\chi\mathrm{d}s\right) \\ & + \mathcal{T}_1(\tau) + \mathcal{T}_2(\tau) + \mathcal{T}_3(\tau),\end{aligned}\tag{4.6}$$

where

$$\begin{aligned}\mathcal{T}_1(\tau) &= 2\mathbb{E}\left(\int_0^\tau\int_0^\Gamma f(\upsilon_h,q_h)\upsilon_h\mathrm{d}\chi\mathrm{d}s\right), \\ \mathcal{T}_2(\tau) &= 2\mathbb{E}\left(\int_0^\tau\int_0^\Gamma \sigma(\upsilon_h,q_h)\upsilon_h\mathrm{d}\chi\mathrm{d}W_s\right), \\ \mathcal{T}_3(\tau) &= \mathbb{E}\left(\int_0^\Gamma\langle\upsilon_h,\upsilon_h\rangle_\tau\mathrm{d}\chi\right).\end{aligned}$$

The terms $\mathcal{T}_1(\tau)$, $\mathcal{T}_2(\tau)$, and $\mathcal{T}_3(\tau)$ are estimated separately as follows:

- **The estimation of $\mathcal{T}_1(\tau)$.** By applying the basic inequality and Eq. (2.5), the estimate of $\mathcal{T}_1(\tau)$ holds

$$\begin{aligned}\mathcal{T}_1(\tau) &\leq \mathbb{E}\left(\int_0^\tau\int_0^\Gamma f^2(\upsilon_h,q_h)\mathrm{d}\chi\mathrm{d}s\right)+\mathbb{E}\left(\int_0^\tau\int_0^\Gamma \upsilon_h^2\mathrm{d}\chi\mathrm{d}s\right)\\&\leq (B_2+1)\mathbb{E}\left(\int_0^\tau\|\upsilon_h(\cdot,s)\|^2\mathrm{d}s\right)\\&\quad+B_2\mathbb{E}\left(\int_0^\tau\|q_h(\cdot,s)\|^2\mathrm{d}s\right)+B_2\Gamma T,\quad \forall\tau\in[0,T].\end{aligned}$$

- **The estimation of $\mathcal{T}_2(\tau)$.** Based on the Cauchy-Schwarz inequality, the following estimate is obtained as

$$\begin{aligned}&\mathbb{E}\left(\left(\int_0^\tau\left(\int_0^\Gamma\sigma(\upsilon_h,q_h)\upsilon_h\mathrm{d}\chi\right)^2\mathrm{d}s\right)^{\frac{1}{2}}\right)\\&\leq\left(\mathbb{E}\left(\int_0^\tau\int_0^\Gamma B_2\left(1+|\upsilon_h|^2+|q_h|^2\right)\mathrm{d}\chi\mathrm{d}s\right)\right)^{\frac{1}{2}}\left(\mathbb{E}\left(\int_0^\tau\|\upsilon_h(\cdot,s)\|^2\mathrm{d}s\right)\right)^{\frac{1}{2}}<\infty.\end{aligned}$$

Since $\int_0^\tau\int_0^\Gamma\sigma(\upsilon_h,q_h)\upsilon_h\mathrm{d}\chi\mathrm{d}W_s$ is a martingale as defined in Lemma 2.1, it follows that $\mathcal{T}_2(\tau)=0$.

- **The estimation of $\mathcal{T}_3(\tau)$.** In the view of Lemma 4.1, it holds that

$$\begin{aligned}\int_0^\Gamma\langle\upsilon_h,\upsilon_h\rangle_\tau\mathrm{d}\chi&=\sum_{\iota=1}^N\int_{I_\iota}\left\langle\upsilon_h,\sum_{k=0}^n\upsilon_{k,\iota}(s)\phi_k^\iota(\chi)\right\rangle_\tau\mathrm{d}\chi\\&=\sum_{\iota=1}^N\sum_{k=0}^n\int_{I_\iota}\phi_k^\iota(\chi)\langle\upsilon_h,\upsilon_{k,\iota}(s)\rangle_\tau\mathrm{d}\chi\\&=\sum_{\iota=1}^N\sum_{k=0}^n\left\langle\int_0^\cdot\int_{I_\iota}\sigma(\upsilon_h,q_h)\phi_k^\iota(\chi)\mathrm{d}\chi\mathrm{d}W_s,\upsilon_{k,\iota}(s)\right\rangle_\tau,\end{aligned}\tag{4.7}$$

where $\{\phi_k^\iota\}_{k=0}^n$ are the basis functions in (3.4). According to (2.2) and (2.6), Eq. (4.7) leads to

$$\begin{aligned}\int_0^\Gamma\langle\upsilon_h,\upsilon_h\rangle_\tau\mathrm{d}\chi&=\sum_{\iota=1}^N\sum_{k=0}^n\int_0^\tau\int_{I_\iota}\sigma(\upsilon_h,q_h)\phi_k^\iota(\chi)\mathrm{d}\chi\mathrm{d}\langle W,\upsilon_{k,\iota}(\cdot)\rangle_s\\&=\sum_{\iota=1}^N\int_{I_\iota}\int_0^\tau\sum_{k=0}^n\mathbf{P}\sigma(\upsilon_h,q_h)\phi_k^\iota(\chi)\mathrm{d}\langle W,\upsilon_{k,\iota}(\cdot)\rangle_s\mathrm{d}\chi\\&=\sum_{\iota=1}^N\int_{I_\iota}\int_0^\tau\mathbf{P}\sigma(\upsilon_h,q_h)\mathrm{d}\left\langle W,\sum_{k=0}^n\upsilon_{k,\iota}(\cdot)\phi_k^\iota(\chi)\right\rangle_s\mathrm{d}\chi\end{aligned}$$

$$= \int_0^\Gamma \left\langle \int_0^\cdot \mathbf{P}\sigma(\upsilon_h, q_h)\mathrm{d}W_s, \upsilon_h \right\rangle_\tau \mathrm{d}\chi. \tag{4.8}$$

Because $\mathbf{P}\sigma(\upsilon_h, q_h) \in S_h$, for any $s \in [0, T]$, it holds that

$$\mathbf{P}\sigma(\upsilon_h, q_h) = \sum_{\iota=1}^{N}\sum_{k=0}^{n} \sigma_{k,\iota}(s)\phi_k^\iota(\chi).$$

By applying Eq. (2.2) twice and using Eq. (4.1), Eq. (4.8) derives

$$\begin{aligned}
\int_{I_\iota} \langle \upsilon_h, \upsilon_h\rangle_\tau \mathrm{d}\chi &= \int_{I_\iota} \left\langle \int_0^\cdot \sum_{k=0}^{n} \sigma_{k,\iota}(s)\phi_k^\iota(\chi)\mathrm{d}W_s, \upsilon_h \right\rangle_\tau \mathrm{d}\chi \\
&= \sum_{k=0}^{n} \int_{I_\iota} \phi_k^\iota(\chi) \left\langle \upsilon_h, \int_0^\cdot \sigma_{k,\iota}(s)\mathrm{d}W_s \right\rangle_\tau \mathrm{d}\chi \\
&= \sum_{k=0}^{n} \left\langle \int_0^\cdot \int_{I_\iota} \sigma(\upsilon_h, q_h)\phi_k^\iota(\chi)\mathrm{d}\chi\mathrm{d}W_s, \int_0^\cdot \sigma_{k,\iota}(s)\mathrm{d}W_s \right\rangle_\tau \\
&= \sum_{k=0}^{n} \int_0^\tau \int_{I_\iota} \sigma(\upsilon_h, q_h)\phi_k^\iota(\chi)\mathrm{d}\chi\sigma_{k,\iota}(s)\mathrm{d}\langle W, W\rangle_s \\
&= \int_0^\tau \int_{I_\iota} \sigma(\upsilon_h, q_h) \sum_{k=0}^{n} \sigma_{k,\iota}(s)\phi_k^\iota(\chi)\mathrm{d}\chi\mathrm{d}s \\
&= \int_0^\tau \int_{I_\iota} \sigma(\upsilon_h, q_h)\mathbf{P}\sigma(\upsilon_h, q_h)\mathrm{d}\chi\mathrm{d}s.
\end{aligned} \tag{4.9}$$

Eq. (4.9) is obtained by applying the Cauchy-Schwarz inequality and summing over ι from 1 to N as follows:

$$\int_0^\Gamma \langle \upsilon_h, \upsilon_h\rangle_\tau \mathrm{d}\chi \le \int_0^\tau \int_0^\Gamma \sigma^2(\upsilon_h, q_h)\mathrm{d}\chi\mathrm{d}s.$$

After taking expectations, the estimation of $\mathcal{T}_3(\tau)$ is

$$\begin{aligned}
\mathcal{T}_3(\tau) &\le \mathbb{E}\left(\int_0^\tau \int_0^\Gamma \sigma^2(\upsilon_h, q_h)\mathrm{d}\chi\mathrm{d}s \right) \\
&\le B_2\mathbb{E}\int_0^\tau \|\upsilon_h(\cdot, s)\|^2\mathrm{d}s + B_2\mathbb{E}\int_0^\tau \|q_h(\cdot, s)\|^2\mathrm{d}s + \Gamma B_2\tau.
\end{aligned}$$

Since the coefficient satisfies the condition $\alpha \ge B_2$, Eq. (4.6) holds

$$\begin{aligned}
\mathbb{E}\left(\|\upsilon_h(\cdot, \tau)\|^2\right) \le\ & \|\upsilon_h(\cdot, 0)\|^2 + (2B_2 - 2\alpha)\mathbb{E}\int_0^\tau \|q_h(\cdot, s)\|^2\mathrm{d}s \\
&+ (1 + 2B_2)\mathbb{E}\int_0^\tau \|\upsilon_h(\cdot, s)\|^2\mathrm{d}s + \Gamma B_2 T \\
\le\ & \|\upsilon_h(\cdot, 0)\|^2 + C\int_0^\tau \mathbb{E}\left(\|\upsilon_h(\cdot, s)\|^2\right)\mathrm{d}s + C, \quad \forall \tau \in [0, T].
\end{aligned}$$

