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Novel Conformal Structure-Preserving Algorithms for Coupled Damped Nonlinear Schrödinger System

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Abstract. This paper introduces two novel conformal structure-preserving algorithms for solving the coupled damped nonlinear Schrödinger (CDNLS) system, which are based on the conformal multi-symplectic Hamiltonian formulation and its conformal conservation laws. The proposed algorithms can preserve corresponding conformal multi-symplectic conservation law and conformal momentum conservation law in any local time-space region, respectively. Moreover, it is further shown that the algorithms admit the conformal charge conservation law, and exactly preserve the dissipation rate of charge under appropriate boundary conditions. Numerical experiments are presented to demonstrate the conformal properties and effectiveness of the proposed algorithms during long-time numerical simulations and validate the analysis.

AMS subject classifications: 35Q55, 37K05, 37M15

Key words: Conformal conservation laws, conformal structure-preserving algorithms, coupled damped nonlinear Schrödinger system, dissipation rate of charge.

1 Introduction

A Bose-Einstein condensate (BEC) is a state of matter of bosons confined in an external potential and cooled to temperatures very near to absolute zero. Under such conditions, a large fraction of the atoms collapse into the lowest quantum state of the external potential, at which point quantum effects become apparent on a macroscopic scale. It was first predicted in 1924. Attention has recently broadened to include exploration of quantized vortex states and their dynamics associated with superfluidity [1, 2], and of systems of two or more condensates [3]. By a mean-field approximation, the state of

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the BEC can be described by the wave function of the condensate to dilute systems. At temperatures much smaller than the critical temperature for a two-component BEC, its wave function can be well described by two coupled nonlinear Schrödinger (CNLS) system [4–7]. The CNLS system also models beam propagation inside crystals or photo refractive as well as water wave interactions. Solitary waves in this system are often called vector solutions in the literature as they generally contain two components. It has been shown that, in addition to passing-through collision, vector solutions can also bounce off each other or trap each other [8]. As for numerical methods focusing on this type of problem, there have been a great deal of methods to solve it [9, 10]. Symplectic and multi-symplectic methods, which can preserve the geometric structures of the original problem under appropriate discretizations, have been paid attentions in recent decades [11–16]. In [17], the authors study the multi-symplectic Preissman scheme for CNLS system. Aydın and Karasözen [18], consider the integration of CNLS equation with soliton solutions by a multi-symplectic six-point scheme. The CNLS system can be split into a linear multi-symplectic subsystem and a nonlinear Hamiltonian subsystem, then authors [19–21] discuss the multi-symplectic splitting methods for the problem. Besides the multi-symplectic conservation law, multi-symplectic Hamiltonian partial differential equations (PDEs) also have the energy and momentum conservation laws which play a crucial role in conservative PDEs. Wang et al. [22] propose the concept of local structure-preserving algorithms for PDEs, which are the natural generalization of the corresponding global structure-preserving algorithms. Then, Cai et al. [23, 24] and Gong et al. [25] generalize the idea of local structure-preserving algorithms and propose a lot of energy-preserving algorithms and momentum-preserving algorithms to solve multi-symplectic PDEs. Later, Li and Wu [26] investigate a general approach to constructing local energy-preserving algorithms which can be of arbitrarily high order in time for solving Hamiltonian PDEs, including the CNLS system.

We consider the two coupled damped nonlinear Schrödinger (CDNLS) system of the form

$$\begin{cases} i(\phi_t+\delta\phi_x)+\alpha\phi_{xx}+(|\phi|^2+\gamma|\psi|^2)\phi+i\dfrac{a}{2}\phi=0,\\ i(\psi_t-\delta\psi_x)+\alpha\psi_{xx}+(\gamma|\phi|^2+|\psi|^2)\psi+i\dfrac{b}{2}\psi=0, \end{cases} \tag{1.1}$$

where ϕ and ψ are complex amplitudes or "envelopes" of two wave packets, i is the imaginary unit, $x\in[x_L,x_R]$ and t are the space and time variables, respectively. The parameter δ is the normalized strength of the linear birefringent, γ is the cross-phase modulation coefficient which describes the minimum approximation of the transmission of light wave, and $a,b\geq 0$ are damping coefficients. As this linearly damped system, McLachlan and Perlmutter develop a reduction conformal theory and show that conformal symplectic methods generally preserve the conformal symplectic structure [27]. Then McLachlan and Quispel [28] extend this theory and construct some numerical conformal methods that preserve conformal properties, such as symplecticity and volume-preservation. Subsequently, Moore generalizes these results to multi-symplectic PDEs, and proposes a conformal multi-symplectic scheme for forced-damped semi-linear wave equation [29]. Later, Moore et al. [30] derive conformal energy, momentum and other quantities that arise from linear symmetries conservation laws and develop conformal Preissman box scheme and conformal discrete gradient methods. These methods are proven to preserve the conformal conservation laws exactly. Comparing with standard structure-preserving algorithms, conformal structure-preserving algorithms preserve the dissipation rate exactly. Our aim in this article is to develop numerical methods that not only preserve a conformal property, but also conserve the dissipation rate for the CDNLS system (1.1).

The rest of the paper is organized as follows: In Section 2, the damped multi-symplectic formulation of CDNLS system and some of its conformal conservation laws are presented. In Section 3, some necessary operators together with their properties are given. Then, we derive a conformal multi-symplectic algorithm and a conformal momentum-preserving algorithm for CDNLS system. In addition, we also prove some conservative properties of the proposed algorithms. Numerical experiments for the solitary wave solutions are presented in Section 4, which show the effectiveness and advantages of these algorithms. Finally, we make conclusions in Section 5.