Using Gronwall's inequality, it is expressed as

$$\sup_{0\le\tau\le T}\mathbb{E}\left(\|\upsilon_h(\cdot,\tau)\|^2\right)\le C\left(1+\|\upsilon_h(\cdot,0)\|^2\right),$$

which proofs the stability of the LDG method. □

5. Error estimation

This section derives the optimal error estimates of the LDG method based on supplementary error ana lysis results concerning υ_h, p_h, q_h, and $\upsilon_{h\tau}$. According to (3.1), υ, p and q are taken as the exact solutions, while υ_h, p_h and q_h denote the corresponding spatially semi-discrete numerical solutions. And the initial condition for $\upsilon_h(\chi,0)$ is assumed to satisfy

$$\mathsf{G}p(\chi,0)=p_h(\chi,0). \tag{5.1}$$

An equivalent formulation of system (3.1) is expressed as

$$\begin{cases} q=\upsilon_\chi,\\ p=q_\chi,\\ \upsilon_\tau=\big(\kappa p_\chi+\alpha q_\chi+f(\upsilon,q)\big)+\sigma(\upsilon,q)\dfrac{\mathrm{d}W_\tau}{\mathrm{d}\tau}. \end{cases} \tag{5.2}$$

The corresponding weak formulation of system (5.2) is described as

$$\begin{cases} \displaystyle\int_0^\Gamma q_h z_h \mathrm{d}\chi=\sum_{\iota=1}^N \hat{J}_\iota^+(\upsilon_h,z_h), & \text{(5.3a)}\\ \displaystyle\int_0^\Gamma p_h w_h \mathrm{d}\chi=\sum_{\iota=1}^N \hat{J}_\iota^-(q_h,w_h), & \text{(5.3b)}\\ \displaystyle\int_0^\Gamma \mathrm{v}_h\upsilon_{h\tau}\mathrm{d}\chi=\kappa\sum_{\iota=1}^N \hat{J}_\iota^-(p_h,\mathrm{v}_h)+\alpha\sum_{\iota=1}^N \hat{J}_\iota^-(q_h,\mathrm{v}_h) & \\ \displaystyle\qquad\qquad +\int_0^\Gamma f(\upsilon_h,q_h)\mathrm{v}_h\mathrm{d}\chi+\int_0^\Gamma \sigma(\upsilon_h,q_h)\mathrm{v}_h\mathrm{d}\chi\frac{\mathrm{d}W_\tau}{\mathrm{d}\tau}. & \text{(5.3c)} \end{cases}$$

By using the projection operators and the numerical fluxes, the total error $e_\upsilon=\upsilon-\upsilon_h$ is decomposed into an approximation error and a discrete error. Define

$$e_\upsilon:=\delta_\upsilon-\theta_\upsilon,\quad \delta_\upsilon:=\mathsf{R}\upsilon-\upsilon_h,\quad \theta_\upsilon:=\mathsf{R}\upsilon-\upsilon,$$

and replace R with G, the definition of e_p and e_q can be given similarly. Throughout this section, $\mathcal{L}^2([0,T];H^m)$ denotes the standard Bochner space, while $\varUpsilon^2([0,T];\mathcal{L}^2)$ refers to the stochastic process space defined in Section 2, which guarantees strong continuity in $\mathcal{L}^2(I)$ and square-integrability in expectation. A detailed and significant result on optimal error estimates is presented in the following theorem.

Theorem 5.1. *Suppose that the exact solution υ of Eq.* (1.1) *with periodic boundary conditions* (1.2) *is sufficiently smooth and satisfies $\upsilon \in \mathcal{L}^2([0,T];H^{n+4}) \cap \Upsilon^2([0,T];\mathcal{L}^2) \cap \mathcal{L}^\infty(0,T;H^{n+1})$ with initial condition $\upsilon_0 \in H^{n+1}$, and assumption* (5.1) *holds. Moreover, the nonlinearities $f(\upsilon,q)$ and $\sigma(\upsilon,q)$ satisfy the Lipschitz condition* (2.4)*. For sufficiently small $\epsilon > 0$, if the parameters κ and α satisfy $\kappa > \max\{\alpha, 2\alpha^2 + 4\epsilon^2\}$ and $\alpha \geq \max\{B_1, (2+\kappa+3\alpha/2+B_1)\epsilon^2\}$, the numerical solution obtained from the LDG method* (5.3) *and* (3.2) *satisfies the following optimal error estimate:*

$$\mathbb{E}\|e_\upsilon(\cdot,\tau)\|^2 + \Big(\frac{\kappa}{2} - \alpha^2 - 2\epsilon^2\Big)\mathbb{E}\|e_p(\cdot,\tau)\|^2 + \mathbb{E}\|\partial_\tau e_\upsilon(\cdot,\tau)\|^2 + \frac{1}{4}\mathbb{E}\|e_q(\cdot,\tau)\|^2 \leq Ch^{2n+2}.$$

Proof. By replacing the numerical solutions q_h, p_h and υ_h in system (5.3) with the exact solutions q, p and υ, and applying the numerical fluxes defined in (3.2), the following error equations are obtained:

$$\left\{\begin{aligned} &\int_0^\Gamma (q-q_h)z_h \mathrm{d}\chi = \sum_{\iota=1}^N \hat{J}_\iota^+(\upsilon-\upsilon_h, z_h), && (5.4a)\\ &\int_0^\Gamma (p-p_h)w_h \mathrm{d}\chi = \sum_{\iota=1}^N \hat{J}_\iota^-(q-q_h, w_h), && (5.4b)\\ &\int_0^\Gamma (\upsilon_\tau - \upsilon_{h\tau})\mathrm{v}_h \mathrm{d}\chi \\ &= \kappa\sum_{\iota=1}^N \hat{J}_\iota^-(p-p_h,\mathrm{v}_h) + \alpha\sum_{\iota=1}^N \hat{J}_\iota^-(q-q_h,\mathrm{v}_h) \\ &\quad + \int_0^\Gamma \big(f(\upsilon,q) - f(\upsilon_h,q_h)\big)\mathrm{v}_h\mathrm{d}\chi \\ &\quad + \int_0^\Gamma \big(\sigma(\upsilon,q)-\sigma(\upsilon_h,q_h)\big)\mathrm{v}_h \mathrm{d}\chi\frac{\mathrm{d}W_\tau}{\mathrm{d}\tau}. && (5.4c)\end{aligned}\right.$$

Firstly, an estimate of expected squared norm $\mathbb{E}\|\delta_\upsilon(\cdot,\tau)\|^2$ is introduced. The test functions $z_h = \kappa\delta_p + \alpha\delta_q$, $w_h = -\kappa\delta_q$ and $\mathrm{v}_h = \delta_\upsilon$ are given, and system (5.4) simplifies as

$$\left\{\begin{aligned} &\int_0^\Gamma (\delta_q - \theta_q)(\kappa\delta_p + \alpha\delta_q)\mathrm{d}\chi = \sum_{\iota=1}^N \hat{J}_\iota^+(\delta_\upsilon - \theta_\upsilon, \kappa\delta_p + \alpha\delta_q), \\ &-\kappa\int_0^\Gamma (\delta_p - \theta_p)\delta_q \mathrm{d}\chi = -\kappa\sum_{\iota=1}^N \hat{J}_\iota^-(\delta_q - \theta_q, \delta_q), \\ &\int_0^\Gamma \delta_\upsilon(\partial_\tau\delta_\upsilon - \partial_\tau\theta_\upsilon)\mathrm{d}\chi \\ &= \kappa\sum_{\iota=1}^N \hat{J}_\iota^-(\delta_p - \theta_p, \delta_\upsilon) + \alpha\sum_{\iota=1}^N \hat{J}_\iota^-(\delta_q-\theta_q,\delta_\upsilon) \\ &\quad + \int_0^\Gamma \big(f(\upsilon,q)-f(\upsilon_h,q_h)\big)\delta_\upsilon\mathrm{d}\chi + \int_0^\Gamma\big(\sigma(\upsilon,q)-\sigma(\upsilon_h,q_h)\big)\delta_\upsilon\mathrm{d}\chi\frac{\mathrm{d}W_\tau}{\mathrm{d}\tau}.\end{aligned}\right. \qquad (5.5)$$

Through multiplying Eqs. (5.5) by $\mathrm{d}\tau$, summing them and applying Lemma 2.2 as well as Lemma 2.3, the following estimate is obtained:

$$\begin{aligned}
&\alpha\|\delta_q(\cdot,\tau)\|^2\mathrm{d}\tau-\kappa\int_0^\Gamma\theta_q\delta_p\mathrm{d}\chi\mathrm{d}\tau-\alpha\int_0^\Gamma\theta_q\delta_q\mathrm{d}\chi\mathrm{d}\tau\\
&\quad+\kappa\int_0^\Gamma\theta_p\delta_q\mathrm{d}\chi\mathrm{d}\tau+\int_0^\Gamma\delta_\upsilon\mathrm{d}\delta_\upsilon\mathrm{d}\chi-\int_0^\Gamma\delta_\upsilon\partial_\tau\theta_\upsilon\mathrm{d}\chi\mathrm{d}\tau\\
&=-\frac{\kappa}{2}\sum_{\iota=1}^N[\delta_q]^2_{\iota-\frac{1}{2}}\mathrm{d}\tau+\int_0^\Gamma\big(f(\upsilon,q)-f(\upsilon_h,q_h)\big)\delta_\upsilon\mathrm{d}\chi\mathrm{d}\tau\\
&\quad+\int_0^\Gamma\big(\sigma(\upsilon,q)-\sigma(\upsilon_h,q_h)\big)\delta_\upsilon\mathrm{d}\chi\mathrm{d}W_\tau.
\end{aligned}\tag{5.6}$$

According to (2.1), the Itô's formula of δ_υ is computed as

$$\|\delta_\upsilon(\cdot,\tau)\|^2=\|\delta_\upsilon(\cdot,0)\|^2+2\int_0^\Gamma\int_0^\tau\delta_\upsilon\mathrm{d}\delta_\upsilon\mathrm{d}\chi+\int_0^\Gamma\langle\delta_\upsilon,\delta_\upsilon\rangle_\tau\mathrm{d}\chi,\tag{5.7}$$

where

$$\begin{aligned}
\int_0^\Gamma\langle\delta_\upsilon,\delta_\upsilon\rangle_\tau\mathrm{d}\chi&=\int_0^\Gamma\int_0^\tau\big(\mathsf{R}\sigma(\upsilon,q)-\sigma(\upsilon_h,q_h)\big)^2\mathrm{d}s\mathrm{d}\chi\\
&\le\left(1+\frac{1}{\eta}\right)\int_0^\tau(\mathsf{R}-I)\|\sigma(\upsilon,q)\|^2\mathrm{d}s\\
&\quad+(1+\eta)\int_0^\tau\|\sigma(\upsilon,q)-\sigma(\upsilon_h,q_h)\|^2\mathrm{d}s\\
&\le Ch^{2n+2}+C\int_0^\tau\|\delta_\upsilon(\cdot,s)\|^2\mathrm{d}s\\
&\quad+C\int_0^\tau\|\delta_q(\cdot,s)\|^2\mathrm{d}s,\quad\forall\eta>0.
\end{aligned}\tag{5.8}$$

After integrating Eq. (5.6) over the time interval $[0,\tau]$, and substitute it into Eq. (5.7), the expected value is taken analogously to the estimation of $\mathcal{T}_2(\tau)$, where the martingale property of the stochastic integral is used to eliminate the stochastic term. Therefore,

$$\begin{aligned}
&\mathbb{E}\|\delta_\upsilon(\cdot,\tau)\|^2+\kappa\sum_{\iota=1}^N\mathbb{E}\int_0^\tau[\delta_q]^2_{\iota-\frac{1}{2}}\mathrm{d}s\\
&\le\mathbb{E}\|\delta_\upsilon(\cdot,0)\|^2-2\alpha\mathbb{E}\int_0^\tau\|\delta_q(\cdot,s)\|^2\mathrm{d}s+2\kappa\mathbb{E}\int_0^\tau\int_0^\Gamma\theta_q\delta_p\mathrm{d}\chi\mathrm{d}s\\
&\quad+2\alpha\mathbb{E}\int_0^\tau\int_0^\Gamma\theta_q\delta_q\mathrm{d}\chi\mathrm{d}s-2\kappa\mathbb{E}\int_0^\tau\int_0^\Gamma\theta_p\delta_q\mathrm{d}\chi\mathrm{d}s+2\mathbb{E}\int_0^\tau\int_0^\Gamma\delta_\upsilon\partial_s\theta_\upsilon\mathrm{d}\chi\mathrm{d}s\\
&\quad+Ch^{2n+2}+C\mathbb{E}\int_0^\tau\|\delta_\upsilon(\cdot,s)\|^2\mathrm{d}s+C\mathbb{E}\int_0^\tau\|\delta_q(\cdot,s)\|^2\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\quad + 2\mathbb{E}\int_0^\tau\int_0^\Gamma \big(f(\upsilon,q)-f(\upsilon_h,q_h)\big)\delta_\upsilon \mathrm{d}\chi\mathrm{d}s \\
&\le \mathbb{E}\|\delta_\upsilon(\cdot,0)\|^2 - 2\alpha\mathbb{E}\int_0^\tau \|\delta_q(\cdot,s)\|^2\mathrm{d}s + C\mathbb{E}\int_0^\tau \|\delta_p(\cdot,s)\|^2\mathrm{d}s + Ch^{2n+2} \\
&\quad + C\mathbb{E}\int_0^\tau \|\delta_q(\cdot,s)\|^2\mathrm{d}s + C\mathbb{E}\int_0^\tau \|\delta_\upsilon(\cdot,s)\|^2\mathrm{d}s \\
&\quad + 2B_1\mathbb{E}\int_0^\tau\int_0^\Gamma \big(|\delta_\upsilon|+|\delta_q|+|\theta_\upsilon|+|\theta_q|\big)\delta_\upsilon\mathrm{d}\chi\mathrm{d}s \\
&\le \mathbb{E}\|\delta_\upsilon(\cdot,0)\|^2 + C\mathbb{E}\int_0^\tau \|\delta_q(\cdot,s)\|^2\mathrm{d}s + C\mathbb{E}\int_0^\tau \|\delta_p(\cdot,s)\|^2\mathrm{d}s \\
&\quad + C\mathbb{E}\int_0^\tau \|\delta_\upsilon(\cdot,s)\|^2\mathrm{d}s + Ch^{2n+2}.
\end{aligned} \tag{5.9}
$$

Then, to estimate the expected squared norm of δ_p, Eqs. (5.4a) and (5.4b) are differentiated with respect to τ. By taking the test functions as $z_h = -\partial_\tau\delta_\upsilon, w_h = \kappa\delta_p + \alpha\delta_q$ and $\mathrm{v}_h = \partial_\tau\delta_q$, the resulting system becomes

$$
\left\{
\begin{aligned}
&-\int_0^\Gamma \partial_\tau(\delta_q-\theta_q)\partial_\tau\delta_\upsilon\mathrm{d}\chi = -\sum_{\iota=1}^N \hat{J}_\iota^+(\partial_\tau\delta_\upsilon-\partial_\tau\theta_\upsilon,\partial_\tau\delta_\upsilon), \\
&\int_0^\Gamma \partial_\tau(\delta_p-\theta_p)(\kappa\delta_p+\alpha\delta_q)\mathrm{d}\chi = \sum_{\iota=1}^N \hat{J}_\iota^-(\partial_\tau\delta_q-\partial_\tau\theta_q,\kappa\delta_p+\alpha\delta_q), \\
&\int_0^\Gamma \partial_\tau\delta_q\partial_\tau(\delta_\upsilon-\theta_\upsilon)\mathrm{d}\chi \\
&= \kappa\sum_{\iota=1}^N \hat{J}_\iota^-(\delta_p-\theta_p,\partial_\tau\delta_q) + \alpha\sum_{\iota=1}^N \hat{J}_\iota^-(\delta_q-\theta_q,\partial_\tau\delta_q) \\
&\quad + \int_0^\Gamma \big(f(\upsilon,q)-f(\upsilon_h,q_h)\big)\partial_\tau\delta_q\mathrm{d}\chi \\
&\quad + \int_0^\Gamma \big(\sigma(\upsilon,q)-\sigma(\upsilon_h,q_h)\big)\partial_\tau\delta_q\mathrm{d}\chi\frac{\mathrm{d}W_\tau}{\mathrm{d}\tau}.
\end{aligned}
\right. \tag{5.10}
$$

Before summing the system (5.10), multiply it by $\mathrm{d}\tau$, then

$$
\begin{aligned}
&\int_0^\Gamma \partial_\tau\theta_q\partial_\tau\delta_\upsilon\mathrm{d}\chi\mathrm{d}\tau + \kappa\int_0^\Gamma \delta_p\mathrm{d}\delta_p\mathrm{d}\chi + \alpha\int_0^\Gamma \partial_\tau\delta_p\delta_q\mathrm{d}\chi\mathrm{d}\tau \\
&\quad - \kappa\int_0^\Gamma \partial_\tau\theta_p\delta_p\mathrm{d}\chi\mathrm{d}\tau - \alpha\int_0^\Gamma \partial_\tau\theta_p\delta_q\mathrm{d}\chi\mathrm{d}\tau - \int_0^\Gamma \partial_\tau\delta_q\partial_\tau\theta_\upsilon\mathrm{d}\chi\mathrm{d}\tau \\
&= \frac{1}{2}\sum_{\iota=1}^N [\partial_\tau\delta_\upsilon]^2_{\iota-\frac{1}{2}}\mathrm{d}\tau + \kappa\sum_{\iota=1}^N [\delta_p]_{\iota-\frac{1}{2}}[\partial_\tau\delta_q]_{\iota-\frac{1}{2}}\mathrm{d}\tau + \alpha\sum_{\iota=1}^N [\delta_q]_{\iota-\frac{1}{2}}[\partial_\tau\delta_q]_{\iota-\frac{1}{2}}\mathrm{d}\tau \\
&\quad + \int_0^\Gamma \big(f(\upsilon,q)-f(\upsilon_h,q_h)\big)\partial_\tau\delta_q\mathrm{d}\chi\mathrm{d}\tau
\end{aligned}
$$

$$+\int_0^\Gamma \big(\sigma(\upsilon,q)-\sigma(\upsilon_h,q_h)\big)\partial_\tau\delta_q \mathrm{d}\chi \mathrm{d}W_\tau. \tag{5.11}$$

With the choice of test function $w_h = \mathrm{d}\delta p$, the difference between the projected and numerical solution of Eq. (5.3b) is expressed as

$$\int_0^\Gamma \delta_p \mathrm{d}\delta_p = \sum_{\iota=1}^N \hat{J}_\iota^-(\delta_q, \mathrm{d}\delta_p),$$

which conforms to the definition of a bounded variation function. Consequently, the norm of δ_p is described as

$$\|\delta_p(\cdot,\tau)\|^2 = \|\delta_p(\cdot,0)\|^2 + 2\int_0^\Gamma\int_0^\tau \delta_p \mathrm{d}\delta_p \mathrm{d}\chi. \tag{5.12}$$