2 Damped multi-symplectic Hamiltonian PDE and conformal conservation laws

By letting

$$\phi=q_1+iq_2,\quad \psi=q_3+iq_4,$$

system (1.1) can be rewritten as a real-value system

$$\begin{cases} -(q_2)_t-\delta(q_2)_x+\alpha(q_1)_{xx}+(|\phi|^2+\gamma|\psi|^2)q_1-\dfrac{a}{2}q_2=0,\\ (q_1)_t+\delta(q_1)_x+\alpha(q_2)_{xx}+(|\phi|^2+\gamma|\psi|^2)q_2+\dfrac{a}{2}q_1=0,\\ -(q_4)_t+\delta(q_4)_x+\alpha(q_3)_{xx}+(\gamma|\phi|^2+|\psi|^2)q_3-\dfrac{b}{2}q_4=0,\\ (q_3)_t-\delta(q_3)_x+\alpha(q_4)_{xx}+(\gamma|\phi|^2+|\psi|^2)q_4+\dfrac{b}{2}q_3=0. \end{cases} \tag{2.1}$$

Introducing the auxiliary variables

$$\alpha\phi_x=p_1+ip_2,\quad \alpha\psi_x=p_3+ip_4,$$

we can rewrite the above system as a first-order system

$$\begin{cases} -(q_2)_t-\delta(q_2)_x+(p_1)_x+(|\phi|^2+\gamma|\psi|^2)q_1-\dfrac{a}{2}q_2=0, \\ (q_1)_t+\delta(q_1)_x+(p_2)_x+(|\phi|^2+\gamma|\psi|^2)q_2+\dfrac{a}{2}q_1=0, \\ -(q_4)_t+\delta(q_4)_x+(p_3)_x+(\gamma|\phi|^2+|\psi|^2)q_3-\dfrac{b}{2}q_4=0, \\ (q_3)_t-\delta(q_3)_x+(p_4)_x+(\gamma|\phi|^2+|\psi|^2)q_4+\dfrac{b}{2}q_3=0, \\ (q_1)_x=\dfrac{1}{\alpha}p_1, \\ (q_2)_x=\dfrac{1}{\alpha}p_2, \\ (q_3)_x=\dfrac{1}{\alpha}p_3, \\ (q_4)_x=\dfrac{1}{\alpha}p_4. \end{cases} \tag{2.2}$$

Actually, system (2.2) can be written as a damped multi-symplectic formulation [30]

$$M\mathbf{z}_t+K\mathbf{z}_x=\nabla_{\mathbf{z}}S(\mathbf{z})+D\mathbf{z}, \tag{2.3}$$

where

$$\begin{aligned} &\mathbf{z}=[q_1,q_2,q_3,q_4,p_1,p_2,p_3,p_4]^T, \\ &S(\mathbf{z})=\frac{1}{4}[(q_1^2+q_2^2)^2+(q_3^2+q_4^2)^2]+\frac{\gamma}{2}(q_1^2+q_2^2)(q_3^2+q_4^2)+\frac{1}{2\alpha}(p_1^2+p_2^2+p_3^2+p_4^2), \end{aligned}$$

and

$$M=\begin{pmatrix} \mathbf{J} & & & \\ & \mathbf{J} & & \\ & & 0 & \\ & & & 0 \end{pmatrix}_{8\times 8}, \quad K=\begin{pmatrix} \delta\mathbf{J} & & -\mathbf{I} & \\ & \delta\mathbf{J} & & -\mathbf{I} \\ \mathbf{I} & & & \\ & \mathbf{I} & & \end{pmatrix}_{8\times 8}, \quad D=\begin{pmatrix} \frac{a}{2}\mathbf{J} & & & \\ & \frac{b}{2}\mathbf{J} & & \\ & & 0 & \\ & & & 0 \end{pmatrix}_{8\times 8},$$

where $\mathbf{I}$ is the 2×2 identity matrix and

$$\mathbf{J}=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Notice that, S dose not depend on the damping coefficients a, b in this case.

The damped multi-symplectic formulation (2.3) can be split into conservative part Ψ (multi-symplectic formulation)

$$M\mathbf{z}_t+K\mathbf{z}_x=\nabla_{\mathbf{z}}S(\mathbf{z}), \tag{2.4}$$

and dissipative part Φ

$$M\mathbf{z}_t = D\mathbf{z}. \tag{2.5}$$

For the conservative part, we apply the standard geometric integrator to discrete it, and obtain $z^{n+1} = \Psi_{\Delta t}(z^n)$. For the dissipative part, we can solve it exactly, which may be stated as $z^{n+1} = \Phi_{\Delta t}(z^n)$. Thus, the formulation (2.3) may be solved by composing flow maps $\Phi_{\Delta t} \circ \Psi_{\Delta t}$.

Following the terminology of McLachlan and Perlmutter [27], the conformal conservation law has a general form with a linear dissipation

$$\partial_t P + \partial_x Q = -a_1 P - a_2 Q, \tag{2.6}$$

where a_1, a_2 are non-negative real numbers, P represents a density and Q is the associated flux. This is a local property, which is independent of the boundary conditions. Next, we give the conformal conservation law of CDNLS system (1.1).

Proposition 2.1. The CDNLS (1.1) system admits the conformal multi-symplectic conservation law (CMSCL)

$$-a dq_1 \wedge dq_2 - b dq_3 \wedge dq_4 = \partial_t \omega + \partial_x \kappa, \tag{2.7}$$

where

$$\begin{aligned} \omega &= d\mathbf{z} \wedge \frac{1}{2} M d\mathbf{z} = dq_1 \wedge dq_2 + dq_3 \wedge dq_4, \\ \kappa &= d\mathbf{z} \wedge \frac{1}{2} K d\mathbf{z} = dq_1 \wedge \delta dq_2 + dq_3 \wedge \delta dq_4 + dp_1 \wedge dq_1 + dp_2 \wedge dq_2 + dp_3 \wedge dq_3 + dp_4 \wedge dq_4, \end{aligned}$$

and the conformal momentum conservation law (CMCL)

$$\partial_t I + \partial_x G = -\frac{a}{2}(q_1 p_2 - q_2 p_1) - \frac{b}{2}(q_3 p_4 - q_4 p_3), \tag{2.8}$$

where

$$\begin{aligned} I &= -\frac{1}{2}\mathbf{z}_x^T M \mathbf{z} = \frac{1}{2}(q_1 p_2 - q_2 p_1 + q_3 p_4 - q_4 p_3), \\ G &= S(z) + \frac{1}{2}\mathbf{z}_t^T M \mathbf{z} = S(\mathbf{z}) + \frac{1}{2}((q_1)_t q_2 - (q_2)_t q_1 + (q_3)_t q_4 - (q_4)_t q_3). \end{aligned}$$

The proof is similar to some references [29, 30] and is thus omitted here.