By combining Eqs. (5.11) with (5.12) and then taking the expectation, it obtains

$$\begin{aligned}
&\frac{\kappa}{2}\mathbb{E}\|\delta_p(\cdot,\tau)\|^2 - \frac{1}{2}\mathbb{E}\sum_{\iota=1}^N\int_0^\tau [\partial_s\delta_\upsilon]^2_{\iota-\frac{1}{2}}\mathrm{d}s \\
&\quad - \kappa\mathbb{E}\sum_{\iota=1}^N\int_0^\tau [\delta_p]_{\iota-\frac{1}{2}}[\partial_s\delta_q]_{\iota-\frac{1}{2}}\mathrm{d}s - \alpha\mathbb{E}\sum_{\iota=1}^N\int_0^\tau [\delta_q]_{\iota-\frac{1}{2}}[\partial_s\delta_q]_{\iota-\frac{1}{2}}\mathrm{d}s \\
&= \frac{\kappa}{2}\mathbb{E}\|\delta_p(\cdot,0)\|^2 - \mathbb{E}\int_0^\tau\int_0^\Gamma \partial_s\theta_q\partial_s\delta_\upsilon \mathrm{d}\chi\mathrm{d}s - \alpha\mathbb{E}\int_0^\tau\int_0^\Gamma \partial_s\delta_p\delta_q\mathrm{d}\chi\mathrm{d}s \\
&\quad + \kappa\mathbb{E}\int_0^\tau\int_0^\Gamma \partial_s\theta_p\delta_p\mathrm{d}\chi\mathrm{d}s + \alpha\mathbb{E}\int_0^\tau\int_0^\Gamma \partial_s\theta_p\delta_q\mathrm{d}\chi\mathrm{d}s \\
&\quad + \mathbb{E}\int_0^\tau\int_0^\Gamma \partial_s\delta_q\partial_s\theta_\upsilon\mathrm{d}\chi\mathrm{d}s + \mathbb{E}\int_0^\tau\int_0^\Gamma \big(f(\upsilon,q)-f(\upsilon_h,q_h)\big)\partial_s\delta_q\mathrm{d}\chi\mathrm{d}s.
\end{aligned} \tag{5.13}$$

Based on Eq. (5.1) and the use of the Cauchy-Schwarz and Young's inequalities, the third term on the right-hand side of Eq. (5.13) is estimated as

$$\begin{aligned}
&-\alpha\mathbb{E}\int_0^\tau\int_0^\Gamma \partial_s\delta_p\delta_q\mathrm{d}\chi\mathrm{d}s \\
&= \alpha\mathbb{E}\int_0^\tau\int_0^\Gamma \delta_p\partial_s\delta_q\mathrm{d}\chi\mathrm{d}s - \alpha\mathbb{E}\int_0^\Gamma \delta_p\delta_q\big|_0^\tau\mathrm{d}\chi \\
&\le \frac{\alpha\epsilon^2}{2}\mathbb{E}\int_0^\tau \|\partial_s\delta_q(\cdot,s)\|^2\mathrm{d}s + \frac{\alpha}{2\epsilon^2}\mathbb{E}\int_0^\tau \|\delta_p(\cdot,s)\|^2\mathrm{d}s \\
&\quad + \alpha^2\mathbb{E}\|\delta_p(\cdot,\tau)\|^2 + \frac{1}{4}\mathbb{E}\|\delta_q(\cdot,\tau)\|^2, \quad \forall\epsilon>0.
\end{aligned}$$

The last term in Eq. (5.13) is estimated by using the Cauchy-Schwarz inequality and condition (2.5), which yields

$$\mathbb{E}\int_0^\tau\int_0^\Gamma \big(f(\upsilon,q)-f(\upsilon_h,q_h)\big)\partial_s\delta_q\mathrm{d}\chi\mathrm{d}s \le \frac{\epsilon^2}{2}\mathbb{E}\int_0^\tau \|\partial_s\delta_q(\cdot,s)\|^2\mathrm{d}s + Ch^{2n+2}$$

$$+C\mathbb{E}\int_0^\tau \|\delta_\upsilon(\cdot,s)\|^2\mathrm{d}s + C\mathbb{E}\int_0^\tau \|\delta_q(\cdot,s)\|^2\mathrm{d}s.$$

Therefore, for sufficiently small $\epsilon > 0$, Eq. (5.13) is computed as

$$\begin{aligned}
&\frac{\kappa}{2}\mathbb{E}\|\delta_p(\cdot,\tau)\|^2 - \frac{1}{2}\mathbb{E}\sum_{\iota=1}^N\int_0^\tau [\partial_s\delta_\upsilon]^2_{\iota-\frac{1}{2}}\mathrm{d}s \\
&\quad - \kappa\mathbb{E}\sum_{\iota=1}^N\int_0^\tau [\delta_p]_{\iota-\frac{1}{2}}[\partial_s\delta_q]_{\iota-\frac{1}{2}}\mathrm{d}s - \alpha\mathbb{E}\sum_{\iota=1}^N\int_0^\tau [\delta_q]_{\iota-\frac{1}{2}}[\partial_s\delta_q]_{\iota-\frac{1}{2}}\mathrm{d}s \\
&\le \frac{\kappa}{2}\mathbb{E}\|\delta_p(\cdot,0)\|^2 + Ch^{2n+2} + C\mathbb{E}\int_0^\tau \|\partial_s\delta_\upsilon(\cdot,s)\|^2\mathrm{d}s + \frac{\alpha\epsilon^2}{2}\mathbb{E}\int_0^\tau \|\partial_s\delta_q(\cdot,s)\|^2\mathrm{d}s \\
&\quad + C\mathbb{E}\int_0^\tau \|\delta_p(\cdot,s)\|^2\mathrm{d}s + \alpha^2\mathbb{E}\|\delta_p(\cdot,\tau)\|^2 + \frac{1}{4}\mathbb{E}\|\delta_q(\cdot,\tau)\|^2 \\
&\quad + C\mathbb{E}\int_0^\tau \|\delta_q(\cdot,s)\|^2\mathrm{d}s + \frac{\epsilon^2}{2}\mathbb{E}\int_0^\tau \|\partial_s\delta_q(\cdot,s)\|^2\mathrm{d}s \\
&\quad + \frac{\epsilon^2}{2}\mathbb{E}\int_0^\tau \|\partial_s\delta_q(\cdot,s)\|^2\mathrm{d}s + C\mathbb{E}\int_0^\tau \|\delta_\upsilon(\cdot,s)\|^2\mathrm{d}s \\
&\le \frac{\kappa}{2}\mathbb{E}\,\|\delta_p(\cdot,0)\|^2 + Ch^{2n+2} + C\mathbb{E}\int_0^\tau \|\partial_s\delta_\upsilon(\cdot,s)\|^2\mathrm{d}s \\
&\quad + \frac{(\alpha+2)\epsilon^2}{2}\mathbb{E}\int_0^\tau \|\partial_s\delta_q(\cdot,s)\|^2\mathrm{d}s + C\mathbb{E}\int_0^\tau \|\delta_p(\cdot,s)\|^2\mathrm{d}s + \alpha^2\mathbb{E}\|\delta_p(\cdot,\tau)\|^2 \\
&\quad + \frac{1}{4}\mathbb{E}\|\delta_q(\cdot,\tau)\|^2 + C\mathbb{E}\int_0^\tau \|\delta_q(\cdot,s)\|^2\mathrm{d}s + C\mathbb{E}\int_0^\tau \|\delta_\upsilon(\cdot,s)\|^2\mathrm{d}s.
\end{aligned} \tag{5.14}$$

Next, the estimation for expected squared norm of $\partial_\tau\delta_\upsilon$ is introduced. By differentiating Eqs. (5.4) with respect to time τ, and selecting the test functions as $z_h = \kappa\partial_\tau\delta_p + \alpha\partial_\tau\delta_q$, $w_h = -\kappa\partial_\tau\delta_q$, and $\mathrm{v}_h = \partial_\tau\delta_\upsilon$, it transforms into

$$\left\{\begin{aligned}
&\int_0^\Gamma (\partial_\tau\delta_q - \partial_\tau\theta_q)(\kappa\partial_\tau\delta_p + \alpha\partial_\tau\delta_q)\mathrm{d} \\
&= \sum_{\iota=1}^N \hat{J}_\iota^+(\partial_\tau\delta_\upsilon - \partial_\tau\theta_\upsilon, \kappa\partial_\tau\delta_p + \alpha\partial_\tau\delta_q), \\
&-\kappa\int_0^\Gamma (\partial_\tau\delta_p - \partial_\tau\theta_p)\partial_\tau\delta_q\mathrm{d}\chi = -\kappa\sum_{\iota=1}^N \hat{J}_\iota^-(\partial_\tau\delta_q - \partial_\tau\theta_q, \partial_\tau\delta_q), \\
&\int_0^\Gamma \partial_\tau\delta_\upsilon(\partial_{\tau\tau}\delta_\upsilon - \partial_{\tau\tau}\theta_\upsilon)\mathrm{d}\chi \\
&= \kappa\sum_{\iota=1}^N \hat{J}_\iota^-(\partial_\tau\delta_p - \partial_\tau\theta_p, \partial_\tau\delta_\upsilon) + \alpha\sum_{\iota=1}^N \hat{J}_\iota^-(\partial_\tau\delta_q - \partial_\tau\theta_q, \partial_\tau\delta_\upsilon) \\
&\quad + \int_0^\Gamma \big(f(\upsilon,q)_\tau - f(\upsilon_h,q_h)_\tau\big)\partial_\tau\delta_\upsilon\mathrm{d}\chi \\
&\quad + \int_0^\Gamma \big(\sigma(\upsilon,q)_\tau - \sigma(\upsilon_h,q_h)_\tau\big)\partial_\tau\delta_\upsilon\mathrm{d}\chi\frac{\mathrm{d}W_\tau}{\mathrm{d}\tau}.
\end{aligned}\right. \tag{5.15}$$