Proposition 2.2. The CDNLS (1.1) system admits the conformal charge conservation law (CCCL)

$$-a(q_1^2 + q_2^2) = \partial_t (q_1^2 + q_2^2) + \partial_x (\delta(q_1^2 + q_2^2) + 2\alpha(q_1 (q_2)_x - q_2 (q_1)_x)), \tag{2.9a}$$

$$-b(q_3^2 + q_4^2) = \partial_t (q_3^2 + q_4^2) + \partial_x (\delta(q_3^2 + q_4^2) + 2\alpha(q_3 (q_4)_x - q_4 (q_3)_x)), \tag{2.9b}$$

or equivalently

$$-a(q_1^2+q_2^2)-b(q_3^2+q_4^2)=\partial_t(q_1^2+q_2^2+q_3^2+q_4^2)+\partial_x[\delta(q_1^2+q_2^2+q_3^2+q_4^2)\\+2\alpha(q_1(q_2)_x-q_2(q_1)_x+q_3(q_4)_x-q_4(q_3)_x)]. \tag{2.10}$$

With periodic boundary conditions, it also admits the global conformal charge conservation law (GCCCL)

$$N_\phi(t)=\int_{x_L}^{x_R}|\phi(x,t)|^2dx=\exp(-at)N_\phi(0), \tag{2.11a}$$

$$N_\psi(t)=\int_{x_L}^{x_R}|\psi(x,t)|^2dx=\exp(-bt)N_\psi(0). \tag{2.11b}$$

Proof. Multiplying the first line of system (2.1) by q_1 and the second line of system (2.1) by $-q_2$ gives

$$0=(q_1)_tq_1+\delta(q_1)_xq_1+\alpha(q_2)_{xx}q_1+(q_1^2+q_2^2+\gamma(q_3^2+q_4^2))q_1q_2+\frac{a}{2}q_1^2, \tag{2.12a}$$

$$0=(q_2)_tq_2+\delta(q_2)_xq_2-\alpha(q_1)_{xx}q_2-(q_1^2+q_2^2+\gamma(q_3^2+q_4^2))q_1q_2+\frac{a}{2}q_2^2. \tag{2.12b}$$

Adding Eq. (2.12a) and Eq. (2.12b) leads to

$$\begin{aligned}-\frac{a}{2}(q_1^2+q_2^2)&=(q_1)_tq_1+(q_2)_tq_2+\delta[(q_1)_xq_1+(q_2)_xq_2]+\alpha[(q_2)_{xx}q_1-(q_1)_{xx}q_2]\\&=\frac{1}{2}\partial_t(q_1^2+q_2^2)+\frac{1}{2}\partial_x[\delta(q_1^2+q_2^2)+2\alpha(q_1(q_2)_x-q_2(q_1)_x)],\end{aligned}$$

so we obtain Eq. (2.9a). Similarly, Eq. (2.9b) can also be obtained.

Integrating Eqs. (2.9a) and (2.9b) with respect to x with periodic boundary conditions gives

$$\partial_t\int_{x_L}^{x_R}|\phi(x,t)|^2dx=-a\int_{x_L}^{x_R}|\phi(x,t)|^2dx, \tag{2.13a}$$

$$\partial_t\int_{x_L}^{x_R}|\psi(x,t)|^2dx=-b\int_{x_L}^{x_R}|\psi(x,t)|^2dx, \tag{2.13b}$$

therefore we obtain GCCCL (2.11a), (2.11b), which completes the proof. □

3 Structure-preserving algorithms for CDNLS system

In this section, some operators together with their properties are briefly introduced. Then a conformal multi-symplectic (CMS) algorithm and a conformal momentum-preserving (CMP) algorithm are developed for CDNLS system (1.1).

3.1 Operator definitions and properties

In order to derive the algorithms conveniently, we give some notations and operator definitions. We introduce a uniform grid $(x_j,t_n)\in R^2$ with mesh-length Δt in time direction and mesh-length Δx in space direction, and denote the value of the function $z(x,t)$ at the mesh point (x_j,t_n) by z_j^n, where $x_j=x_L+j\Delta x$, $j=0,\cdots,J$, $\Delta x=(x_R-x_L)/J$ and $t_n=n\Delta t$, $n=0,\cdots,N$. Define finite difference operators

$$D_t^{\frac{\beta}{2}}z^n=\frac{z^{n+1}-\exp(-\frac{\beta}{2}\Delta t)z^n}{\Delta t},\qquad D_x^{\frac{\beta}{2}}z_j=\frac{z_{j+1}-\exp(-\frac{\beta}{2}\Delta x)z_j}{\Delta x},$$

and averaging operators

$$A_t^{\frac{\beta}{2}}z^n=\frac{z^{n+1}+\exp(-\frac{\beta}{2}\Delta t)z^n}{2},\qquad A_x^{\frac{\beta}{2}}z_j=\frac{z_{j+1}+\exp(-\frac{\beta}{2}\Delta x)z_j}{2},$$

where $\beta\geq 0$. If we take $\beta=0$, then

$$D_t^{\frac{\beta}{2}}=D_t,\quad D_x^{\frac{\beta}{2}}=D_x,$$

are standard finite difference operators and

$$A_t^{\frac{\beta}{2}}=A_t,\quad A_x^{\frac{\beta}{2}}=A_x,$$

are standard averaging operators. Thus, we have the following properties [30,31]:

(i) Commutative law

$$\begin{aligned}
&D_t^{\frac{\beta}{2}}D_x^{\frac{\eta}{2}}z_j^n=D_x^{\frac{\eta}{2}}D_t^{\frac{\beta}{2}}z_j^n, &&A_t^{\frac{\beta}{2}}A_x^{\frac{\eta}{2}}z_j^n=A_x^{\frac{\eta}{2}}A_t^{\frac{\beta}{2}}z_j^n,\\
&D_t^{\frac{\beta}{2}}A_x^{\frac{\eta}{2}}z_j^n=A_x^{\frac{\eta}{2}}D_t^{\frac{\beta}{2}}z_j^n, &&A_t^{\frac{\beta}{2}}D_x^{\frac{\eta}{2}}z_j^n=D_x^{\frac{\eta}{2}}A_t^{\frac{\beta}{2}}z_j^n.
\end{aligned}$$

(ii) Generalized discrete Leibnitz rule

$$\begin{aligned}
D_x^{\beta}(f\cdot g)_j&=D_x^{\frac{\beta}{2}}f_j\cdot A_x^{\frac{\beta}{2}}g_j+A_x^{\frac{\beta}{2}}f_j\cdot D_x^{\frac{\beta}{2}}g_j,\\
D_t^{\beta}(f\cdot g)_j&=D_t^{\frac{\beta}{2}}f_j\cdot A_t^{\frac{\beta}{2}}g_j+A_t^{\frac{\beta}{2}}f_j\cdot D_t^{\frac{\beta}{2}}g_j.
\end{aligned}$$

These discrete Leibnitz rules are essential for proving a method preserves any of the conformal conservation laws mentioned in this article.