After summing all the terms in Eqs. (5.15) and multiplying by $\mathrm{d}\tau$, it becomes

$$\begin{aligned}
&\alpha\|\partial_\tau\delta_q(\cdot,\tau)\|^2\mathrm{d}\tau-\kappa\int_0^\Gamma\partial_\tau\theta_q\partial_\tau\delta_p\mathrm{d}\chi\mathrm{d}\tau-\alpha\int_0^\Gamma\partial_\tau\theta_q\partial_\tau\delta_q\mathrm{d}\chi\mathrm{d}\tau\\
&\quad+\kappa\int_0^\Gamma\partial_\tau\theta_p\partial_\tau\delta_q\mathrm{d}\chi\mathrm{d}\tau+\int_0^\Gamma\partial_\tau\delta_v\mathrm{d}\partial_\tau\delta_v\mathrm{d}\chi-\int_0^\Gamma\partial_\tau\delta_v\partial_{\tau\tau}\theta_v\mathrm{d}\chi\mathrm{d}\tau\\
&=-\frac{\kappa}{2}\sum_{\iota=1}^N[\partial_\tau\delta_q]^2_{\iota-\frac12}\mathrm{d}\tau+\int_0^\Gamma\big(f(v,q)_\tau-f(v_h,q_h)_\tau\big)\partial_\tau\delta_v\mathrm{d}\chi\mathrm{d}\tau\\
&\quad+\int_0^\Gamma\big(\sigma(v,q)_\tau-\sigma(v_h,q_h)_\tau\big)\partial_\tau\delta_v\mathrm{d}\chi\mathrm{d}W_\tau.
\end{aligned}\tag{5.16}$$

Based on Eq. (2.1), the Itô's formula for function $\partial_\tau\delta_v$ implies

$$\begin{aligned}
\|\partial_\tau\delta_v(\cdot,\tau)\|^2=\|\partial_\tau\delta_v(\cdot,0)\|^2+2\int_0^\Gamma\int_0^\tau\partial_s\delta_v\mathrm{d}\partial_s\delta_v\mathrm{d}\chi\\
+\int_0^\Gamma\langle\partial_s\delta_v,\partial_s\delta_v\rangle_\tau\mathrm{d}\chi.
\end{aligned}\tag{5.17}$$

Being similar to the computation of Eq. (5.8), the last term of Eq. (5.17) satisfies that

$$\begin{aligned}
\int_0^\Gamma\langle\partial_s\delta_v,\partial_s\delta_v\rangle_\tau\mathrm{d}\chi\le Ch^{2n+2}+C\int_0^\tau\|\partial_s\delta_v(\cdot,s)\|^2\mathrm{d}s\\
+(1+\eta)^3B_1\int_0^\tau\|\partial_s\delta_q(\cdot,s)\|^2\mathrm{d}s,
\end{aligned}\tag{5.18}$$

where we set $\eta=(\alpha/B_1)^{1/3}-1>0$ under the assumption $\alpha>B_1$. By the combination of Eq. (5.16) with (5.18), and the expectation of Eq. (5.17), it follows that

$$\begin{aligned}
&\mathbb{E}\|\partial_\tau\delta_v(\cdot,\tau)\|^2+\kappa\mathbb{E}\sum_{\iota=1}^N\int_0^\tau[\partial_s\delta_q]^2_{\iota-\frac12}\mathrm{d}s+2\alpha\mathbb{E}\int_0^\tau\|\partial_s\delta_q(\cdot,s)\|^2\mathrm{d}s\\
&\le\mathbb{E}\|\partial_\tau\delta_v(\cdot,0)\|^2+2\kappa\mathbb{E}\int_0^\tau\int_0^\Gamma\partial_s\theta_q\partial_s\delta_p\mathrm{d}\chi\mathrm{d}s+2\alpha\mathbb{E}\int_0^\tau\int_0^\Gamma\partial_s\theta_q\partial_s\delta_q\mathrm{d}\chi\mathrm{d}s\\
&\quad-2\kappa\mathbb{E}\int_0^\tau\int_0^\Gamma\partial_s\theta_p\partial_s\delta_q\mathrm{d}\chi\mathrm{d}s+2\mathbb{E}\int_0^\tau\int_0^\Gamma\partial_s\delta_v\partial_{ss}\theta_v\mathrm{d}\chi\mathrm{d}s\\
&\quad+2\mathbb{E}\int_0^\tau\int_0^\Gamma\big(f(v,q)_s-f(v_h,q_h)_s\big)\partial_s\delta_v\mathrm{d}\chi\mathrm{d}s+Ch^{2n+2}\\
&\quad+C\mathbb{E}\int_0^\tau\|\partial_s\delta_v(\cdot,s)\|^2\mathrm{d}s+\alpha\mathbb{E}\int_0^\tau\|\partial_s\delta_q(\cdot,s)\|^2\mathrm{d}s,
\end{aligned}\tag{5.19}$$

where in the view of (5.1) it follows that

$$\begin{aligned}
&2\kappa\mathbb{E}\int_0^\tau\int_0^\Gamma\partial_s\theta_q\partial_s\delta_p\mathrm{d}\chi\mathrm{d}s\\
&=-2\kappa\mathbb{E}\int_0^\tau\int_0^\Gamma\partial_{ss}\theta_q\delta_p\mathrm{d}\chi\mathrm{d}s+2\kappa\mathbb{E}\int_0^\Gamma\partial_s\theta_q\delta_p\big|_0^\tau\mathrm{d}\chi
\end{aligned}$$

$$\leq Ch^{2n+2} + C\mathbb{E}\int_0^\tau \|\delta_p(\cdot,s)\|^2\mathrm{d}s + C\mathbb{E}\|\partial_s\theta_q(\cdot,\tau)\|^2 + 2\epsilon^2\mathbb{E}\|\delta_p(\cdot,\tau)\|^2,$$

and the estimate of the sixth term on the right-hand side is as follows:

$$\begin{aligned}
&\mathbb{E}\int_0^\tau\int_0^\Gamma \big(f(\upsilon,q)_s - f(\upsilon_h,q_h)_s\big)\partial_s\delta_\upsilon \mathrm{d}\chi\mathrm{d}s \\
&\leq Ch^{2n+2} + C\mathbb{E}\int_0^\tau \|\partial_s\delta_\upsilon(\cdot,s)\|^2\mathrm{d}s + \frac{B_1\epsilon^2}{2}\mathbb{E}\int_0^\tau \|\partial_s\delta_q(\cdot,s)\|^2\mathrm{d}s, \quad \forall\epsilon>0.
\end{aligned}$$

Therefore, Eq. (5.19) is computed as

$$\begin{aligned}
&\mathbb{E}\|\partial_\tau\delta_\upsilon(\cdot,\tau)\|^2 + \kappa\mathbb{E}\sum_{\iota=1}^N\int_0^\tau [\partial_s\delta_q]^2_{\iota-\frac{1}{2}}\mathrm{d}s \\
&\quad + 2\alpha\mathbb{E}\int_0^\tau \|\partial_s\delta_q(\cdot,s)\|^2\mathrm{d}s - 2\epsilon^2\mathbb{E}\|\delta_p(\cdot,\tau)\|^2 \\
&\leq \mathbb{E}\|\partial_\tau\delta_\upsilon(\cdot,0)\|^2 + C\mathbb{E}\int_0^\tau \|\delta_p(\cdot,s)\|^2\mathrm{d}s \\
&\quad + (\kappa+\alpha)\epsilon^2\mathbb{E}\int_0^\tau \|\partial_s\delta_q(\cdot,s)\|^2\mathrm{d}s + C\mathbb{E}\int_0^\tau \|\partial_s\delta_\upsilon(\cdot,s)\|^2\mathrm{d}s \\
&\quad + (B_1\epsilon^2+\alpha)\mathbb{E}\int_0^\tau \|\partial_s\delta_q(\cdot,s)\|^2\mathrm{d}s + Ch^{2n+2}.
\end{aligned} \tag{5.20}$$

Furthermore, we estimate the expected squared norm of δ_q. By differentiating Eq. (5.4a) with respect to τ and applying the test functions $z_h = \delta_q, w_h = \partial_\tau\delta_\upsilon$ and $\mathrm{v}_h = 0$, it holds

$$\begin{cases}
\displaystyle\int_0^\Gamma (\partial_\tau\delta_q - \partial_\tau\theta_q)\delta_q\mathrm{d}\chi = \sum_{\iota=1}^N \hat{J}_\iota^+(\partial_\tau\delta_\upsilon - \partial_\tau\theta_\upsilon,\delta_q), \\
\displaystyle\int_0^\Gamma (\delta_p - \theta_p)\partial_\tau\delta_\upsilon\mathrm{d}\chi = \sum_{\iota=1}^N \hat{J}_\iota^-(\delta_q - \theta_q,\partial_\tau\delta_\upsilon).
\end{cases} \tag{5.21}$$

Eqs. (5.21) are summed and then multiplied by $\mathrm{d}\tau$, which yields

$$\int_0^\Gamma \delta_q\mathrm{d}\delta_q\mathrm{d}\chi = \int_0^\Gamma \partial_\tau\theta_q\delta_q\mathrm{d}\chi\mathrm{d}\tau - \int_0^\Gamma \delta_p\partial_\tau\delta_\upsilon\mathrm{d}\chi\mathrm{d}\tau + \int_0^\Gamma \theta_p\partial_\tau\delta_\upsilon\mathrm{d}\chi\mathrm{d}\tau. \tag{5.22}$$

With the choice of test function $z_h = \mathrm{d}\delta_q$, the difference between the projected and numerical solution of Eq. (5.3a) is expressed as

$$\int_0^\Gamma \delta_q\mathrm{d}\delta_q\mathrm{d}\chi = \sum_{\iota=1}^N \hat{J}_\iota^+(\delta_\upsilon,\mathrm{d}\delta_q),$$

which satisfies the definition of a bounded variation function. Therefore, the norm of δ_q holds

$$\|\delta_q(\cdot,\tau)\|^2=\|\delta_q(\cdot,0)\|^2+2\int_0^\Gamma\int_0^\tau\delta_q\mathrm{d}\delta_q\mathrm{d}\chi. \tag{5.23}$$