3.2 Conformal multi-symplectic integrator

In this section, employing the difference and averaging operators, the implicit midpoint rule applied to time and space derivatives of system (2.2) yields

$$\left\{\begin{array}{l} D_t^{\frac{a}{2}}A_x(q_1)_j^n+\delta A_t^{\frac{a}{2}}D_x(q_1)_j^n+A_t^{\frac{a}{2}}D_x(p_2)_j^n+(|A_t^{\frac{a}{2}}A_x\phi_j^n|^2+\gamma|A_t^{\frac{b}{2}}A_x\psi_j^n|^2)\cdot A_t^{\frac{a}{2}}A_x(q_2)_j^n=0,\\ -D_t^{\frac{a}{2}}A_x(q_2)_j^n-\delta A_t^{\frac{a}{2}}D_x(q_2)_j^n+A_t^{\frac{a}{2}}D_x(p_1)_j^n+(|A_t^{\frac{a}{2}}A_x\phi_j^n|^2+\gamma|A_t^{\frac{b}{2}}A_x\psi_j^n|^2)\cdot A_t^{\frac{a}{2}}A_x(q_1)_j^n=0,\\ D_t^{\frac{b}{2}}A_x(q_3)_j^n-\delta A_t^{\frac{b}{2}}D_x(q_3)_j^n+A_t^{\frac{b}{2}}D_x(p_4)_j^n+(|A_t^{\frac{b}{2}}A_x\psi_j^n|^2+\gamma|A_t^{\frac{a}{2}}A_x\phi_j^n|^2)\cdot A_t^{\frac{b}{2}}A_x(q_4)_j^n=0,\\ -D_t^{\frac{b}{2}}A_x(q_4)_j^n+\delta A_t^{\frac{b}{2}}D_x(q_4)_j^n+A_t^{\frac{b}{2}}D_x(p_3)_j^n+(|A_t^{\frac{b}{2}}A_x\psi_j^n|^2+\gamma|A_t^{\frac{a}{2}}A_x\phi_j^n|^2)\cdot A_t^{\frac{b}{2}}A_x(q_3)_j^n=0,\\ A_t^{\frac{a}{2}}A_x(p_1)_j^n=\alpha A_t^{\frac{a}{2}}D_x(q_1)_j^n,\\ A_t^{\frac{a}{2}}A_x(p_2)_j^n=\alpha A_t^{\frac{a}{2}}D_x(q_2)_j^n,\\ A_t^{\frac{b}{2}}A_x(p_3)_j^n=\alpha A_t^{\frac{b}{2}}D_x(q_3)_j^n,\\ A_t^{\frac{b}{2}}A_x(p_4)_j^n=\alpha A_t^{\frac{b}{2}}D_x(q_4)_j^n, \end{array}\right. \tag{3.1}$$

where

$$|A_t^{\frac{a}{2}}A_x\phi_j^n|^2=(A_t^{\frac{a}{2}}A_x(q_1)_j^n)^2+(A_t^{\frac{a}{2}}A_x(q_2)_j^n)^2,$$
$$|A_t^{\frac{b}{2}}A_x\psi_j^n|^2=(A_t^{\frac{b}{2}}A_x(q_3)_j^n)^2+(A_t^{\frac{b}{2}}A_x(q_4)_j^n)^2.$$

We can rewrite (3.1) in the compact form

$$MD_t^{\frac{\eta}{2}}A_x\mathbf{z}_j^n+KA_t^{\frac{\eta}{2}}D_x\mathbf{z}_j^n=\nabla_z S(A_t^{\frac{\eta}{2}}A_x\mathbf{z}_j^n), \tag{3.2}$$

where

$$D_t^{\frac{\eta}{2}}\mathbf{z}_j^n=(D_t^{\frac{a}{2}}(q_1)_j^n,D_t^{\frac{a}{2}}(q_2)_j^n,D_t^{\frac{b}{2}}(q_3)_j^n,D_t^{\frac{b}{2}}(q_4)_j^n,D_t^{\frac{a}{2}}(p_1)_j^n,D_t^{\frac{a}{2}}(p_2)_j^n,D_t^{\frac{b}{2}}(p_3)_j^n,D_t^{\frac{b}{2}}(p_4)_j^n)^T,$$
$$A_t^{\frac{\eta}{2}}\mathbf{z}_j^n=(A_t^{\frac{a}{2}}(q_1)_j^n,A_t^{\frac{a}{2}}(q_2)_j^n,A_t^{\frac{b}{2}}(q_3)_j^n,A_t^{\frac{b}{2}}(q_4)_j^n,A_t^{\frac{a}{2}}(p_1)_j^n,A_t^{\frac{a}{2}}(p_2)_j^n,A_t^{\frac{b}{2}}(p_3)_j^n,A_t^{\frac{b}{2}}(p_4)_j^n)^T.$$

Now, we analyze the local and global conformal properties of the proposed scheme.

Theorem 3.1. *The discrete scheme* (3.1) *is a conformal multi-symplectic algorithm and satisfies the discrete conformal multi-symplectic conservation law*

$$D_t^{\alpha}\omega_{j+\frac{1}{2}}^n+D_x\kappa_j^{n+\frac{1}{2}}=0, \tag{3.3}$$

where

$$\omega_{j+\frac{1}{2}}^n=A_x dz_j^n\wedge\frac{1}{2}MA_x dz_j^n,\quad \kappa_j^{n+\frac{1}{2}}=A_t^{\eta}dz_j^n\wedge\frac{1}{2}KA_t^{\eta}dz_j^n.$$

Proof. The variation equation associated of Eq. (3.2) is

$$MD_t^{\frac{\eta}{2}}A_x d\mathbf{z}_j^n + KA_t^{\frac{\eta}{2}}D_x d\mathbf{z}_j^n = \nabla_{\mathbf{zz}}S(A_t^{\frac{\eta}{2}}A_x\mathbf{z}_j^n)A_t^{\frac{\eta}{2}}A_x d\mathbf{z}_j^n. \tag{3.4}$$

Taking the wedge product of Eq. (3.4) with $A_t^{\frac{\eta}{2}}A_x d\mathbf{z}_j^n$ yields

$$A_t^{\frac{\eta}{2}}A_x d\mathbf{z}_j^n \wedge MD_t^{\frac{\eta}{2}}A_x d\mathbf{z}_j^n + A_t^{\frac{\eta}{2}}A_x d\mathbf{z}_j^n \wedge KA_t^{\frac{\eta}{2}}D_x d\mathbf{z}_j^n = 0.$$

Noting that

$$\begin{aligned}
&A_t^{\frac{\eta}{2}}A_x d\mathbf{z}_j^n \wedge MD_t^{\frac{\eta}{2}}A_x d\mathbf{z}_j^n \\
=&\frac{A_x d\mathbf{z}_j^{n+1} + e^{-\frac{\eta}{2}\Delta t}A_x d\mathbf{z}_j^n}{2} \wedge M\frac{A_x d\mathbf{z}_j^{n+1} - e^{-\frac{\eta}{2}\Delta t}A_x d\mathbf{z}_j^n}{\Delta t} \\
=&\frac{1}{2\Delta t}(A_x d\mathbf{z}_j^{n+1} \wedge MA_x d\mathbf{z}_j^{n+1} - e^{-\eta\Delta t}A_x d\mathbf{z}_j^n \wedge MA_x d\mathbf{z}_j^n) \\
=&\frac{1}{2}D_t^{\eta}\omega_{j+\frac{1}{2}}^n.
\end{aligned}$$

Similarly, we have

$$A_t^{\frac{\eta}{2}}A_x d\mathbf{z}_j^n \wedge KA_t^{\frac{\eta}{2}}D_x d\mathbf{z}_j^n = \frac{1}{2}D_x\kappa_j^{n+\frac{1}{2}}.$$

So we complete the proof. □

In general, as a quadratic invariant, the conformal charge conservation law plays an important role in self-focusing of laser in dielectrics, propagation of signals in optical fibers, 1D Heisenberg magnets and so on. Therefore, it is necessary to discuss whether it can be captured.

Theorem 3.2. *The discrete scheme* (3.1) *is a conformal charge-preserving algorithm, which exactly conserves the discrete CCCL*

$$\begin{aligned}
0=&R_N(x_j,t_n) \\
=&D_t^a[(A_x(q_1)_j^n)^2 + (A_x(q_2)_j^n)^2] + D_t^b[(A_x(q_3)_j^n)^2 \\
&+(A_x(q_4)_j^n)^2] + \delta D_x[(A_t^{\frac{a}{2}}(q_1)_j^n)^2 + (A_t^{\frac{a}{2}}(q_2)_j^n)^2 \\
&+(A_t^{\frac{b}{2}}(q_3)_j^n)^2 + (A_t^{\frac{b}{2}}(q_4)_j^n)^2] + 2D_x\left(A_t^{\frac{a}{2}}(q_1)_j^n \cdot A_t^{\frac{a}{2}}(p_2)_j^n - A_t^{\frac{a}{2}}(q_2)_j^n \cdot A_t^{\frac{a}{2}}(p_1)_j^n\right) \\
&+2D_x\left(A_t^{\frac{b}{2}}(q_3)_j^n \cdot A_t^{\frac{b}{2}}(p_4)_j^n - A_t^{\frac{b}{2}}(q_4)_j^n \cdot A_t^{\frac{b}{2}}(p_3)_j^n\right).
\end{aligned} \tag{3.5}$$

Proof. In Eq. (3.1), multiplying the first and second lines by $2A_t^{\frac{a}{2}}A_x(q_1)_j^n$, $2A_t^{\frac{a}{2}}A_x(q_2)_j^n$, respectively, and then, adding them together gives

$$\begin{aligned}0=&2D_t^{\frac{a}{2}}A_x(q_1)_j^n\cdot A_t^{\frac{a}{2}}A_x(q_1)_j^n+2\delta A_t^{\frac{a}{2}}D_x(q_1)_j^n\cdot A_t^{\frac{a}{2}}A_x(q_1)_j^n\\&+2A_t^{\frac{a}{2}}D_x(p_2)_j^n\cdot A_t^{\frac{a}{2}}A_x(q_1)_j^n+2D_t^{\frac{a}{2}}A_x(q_2)_j^n\cdot A_t^{\frac{a}{2}}A_x(q_2)_j^n\\&+2\delta A_t^{\frac{a}{2}}D_x(q_2)_j^n\cdot A_t^{\frac{a}{2}}A_x(q_2)_j^n-2A_t^{\frac{a}{2}}D_x(p_1)_j^n\cdot A_t^{\frac{a}{2}}A_x(q_2)_j^n.\end{aligned}$$

Applying the discrete Leibniz rule leads to

$$\begin{aligned}0=&D_t^a[(A_x(q_1)_j^n)^2+(A_x(q_2)_j^n)^2]+\delta D_x[(A_t^{\frac{a}{2}}(q_1)_j^n)^2\\&+(A_t^{\frac{a}{2}}(q_2)_j^n)^2]+2D_x[A_t^{\frac{a}{2}}(q_1)_j^n\cdot A_t^{\frac{a}{2}}(p_2)_j^n-A_t^{\frac{a}{2}}(q_2)_j^n\cdot A_t^{\frac{a}{2}}(p_1)_j^n].\end{aligned}\tag{3.6}$$

Similarly, multiplying the third and forth lines by $2A_t^{\frac{b}{2}}A_x(q_3)_j^n$, $2A_t^{\frac{b}{2}}A_x(q_4)_j^n$, respectively, we obtain

$$\begin{aligned}0=&D_t^b[(A_x(q_3)_j^n)^2+(A_x(q_4)_j^n)^2]+\delta D_x[(A_t^{\frac{b}{2}}(q_3)_j^n)^2\\&+(A_t^{\frac{b}{2}}(q_4)_j^n)^2]+2D_x[A_t^{\frac{b}{2}}(q_3)_j^n\cdot A_t^{\frac{b}{2}}(p_4)_j^n-A_t^{\frac{b}{2}}(q_4)_j^n\cdot A_t^{\frac{b}{2}}(p_3)_j^n].\end{aligned}\tag{3.7}$$

Adding Eq. (3.6) on Eq. (23), we complete the proof. □

Corollary 3.1. *With the periodic boundary conditions, the algorithm* (3.1) *preserves the global conformal charge exactly, namely*

$$N_\phi^{n+1}=\sum_j\|A_x\phi_j^{n+1}\|^2=e^{-a(n+1)\Delta t}N_\phi^0,\tag{3.8a}$$

$$N_\psi^{n+1}=\sum_j\|A_x\psi_j^{n+1}\|^2=e^{-b(n+1)\Delta t}N_\psi^0.\tag{3.8b}$$

That is to say, the conformal charge-preserving method preserves the dissipation rate of charge exactly.

In order to prove the Corollary 3.1, we only need to sum the discrete conformal conservation law (3.5) over all space index j and apply the periodic boundary conditions. Because of the exponential dissipation of the global conformal charge, we use

$$r_\phi=a+\frac{1}{\Delta t}\ln\frac{N_\phi^{n+1}}{N_\phi^n},\quad r_\psi=b+\frac{1}{\Delta t}\ln\frac{N_\psi^{n+1}}{N_\psi^n},\tag{3.9}$$

to measure the dissipation rate of charge. If $r_\phi=r_\psi=0$, we say the numerical method preserves the dissipation rate of charge. Thus the algorithm we propose conserves the dissipation rate of charge exactly. But the standard structure-preserving algorithms do not, in general, preserve the conformal conservation law and dissipation rate; this is explained and demonstrated as [30].