From Eq. (5.22) and the expectation of Eq. (5.23), the following equation is obtained:

$$\begin{aligned}\frac{1}{2}\mathbb{E}\|\delta_q(\cdot,\tau)\|^2=&\frac{1}{2}\mathbb{E}\|\delta_q(\cdot,0)\|^2+\mathbb{E}\int_0^\tau\int_0^\Gamma\partial_s\theta_q\delta_q\mathrm{d}\chi\mathrm{d}s\\&-\mathbb{E}\int_0^\tau\int_0^\Gamma\delta_p\partial_s\delta_\upsilon\mathrm{d}\chi\mathrm{d}s+\mathbb{E}\int_0^\tau\int_0^\Gamma\theta_p\partial_s\delta_\upsilon\mathrm{d}\chi\mathrm{d}s\\ \le&\frac{1}{2}\mathbb{E}\|\delta_q(\cdot,0)\|^2+Ch^{2n+2}+C\mathbb{E}\int_0^\tau\|\delta_q(\cdot,s)\|^2\mathrm{d}s\\&+C\mathbb{E}\int_0^\tau\|\delta_p(\cdot,s)\|^2\mathrm{d}s+C\mathbb{E}\int_0^\tau\|\partial_s\delta_\upsilon(\cdot,s)\|^2\mathrm{d}s.\end{aligned} \tag{5.24}$$

Finally, Eq. (5.4a) is differentiated with respect to time τ, and the test functions are chosen as $z_h=\partial_\tau\delta_\upsilon$, $w_h=\alpha\delta_p$, and $\mathrm{v}_h=-\delta_p$, leading to

$$\left\{\begin{aligned}&\int_0^\Gamma(\partial_\tau\delta_q-\partial_\tau\theta_q)\partial_\tau\delta_\upsilon\mathrm{d}\chi=\sum_{\iota=1}^N\hat{J}_\iota^+(\partial_\tau\delta_\upsilon-\partial_\tau\theta_\upsilon,\partial_\tau\delta_\upsilon),\\&\alpha\int_0^\Gamma(\delta_p-\theta_p)\delta_p\mathrm{d}\chi=\alpha\sum_{\iota=1}^N\hat{J}_\iota^-(\delta_q-\theta_q,\delta_p),\\&-\int_0^\Gamma\delta_p(\partial_\tau\delta_\upsilon-\partial_\tau\theta_\upsilon)\mathrm{d}\chi\\=&-\kappa\sum_{\iota=1}^N\hat{J}_\iota^-(\delta_p-\theta_p,\delta_p)-\alpha\sum_{\iota=1}^N\hat{J}_\iota^-(\delta_q-\theta_q,\delta_p)\\&\quad-\int_0^\Gamma\big(f(\upsilon,q)-f(\upsilon_h,q_h)\big)\delta_p\mathrm{d}\chi-\int_0^\Gamma\big(\sigma(\upsilon,q)-\sigma(\upsilon_h,q_h)\big)\delta_p\mathrm{d}\chi\frac{\mathrm{d}W_\tau}{\mathrm{d}\tau}.\end{aligned}\right. \tag{5.25}$$

After summing Eqs. (5.25) and multiplying by $\mathrm{d}\tau$, we apply Lemmas 2.2 and 2.3, which leads to

$$\begin{aligned}&\frac{1}{2}\sum_{\iota=1}^N[\partial_\tau\delta_\upsilon]^2_{\iota-\frac{1}{2}}\mathrm{d}\tau+\frac{\kappa}{2}\sum_{\iota=1}^N[\delta_p]^2_{\iota-\frac{1}{2}}\mathrm{d}\tau\\=&-\int_0^\Gamma(\partial_\tau\delta_q-\partial_\tau\theta_q)\partial_\tau\delta_\upsilon\mathrm{d}\chi\mathrm{d}\tau-\alpha\int_0^\Gamma\delta_p\delta_p\mathrm{d}\chi\mathrm{d}\tau+\alpha\int_0^\Gamma\theta_p\delta_p\mathrm{d}\chi\mathrm{d}\tau\\&+\int_0^\Gamma\delta_p(\partial_\tau\delta_\upsilon-\partial_\tau\theta_\upsilon)\mathrm{d}\chi\mathrm{d}\tau-\int_0^\Gamma\big(f(\upsilon,q)-f(\upsilon_h,q_h)\big)\delta_p\mathrm{d}\chi\mathrm{d}\tau\\&-\int_0^\Gamma\big(\sigma(\upsilon,q)-\sigma(\upsilon_h,q_h)\big)\delta_p\mathrm{d}\chi\mathrm{d}W_\tau.\end{aligned}$$

After applying the time integration and taking the expectation, it follows that

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\sum_{\iota=1}^{N}\int_0^\tau [\partial_s\delta_v]^2_{\iota-\frac{1}{2}}\mathrm{d}s+\frac{\kappa}{2}\mathbb{E}\sum_{\iota=1}^{N}\int_0^\tau [\delta_p]^2_{\iota-\frac{1}{2}}\mathrm{d}s\\
&=-\mathbb{E}\int_0^\tau\int_0^\Gamma(\partial_s\delta_q-\partial_s\theta_q)\partial_s\delta_v\mathrm{d}\chi\mathrm{d}s-\alpha\mathbb{E}\int_0^\tau\|\delta_p(\cdot,s)\|^2\mathrm{d}s+\alpha\mathbb{E}\int_0^\tau\int_0^\Gamma\theta_p\delta_p\mathrm{d}\chi\mathrm{d}s\\
&\quad+\mathbb{E}\int_0^\tau\int_0^\Gamma\delta_p(\partial_s\delta_v-\partial_s\theta_v)\mathrm{d}\chi\mathrm{d}s-\mathbb{E}\int_0^\tau\int_0^\Gamma\big(f(v,q)-f(v_h,q_h)\big)\delta_p\mathrm{d}\chi\mathrm{d}s\\
&\le Ch^{2n+2}+\epsilon^2\mathbb{E}\int_0^\tau\|\partial_s\delta_q(\cdot,s)\|^2\mathrm{d}s+C\mathbb{E}\int_0^\tau\|\partial_s\delta_v(\cdot,s)\|^2\mathrm{d}s\\
&\quad+C\mathbb{E}\int_0^\tau\|\delta_p(\cdot,s)\|^2\mathrm{d}s+C\mathbb{E}\int_0^\tau\|\delta_q(\cdot,s)\|^2\mathrm{d}s+C\mathbb{E}\int_0^\tau\|\delta_v(\cdot,s)\|^2\mathrm{d}s. \qquad (5.26)
\end{aligned}$$

By summing Eqs. (5.9), (5.14), (5.20), (5.24) and (5.26), the following estimate is obtained:

$$\begin{aligned}
&\mathbb{E}\|\delta_v(\cdot,\tau)\|^2+\Big(\frac{\kappa}{2}-\alpha^2-2\epsilon^2\Big)\mathbb{E}\|\delta_p(\cdot,\tau)\|^2+\mathbb{E}\|\partial_\tau\delta_v(\cdot,\tau)\|^2+\frac{1}{4}\mathbb{E}\|\delta_q(\cdot,\tau)\|^2\\
&\quad+\Big(\alpha-2\epsilon^2-\kappa\epsilon^2-\frac{3\alpha\epsilon^2}{2}-B_1\epsilon^2\Big)\mathbb{E}\int_0^\tau\|\partial_s\delta_q(\cdot,s)\|^2\mathrm{d}s+\kappa\mathbb{E}\sum_{\iota=1}^{N}\int_0^\tau[\delta_q]^2_{\iota-\frac{1}{2}}\mathrm{d}s\\
&\quad-\kappa\mathbb{E}\sum_{\iota=1}^{N}\int_0^\tau[\delta_p]_{\iota-\frac{1}{2}}[\partial_s\delta_q]_{\iota-\frac{1}{2}}\mathrm{d}s-\alpha\mathbb{E}\sum_{\iota=1}^{N}\int_0^\tau[\delta_q]_{\iota-\frac{1}{2}}[\partial_s\delta_q]_{\iota-\frac{1}{2}}\mathrm{d}s\\
&\quad+\kappa\mathbb{E}\sum_{\iota=1}^{N}\int_0^\tau[\partial_s\delta_q]^2_{\iota-\frac{1}{2}}\mathrm{d}s+\frac{\kappa}{2}\mathbb{E}\sum_{\iota=1}^{N}\int_0^\tau[\delta_p]^2_{\iota-\frac{1}{2}}\mathrm{d}s\\
&\le\mathbb{E}\|\delta_v(\cdot,0)\|^2+\frac{\kappa}{2}\mathbb{E}\|\delta_p(\cdot,0)\|^2+\mathbb{E}\|\partial_\tau\delta_v(\cdot,0)\|^2+\frac{1}{2}\mathbb{E}\|\delta_q(\cdot,0)\|^2+Ch^{2n+2}\\
&\quad+C\mathbb{E}\int_0^\tau\|\delta_q(\cdot,s)\|^2\mathrm{d}s+C\mathbb{E}\int_0^\tau\|\delta_p(\cdot,s)\|^2\mathrm{d}s+C\mathbb{E}\int_0^\tau\|\delta_v(\cdot,s)\|^2\mathrm{d}s\\
&\quad+C\mathbb{E}\int_0^\tau\|\partial_s\delta_v(\cdot,s)\|^2\mathrm{d}s,\quad\forall\epsilon>0. \qquad (5.27)
\end{aligned}$$