3.3 Conformal momentum-preserving algorithm

As we know, momentum conservation law is also an important invariant in physics. But there are few conformal momentum-preserving algorithms in the literatures. Therefore, to construct algorithms, which conserve the conformal momentum property, is very essential. Next, we give a conformal momentum-preserving algorithm.

In first order system (2.2), employing the difference and average operators, and applying implicit midpoint rule to time and space derivatives, give the scheme

$$\begin{cases} D_t^{\frac{a}{2}} A_x(q_1)_j^n + \delta A_t^{\frac{a}{2}} D_x(q_1)_j^n + A_t^{\frac{a}{2}} D_x(p_2)_j^n + A_x(|A_t^{\frac{a}{2}}\phi_j^n|^2 + \gamma|A_t^{\frac{b}{2}}\psi_j^n|^2) A_t^{\frac{a}{2}} A_x(q_2)_j^n = 0, \\ -D_t^{\frac{a}{2}} A_x(q_2)_j^n - \delta A_t^{\frac{a}{2}} D_x(q_2)_j^n + A_t^{\frac{a}{2}} D_x(p_1)_j^n + A_x(|A_t^{\frac{a}{2}}\phi_j^n|^2 + \gamma|A_t^{\frac{b}{2}}\psi_j^n|^2) A_t^{\frac{a}{2}} A_x(q_1)_j^n = 0, \\ D_t^{\frac{b}{2}} A_x(q_3)_j^n - \delta A_t^{\frac{b}{2}} D_x(q_3)_j^n + A_t^{\frac{b}{2}} D_x(p_4)_j^n + A_x(|A_t^{\frac{b}{2}}\psi_j^n|^2 + \gamma|A_t^{\frac{a}{2}}\phi_j^n|^2) A_t^{\frac{b}{2}} A_x(q_4)_j^n = 0, \\ -D_t^{\frac{b}{2}} A_x(q_4)_j^n + \delta A_t^{\frac{b}{2}} D_x(q_4)_j^n + A_t^{\frac{b}{2}} D_x(p_3)_j^n + A_x(|A_t^{\frac{b}{2}}\psi_j^n|^2 + \gamma|A_t^{\frac{a}{2}}\phi_j^n|^2) A_t^{\frac{b}{2}} A_x(q_3)_j^n = 0, \\ A_t^{\frac{a}{2}} A_x(p_1)_j^n = \alpha A_t^{\frac{a}{2}} D_x(q_1)_j^n, \\ A_t^{\frac{a}{2}} A_x(p_2)_j^n = \alpha A_t^{\frac{a}{2}} D_x(q_2)_j^n, \\ A_t^{\frac{b}{2}} A_x(p_3)_j^n = \alpha A_t^{\frac{b}{2}} D_x(q_3)_j^n, \\ A_t^{\frac{b}{2}} A_x(p_4)_j^n = \alpha A_t^{\frac{b}{2}} D_x(q_4)_j^n, \end{cases} \tag{3.10}$$

where

$$|A_t^{\frac{a}{2}}\phi_j^n|^2 = (A_t^{\frac{a}{2}}(q_1)_j^n)^2 + (A_t^{\frac{a}{2}}(q_2)_j^n)^2, \quad |A_t^{\frac{b}{2}}\psi_j^n|^2 = (A_t^{\frac{b}{2}}(q_3)_j^n)^2 + (A_t^{\frac{b}{2}}(q_4)_j^n)^2.$$

Remark 3.1. The conformal multi-symplecitc scheme (3.1) and the conformal momentum-preserving scheme (3.10) only differ in the nonlinear term.

Now, we analyze the local and global conformal properties of the proposed scheme (3.10).

Theorem 3.3. *The discrete scheme* (3.10) *is a conformal momentum-preserving algorithm, which exactly conserves the discrete CMCL*

$$R_M(x_j,t_n) = D_t^{\alpha} I_{j+\frac{1}{2}}^n + D_x G_j^{n+\frac{1}{2}} = 0, \tag{3.11}$$

where

$$I_{j+\frac{1}{2}}^n = A_x(q_1)_j^n \cdot D_x(q_2)_j^n - A_x(q_2)_j^n \cdot D_x(q_1)_j^n + A_x(q_3)_j^n \cdot D_x(q_4)_j^n - A_x(q_4)_j^n \cdot D_x(q_3)_j^n,$$

$$
\begin{aligned}
G_j^{n+\frac{1}{2}} =& A_t^{\frac{a}{2}}(q_2)_j^n \cdot D_t^{\frac{a}{2}}(q_1)_j^n - A_t^{\frac{a}{2}}(q_1)_j^n \cdot D_t^{\frac{a}{2}}(q_2)_j^n + A_t^{\frac{b}{2}}(q_4)_j^n \cdot D_t^{\frac{b}{2}}(q_3)_j^n - A_t^{\frac{b}{2}}(q_3)_j^n \cdot D_t^{\frac{b}{2}}(q_4)_j^n \\
&+\frac{1}{\alpha}[(A_t^{\frac{a}{2}}(p_1)_j^n)^2+(A_t^{\frac{a}{2}}(p_2)_j^n)^2+(A_t^{\frac{b}{2}}(p_3)_j^n)^2+(A_t^{\frac{b}{2}}(p_4)_j^n)^2] \\
&+\frac{1}{2}(|A_t^{\frac{a}{2}}\phi_j^n|^4+|A_t^{\frac{b}{2}}\psi_j^n|^4)+\gamma|A_t^{\frac{b}{2}}\psi_j^n|^2\cdot|A_t^{\frac{a}{2}}\phi_j^n|^2.
\end{aligned}
$$