From the Cauchy-Schwartz inequality and the condition $\kappa\ge\alpha$, the estimation of the jump operators in (5.27) is

$$\begin{aligned}
&\kappa\mathbb{E}\sum_{\iota=1}^{N}\int_0^\tau[\delta_q]^2_{\iota-\frac{1}{2}}\mathrm{d}s-\kappa\mathbb{E}\sum_{\iota=1}^{N}\int_0^\tau[\delta_p]_{\iota-\frac{1}{2}}[\partial_s\delta_q]_{\iota-\frac{1}{2}}\mathrm{d}s\\
&\quad-\alpha\mathbb{E}\sum_{\iota=1}^{N}\int_0^\tau[\delta_q]_{\iota-\frac{1}{2}}[\partial_s\delta_q]_{\iota-\frac{1}{2}}\mathrm{d}s+\kappa\mathbb{E}\sum_{\iota=1}^{N}\int_0^\tau[\partial_s\delta_q]^2_{\iota-\frac{1}{2}}\mathrm{d}s+\frac{\kappa}{2}\mathbb{E}\sum_{\iota=1}^{N}\int_0^\tau[\delta_p]^2_{\iota-\frac{1}{2}}\mathrm{d}s\\
&\ge\kappa\mathbb{E}\sum_{\iota=1}^{N}\int_0^\tau[\delta_q]^2_{\iota-\frac{1}{2}}\mathrm{d}s-\frac{\kappa}{2}\mathbb{E}\sum_{\iota=1}^{N}\int_0^\tau[\delta_p]^2_{\iota-\frac{1}{2}}\mathrm{d}s-\frac{\kappa}{2}\mathbb{E}\sum_{\iota=1}^{N}\int_0^\tau[\partial_s\delta_q]^2_{\iota-\frac{1}{2}}\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
& -\frac{\alpha}{2}\mathbb{E}\sum_{\iota=1}^{N}\int_0^{\tau}[\delta_q]^2_{\iota-\frac{1}{2}}\mathrm{d}s-\frac{\alpha}{2}\mathbb{E}\sum_{\iota=1}^{N}\int_0^{\tau}[\partial_s\delta_q]^2_{\iota-\frac{1}{2}}\mathrm{d}s \\
& +\kappa\mathbb{E}\sum_{\iota=1}^{N}\int_0^{\tau}[\partial_s\delta_q]^2_{\iota-\frac{1}{2}}\mathrm{d}s+\frac{\kappa}{2}\mathbb{E}\sum_{\iota=1}^{N}\int_0^{\tau}[\delta_p]^2_{\iota-\frac{1}{2}}\mathrm{d}s \\
= & \left(\kappa-\frac{\alpha}{2}\right)\mathbb{E}\sum_{\iota=1}^{N}\int_0^{\tau}[\delta_q]^2_{\iota-\frac{1}{2}}\mathrm{d}s+\frac{\kappa-\alpha}{2}\mathbb{E}\sum_{\iota=1}^{N}\int_0^{\tau}[\partial_s\delta_q]^2_{\iota-\frac{1}{2}}\,\mathrm{d}s\geq 0.
\end{aligned}
$$

According to Gronwall's inequality and $\alpha \geq (2+\kappa+3\alpha/2+B_1)\epsilon^2$, Eq. (5.27) simplifies as

$$
\begin{aligned}
& \mathbb{E}\|\delta_\upsilon(\cdot,\tau)\|^2+\left(\frac{\kappa}{2}-\alpha^2-2\epsilon^2\right)\mathbb{E}\|\delta_p(\cdot,\tau)\|^2 \\
& \quad +\mathbb{E}\|\partial_\tau\delta_\upsilon(\cdot,\tau)\|^2+\frac{1}{4}\mathbb{E}\|\delta_q(\cdot,\tau)\|^2\leq Ch^{2n+2}.
\end{aligned} \tag{5.28}
$$

By using the projection properties, the projection errors also obtain

$$
\begin{aligned}
& \mathbb{E}\|\theta_\upsilon(\cdot,\tau)\|^2+\left(\frac{\kappa}{2}-\alpha^2-2\epsilon^2\right)\mathbb{E}\|\theta_p(\cdot,\tau)\|^2 \\
& \quad +\mathbb{E}\|\partial_\tau\theta_\upsilon(\cdot,\tau)\|^2+\frac{1}{4}\mathbb{E}\|\theta_q(\cdot,\tau)\|^2\leq Ch^{2n+2}.
\end{aligned} \tag{5.29}
$$

By combining Eq. (5.28) with Eq. (5.29), the desired estimate is obtained, which completes the proof. $\square$

6. Numerical experiments

This section presents two numerical examples of stochastic linear and nonlinear KdV equations to demonstrate the efficiency and optimal error estimates of the proposed LDG method.

The total error is separated as follows, where the first term on the right-hand side represents the spatial error, and the second term corresponds to the temporal error

$$
\|\upsilon(\cdot,\tau_m)-\upsilon_h^m\|\leq\|\upsilon(\cdot,\tau_m)-\upsilon_h(\cdot,\tau_m)\|+\|\upsilon_h(\tau_m)-\upsilon_h^m\|.
$$

Time discretization is carried out by using the implicit midpoint method for the matrix-form stochastic ordinary differential equation obtained by the spatial semi-discretization of the SPDE with the LDG method

$$
Y_h^{m+1}(\tau+\Delta\tau)=Y^m(\tau)+\Delta\tau F\left(\tau_{m+\frac{1}{2}},Y^{m+\frac{1}{2}}(\tau)\right)+\Delta W_m G\left(\tau_{m+\frac{1}{2}},Y^{m+\frac{1}{2}}(\tau)\right),
$$

where $\Delta\tau$ represents time size, then $M=T/\Delta\tau$ represents the number of time points.

In these numerical experiments, a specific realization of the Gaussian-Legendre basis functions in Eq. (3.4) is adopted on $\chi \in I_\iota$, with the expressions for $k = 0, 1, 2$ providing as follows:

$$\phi_0^\iota(\chi) = 1, \quad \phi_1^\iota(\chi) = \frac{\chi - \chi_\iota}{h}, \quad \phi_2^\iota(\chi) = \frac{(\chi - \chi_\iota)^2}{h^2} - \frac{1}{12}.$$

Example 6.1. The stochastic linear KdV equation under consideration is

$$\begin{cases} \mathrm{d}\upsilon = (\kappa \upsilon_{\chi\chi\chi} + \alpha \upsilon_{\chi\chi} + \beta \upsilon_\chi + \gamma \upsilon)\mathrm{d}\tau \\ \qquad +(a\upsilon + b\upsilon_\chi)\mathrm{d}W_\tau, & (\chi, \tau) \in [0, \Gamma] \times [0, T], \\ \upsilon(\chi, 0) = \sin(\chi), & \chi \in [0, \Gamma], \end{cases} \tag{6.1}$$

where $\Gamma = 2\pi$, $T = 100$. Although this work pay less attention on preserving structure properties to some extend, the following parameter relations introduced in [17] are adopted, under which energy conservation in the expected sense was demonstrated

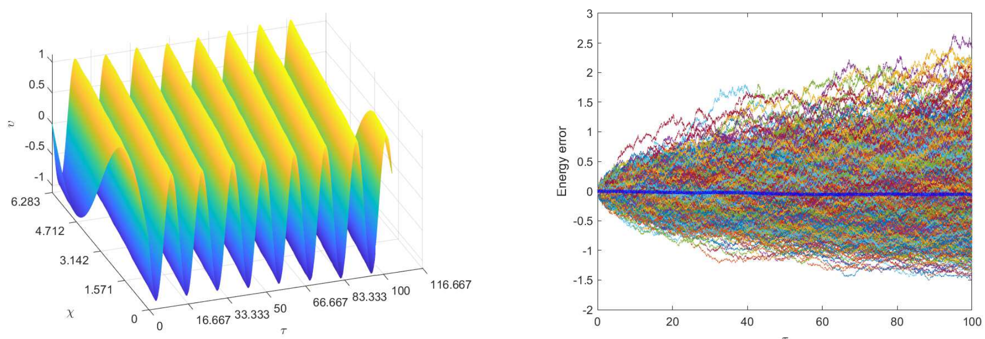
$$\alpha = \frac{1}{2}b^2, \quad \gamma = -\frac{1}{2}a^2, \quad \beta = -ab.$$

Taking $\kappa = 0.5$, and the noise coefficients $a = 0.01$ and $b = 0.01$, the remaining parameters are determined accordingly. The above parameters satisfy the theoretical constraints in Theorem 5.1, ensuring consistency between the theoretical assumptions and simulations. Then the LDG method is applied to solve Eq. (6.1) with polynomial degrees $n = 1$ and $n = 2$, by using a time step size $\Delta\tau = 1/125$ and spatial step size $h = \pi/40$.

Figs. 1 and 2 present the phase diagrams and the evolution of energy error over time for the cases $n = 1$ and $n = 2$ respectively. In particular, the thick blue line in Fig. 1(b) illustrates the evolution of the average energy error, defined as

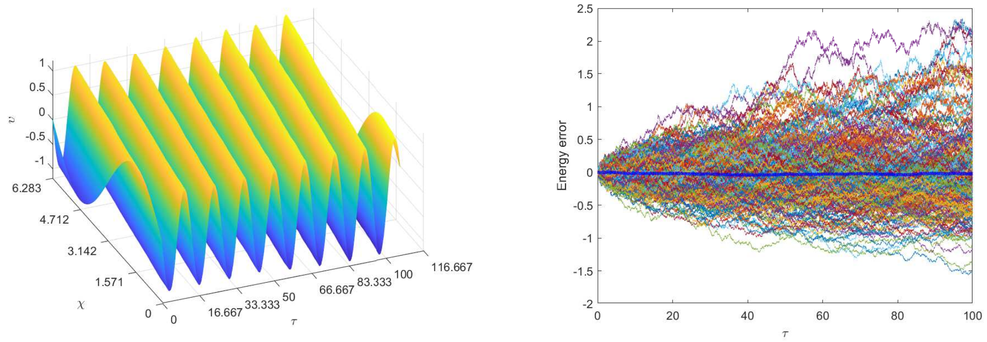
$$H_{error} = \mathbb{E}\left(\|\upsilon_h(\cdot, \tau)\|^2\right) - \|\upsilon_h(\cdot, 0)\|^2,$$

computed over 10^3 sample paths of Brownian motion.



(a) Phase space trajectory (b) Evolution of energy norm error

Figure 1: Phase trajectory and energy error obtained by the LDG method for Eq. (6.1) with $n = 1$, $h = \pi/40$, $\Delta\tau = 1/125$.



(a) Phase space trajectory (b) Evolution of energy norm error

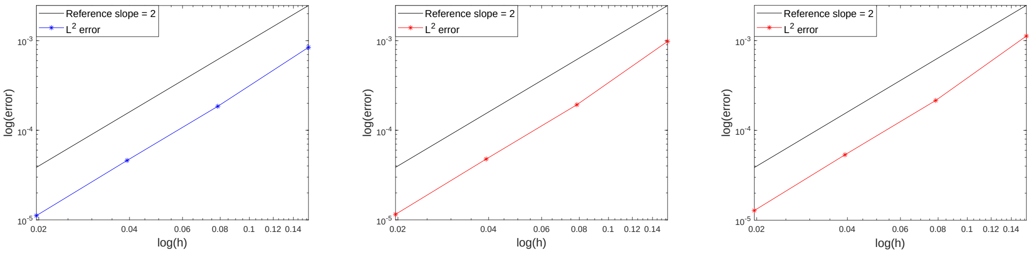
Figure 2: Phase trajectory and energy error obtained by the LDG method for Eq. (6.1) with $n = 2$, $h = \pi/40$, $\Delta\tau = 1/125$.

From Figs. 1 and 2, it is observed that the numerical solution preserves a nearly constant average energy over time, indicating that the numerical method (3.3) preserves the energy in the expected sense.

In Table 1, the $\mathcal{L}^2$ errors for spatial discretizations with mesh sizes $\pi/20$, $\pi/40$, $\pi/80$ and $\pi/160$ are listed, where the numerical solution computed with $h = \pi/320$ serves as the reference. From Table 1, numerical results are presented for the $\mathcal{L}^2$ errors and corresponding convergence orders for polynomial degrees $n = 1$ and $n = 2$ at different final times $T = 0.25, 0.5$, and 1. In all cases, the numerical method achieves the expected convergence rates that second order is for $n = 1$ and third order is for $n = 2$, demonstrating the optimal performance of the LDG method across different time horizons.

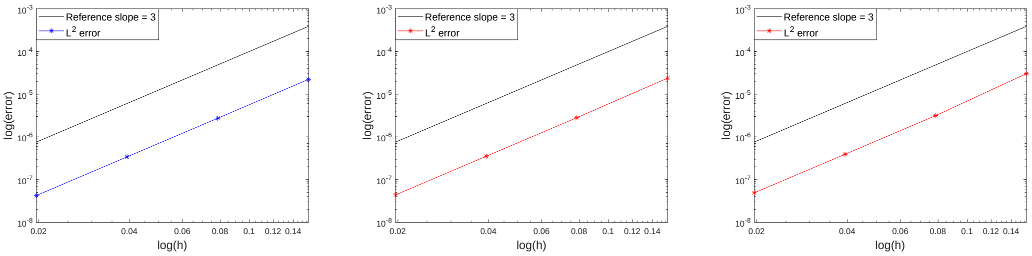
Table 1: Mean-square convergence rates of the LDG method for Eq. (6.1).

T	N	$n=1$ $\mathcal{L}^2$ error	order	$n=2$ $\mathcal{L}^2$ error	order
$T=0.25$	20	8.4240E-04	/	2.2183E-05	/
	40	1.8511E-04	2.1861	2.7343E-06	3.0202
	80	4.6053E-05	2.0070	3.4176E-07	3.0001
	160	1.1154E-05	2.0458	4.2433E-08	3.0097
$T=0.5$	20	9.8217E-04	/	2.3769E-05	/
	40	1.9279E-04	2.3490	2.8224E-06	3.0741
	80	4.7731E-05	2.0140	3.5256E-07	3.0010
	160	1.1521E-05	2.0506	4.3890E-08	3.0059
$T=1$	20	1.1234E-03	/	3.0204E-05	/
	40	2.1530E-04	2.3835	3.1520E-06	3.2604
	80	5.3495E-05	2.0089	3.9200E-07	3.0074
	160	1.2814E-05	2.0617	4.9146E-08	2.9957



(a) $T = 0.25$ (b) $T = 0.5$ (c) $T = 1$

Figure 3: Mean-square convergence curve of LDG method for different T with $n = 1$.



(a) $T = 0.25$ (b) $T = 0.5$ (c) $T = 1$

Figure 4: Mean-square convergence curve of LDG method for different T with $n = 2$.

In Figs. 3 and 4, the mean-square convergence curves of the LDG method for different final times T are displayed, corresponding to polynomial degrees $n = 1$ and $n = 2$ respectively. From Figs. 3 and 4, it is observed that LDG method achieves expected $(n+1)$ order accuracy, which is consistent with Theorem 5.1.

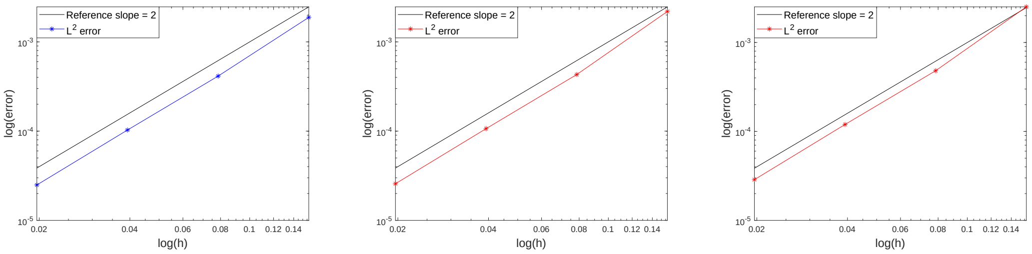
Example 6.2. Considering stochastic nonlinear KdV equation

$$\begin{cases} \mathrm{d}\upsilon = (\kappa\upsilon_{\chi\chi\chi} + \alpha\upsilon_{\chi\chi} + \beta\sin\upsilon + \gamma\upsilon)\mathrm{d}\tau & \\ \qquad +(a\upsilon + b\upsilon_{\chi})\mathrm{d}W_{\tau}, & (\chi,\tau) \in (0,\Gamma)\times[0,T], \\ \upsilon_0(\chi) = \sin(\chi), & \chi \in (0,\Gamma), \end{cases} \tag{6.2}$$

where $\Gamma = 2\pi$, the parameters for (6.2) are selected to be the same as in Example 6.1, which also satisfy the theoretical constraints in Theorem 5.1.

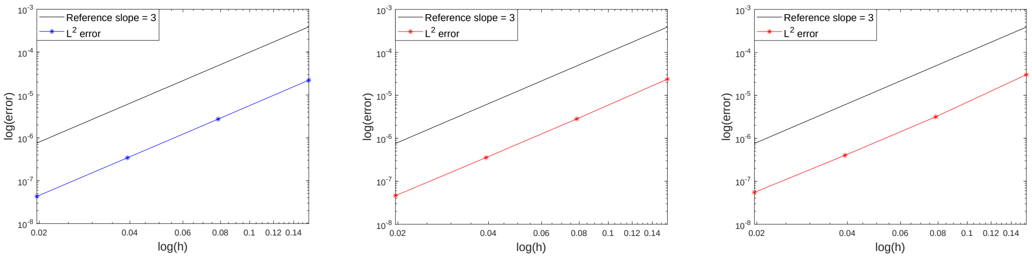
In Table 2, the $\mathcal{L}^2$ errors for spatial discretizations with mesh sizes $\pi/20$, $\pi/40$, $\pi/80$ and $\pi/160$ are presented, where the numerical solution computed with $h = \pi/320$ serves as the reference. Table 2 confirms that the LDG method achieves optimal convergence rates for various final times: second order when $n = 1$ and third order when $n = 2$.

In Figs. 5 and 6, the mean-square convergence curves of the LDG method are displayed for different final times T, with polynomial degrees $n = 1$ and $n = 2$ respectively. From Figs. 5 and 6, LDG method also has $(n+1)$ order convergence, which complies with Theorem 5.1.



(a) $T = 0.25$ (b) $T = 0.5$ (c) $T = 1$

Figure 5: Mean-square convergence curve of LDG method for different T with $n = 1$.



(a) $T = 0.25$ (b) $T = 0.5$ (c) $T = 1$

Figure 6: Mean-square convergence curve of LDG method for different T with $n = 2$.

Table 2: Mean-square convergence rates of the LDG method for Eq. (6.2).

T	$n=1$		$n=2$		
	N	$\mathcal{L}^2$ error	order	$\mathcal{L}^2$ error	order
$T = 0.25$	20	1.8818E-03	/	2.2192E-05	/
	40	4.1362E-04	2.1857	2.7363E-06	3.0197
	80	1.0292E-04	2.0068	3.4291E-07	2.9963
	160	2.4927E-05	2.0457	4.3121E-08	2.9914
$T = 0.5$	20	2.1929E-03	/	2.3771E-05	/
	40	4.3065E-04	2.3483	2.8253E-06	3.0727
	80	1.0667E-04	2.0134	3.5608E-07	2.9882
	160	2.5752E-05	2.0503	4.6134E-08	2.9483
$T = 1$	20	2.5156E-03	/	3.0214E-05	/
	40	4.8207E-04	2.3836	3.1612E-06	3.2567
	80	1.1978E-04	2.0089	4.0227E-07	2.9743
	160	2.8710E-05	2.0608	5.5091E-08	2.8683

7. Conclusions

This work develops a spatially semi-discrete LDG method for solving the stochastic nonlinear KdV equation. The proposed method is shown to be $\mathcal{L}^2$-stable, and its optimal mean-square convergence rate of order $n+1$ is established. By incorporating an implicit midpoint method for time integration, we conduct a series of numerical experiments whose results confirm both the accuracy of the method and its consistency with the error estimates. This framework provides a reliable tool for solving high-order SPDEs, and the flexibility of the LDG method contributes to potential extensions to broader classes of stochastic systems.

Acknowledgements

This work was supported by the Natural Science Foundation of Shandong Province of China (Grant No. ZR2022QA051) and by the China Scholarship Council (Grant No. 202406120186).

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