Proof. Multiplying the first and second lines of Eq. (3.10) by $2A_t^{\frac{a}{2}}D_x(q_2)_j^n$, $2A_t^{\frac{a}{2}}D_x(q_1)_j^n$, respectively, and then adding them together, we have

$$
\begin{aligned}
0=&2[D_t^{\frac{a}{2}}A_x(q_1)_j^n\cdot A_t^{\frac{a}{2}}D_x(q_2)_j^n - D_t^{\frac{a}{2}}A_x(q_2)_j^n\cdot A_t^{\frac{a}{2}}D_x(q_1)_j^n \\
&+A_t^{\frac{a}{2}}D_x(p_2)_j^n\cdot A_t^{\frac{a}{2}}D_x(q_2)_j^n + A_t^{\frac{a}{2}}D_x(p_1)_j^n\cdot A_t^{\frac{a}{2}}D_x(q_1)_j^n \\
&+A_x(|A_t^{\frac{a}{2}}\phi_j^n|^2+\gamma|A_t^{\frac{b}{2}}\psi_j^n|^2)(A_t^{\frac{a}{2}}A_x(q_1)_j^n\cdot A_t^{\frac{a}{2}}D_x(q_1)_j^n \\
&+A_t^{\frac{a}{2}}A_x(q_2)_j^n\cdot A_t^{\frac{a}{2}}D_x(q_2)_j^n)].
\end{aligned}
\tag{3.12}
$$

By applying the discrete Leibnitz rule and commutative law, we obtain the first term

$$
\begin{aligned}
&2D_t^{\frac{a}{2}}A_x(q_1)_j^n\cdot A_t^{\frac{a}{2}}D_x(q_2)_j^n - 2D_t^{\frac{a}{2}}A_x(q_2)_j^n\cdot A_t^{\frac{a}{2}}D_x(q_1)_j^n \\
=&D_t^{\frac{a}{2}}A_x(q_1)_j^n\cdot A_t^{\frac{a}{2}}D_x(q_2)_j^n - D_t^{\frac{a}{2}}A_x(q_2)_j^n\cdot A_t^{\frac{a}{2}}D_x(q_1)_j^n \\
&+D_t^{\frac{a}{2}}A_x(q_1)_j^n\cdot A_t^{\frac{a}{2}}D_x(q_2)_j^n - D_t^{\frac{a}{2}}A_x(q_2)_j^n\cdot A_t^{\frac{a}{2}}D_x(q_1)_j^n \\
=&D_t^{\frac{a}{2}}(A_x(q_1)_j^n\cdot D_x(q_2)_j^n - A_x(q_2)_j^n\cdot D_x(q_1)_j^n) \\
&+D_x(A_t^{\frac{a}{2}}(q_2)_j^n\cdot D_t^{\frac{a}{2}}(q_1)_j^n - A_t^{\frac{a}{2}}(q_1)_j^n\cdot D_t^{\frac{a}{2}}(q_2)_j^n),
\end{aligned}
$$

the second term

$$
\begin{aligned}
&2A_t^{\frac{a}{2}}D_x(p_2)_j^n\cdot A_t^{\frac{a}{2}}D_x(q_2)_j^n + 2A_t^{\frac{a}{2}}D_x(p_1)_j^n\cdot A_t^{\frac{a}{2}}D_x(q_1)_j^n \\
=&\frac{2}{\alpha}A_t^{\frac{a}{2}}D_x(p_2)_j^n\cdot A_t^{\frac{a}{2}}A_x(p_2)_j^n + \frac{2}{\alpha}A_t^{\frac{a}{2}}D_x(p_1)_j^n\cdot A_t^{\frac{a}{2}}A_x(p_1)_j^n \\
=&\frac{1}{\alpha}D_x[(A_t^{\frac{a}{2}}(p_1)_j^n)^2+(A_t^{\frac{a}{2}}(p_2)_j^n)^2],
\end{aligned}
$$

and the last term

$$
\begin{aligned}
&2A_x(|A_t^{\frac{a}{2}}\phi_j^n|^2+\gamma|A_t^{\frac{b}{2}}\psi_j^n|^2)(A_t^{\frac{a}{2}}A_x(q_1)_j^n\cdot A_t^{\frac{a}{2}}D_x(q_1)_j^n + A_t^{\frac{a}{2}}A_x(q_2)_j^n\cdot A_t^{\frac{a}{2}}D_x(q_2)_j^n) \\
=&A_x(|A_t^{\frac{a}{2}}\phi_j^n|^2+\gamma|A_t^{\frac{b}{2}}\psi_j^n|^2)\cdot D_x|A_t^{\frac{a}{2}}\phi_j^n|^2 \\
=&\frac{1}{2}D_x|A_t^{\frac{a}{2}}\phi_j^n|^4+\gamma A_x|A_t^{\frac{b}{2}}\psi_j^n|^2\cdot D_x|A_t^{\frac{a}{2}}\phi_j^n|^2.
\end{aligned}
$$

Therefore, the Eq. (3.12) can be reduced to

$$\begin{aligned}0=&D_t^{\frac{a}{2}}[A_x(q_1)_j^n\cdot D_x(q_2)_j^n-A_x(q_2)_j^n\cdot D_x(q_1)_j^n]+D_x[A_t^{\frac{a}{2}}(q_2)_j^n\cdot D_t^{\frac{a}{2}}(q_1)_j^n-A_t^{\frac{a}{2}}(q_1)_j^n\cdot D_t^{\frac{a}{2}}(q_2)_j^n\\&+\frac{1}{\alpha}\left((A_t^{\frac{a}{2}}(p_1)_j^n)^2+(A_t^{\frac{a}{2}}(p_2)_j^n)^2\right)+\frac{1}{2}|A_t^{\frac{a}{2}}\phi_j^n|^4]+\gamma A_x|A_t^{\frac{b}{2}}\psi_j^n|^2\cdot D_x|A_t^{\frac{a}{2}}\phi_j^n|^2.\end{aligned}\tag{3.13}$$

Similarly, multiplying the third and forth lines of Eq. (3.10) by $2A_t^{\frac{b}{2}}D_x(q_4)_j^n$, $2A_t^{\frac{b}{2}}D_x(q_3)_j^n$ respectively, leads to

$$\begin{aligned}0=&D_t^{\frac{b}{2}}[A_x(q_3)_j^n\cdot D_x(q_4)_j^n-A_x(q_4)_j^n\cdot D_x(q_3)_j^n]+D_x[A_t^{\frac{b}{2}}(q_4)_j^n\cdot D_t^{\frac{b}{2}}(q_3)_j^n-A_t^{\frac{b}{2}}(q_3)_j^n\cdot D_t^{\frac{b}{2}}(q_4)_j^n\\&+\frac{1}{\alpha}\left((A_t^{\frac{b}{2}}(p_3)_j^n)^2+(A_t^{\frac{b}{2}}(p_4)_j^n)^2\right)+\frac{1}{2}|A_t^{\frac{b}{2}}\psi_j^n|^4]+\gamma D_x|A_t^{\frac{b}{2}}\psi_j^n|^2\cdot A_x|A_t^{\frac{a}{2}}\phi_j^n|^2.\end{aligned}\tag{3.14}$$

Adding Eq. (3.13) on Eq. (3.14), we complete the proof. ☐

Next, we present the discrete conformal charge conservation law of the CMP algorithm.

Theorem 3.4. *The discrete scheme* (3.10) *is a conformal charge-preserving algorithm, which exactly conserves the discrete CCCL*

$$\begin{aligned}0=&R_N(x_j,t_n)\\=&D_t^a[(A_x(q_1)_j^n)^2+(A_x(q_2)_j^n)^2]+D_t^b[(A_x(q_3)_j^n)^2+(A_x(q_4)_j^n)^2]+\delta D_x[(A_t^{\frac{a}{2}}(q_1)_j^n)^2\\&+(A_t^{\frac{a}{2}}(q_2)_j^n)^2+(A_t^{\frac{b}{2}}(q_3)_j^n)^2+(A_t^{\frac{b}{2}}(q_4)_j^n)^2]+2D_x\left(A_t^{\frac{a}{2}}(q_1)_j^n\cdot A_t^{\frac{a}{2}}(p_2)_j^n\right.\\&\left.-A_t^{\frac{a}{2}}(q_2)_j^n\cdot A_t^{\frac{a}{2}}(p_1)_j^n\right)+2D_x\left(A_t^{\frac{b}{2}}(q_3)_j^n\cdot A_t^{\frac{b}{2}}(p_4)_j^n-A_t^{\frac{b}{2}}(q_4)_j^n\cdot A_t^{\frac{b}{2}}(p_3)_j^n\right).\end{aligned}\tag{3.15}$$

The proof of this theorem is similar to Theorem 3.2 and is thus omitted here. Summing the discrete conformal charge conservation law (3.15) over all space index j and applying the periodic boundary conditions, give the following corollary.

Corollary 3.2. *With the periodic boundary conditions, the scheme* (3.10) *preserves the global conformal charge exactly, namely*

$$N_\phi^{n+1}=\sum_j\|A_x\phi_j^{n+1}\|^2=e^{-a(n+1)\Delta t}N_\phi^0,\tag{3.16a}$$

$$N_\psi^{n+1}=\sum_j\|A_x\psi_j^{n+1}\|^2=e^{-b(n+1)\Delta t}N_\psi^0.\tag{3.16b}$$

That is to say, the conformal momentum-preserving algorithm (3.10) *preserves the dissipation rate of charge exactly.*

4 Numerical experiments

In this section, we conduct some typical experiments to illustrate the performance of the proposed algorithms. All of the solutions are computed with time-step $\Delta t = 0.01$ and periodic boundary conditions in $[-40,40]$. We choose grid number $N=499$ and the time interval is taken as $[0,50]$ for all experiments. To test the conservative properties, we conduct long time simulation for two solitons collision propagation. For the purpose of numerical comparison, we apply the standard multi-symplectic Preissman (MSP) scheme in [17] to the CDNLS system (1.1). In addition, we solve the implicit nonlinear equations by using the fixed-point iteration method with tolerance $\varepsilon = 10^{-13}$. Let the parameters $\delta=0$ and $\alpha=1$.

As the fibre technology advanced, the interest in optical solitons grows rapidly. Various soliton collision scenarios such as transmission, reflection and creation of a new soliton have been reported. If a system is integrable, solitary waves collide elastically; that is, they preserve their shape after collision. However, if the system is non-integrable, the collision may be highly non-trivial and inelastic; that is, the shapes of the solitons change after collision. Additionally, the soliton interactions can lead to large and rapidly decaying oscillating radiative tails. In order to describe the different soliton behaviour, we use the following initial conditions

$$\phi(x,0)=\sqrt{2}r_1\operatorname{sech}(r_1x+D_0/2)\exp(iv_1x/4),$$
$$\psi(x,0)=\sqrt{2}r_2\operatorname{sech}(r_1x-D_0/2)\exp(-iv_2x/4).$$

Here we choose the equal amplitudes $r_1=r_2=1$, the equal velocities $v_1=v_2=1$, and the initial phase $D_0=20$.

We exhibit in Tables 1 and 2 the L_2, L_∞ numerical errors of ϕ and ψ, and the convergent rate order in time with the fixed spatial step length $\Delta x=\frac{80}{999}$ by CMS and CMP, respectively. The parameters are fixed by $a=0.02$, $b=0.01$, $\delta=0$, $T=1$ and $\gamma=1$. As is expected, the convergent rate is consistent with our academic analysis of $\mathcal{O}(\Delta t)$.

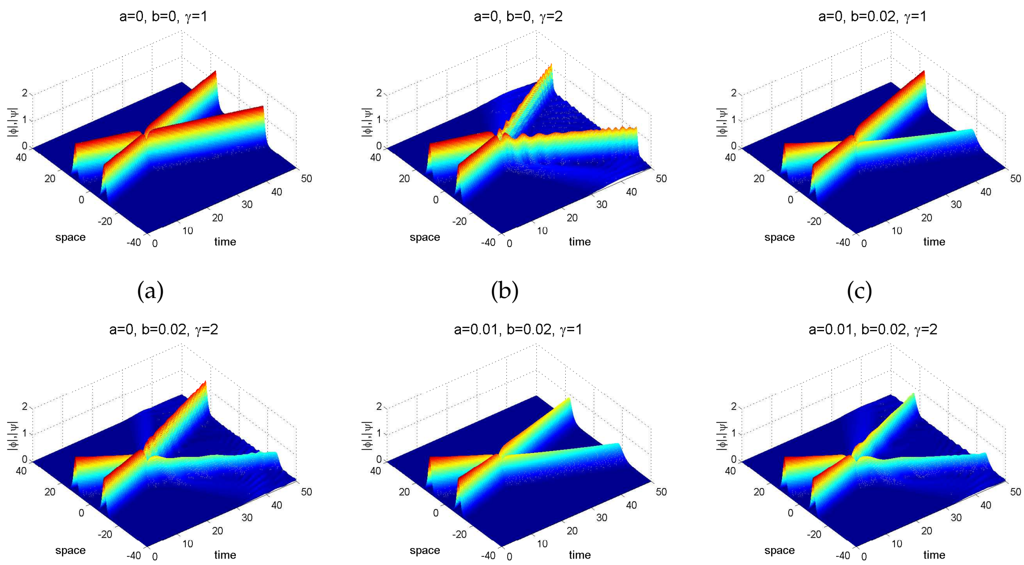
Table 1: Temporal convergent rate of ϕ and ψ with $\Delta x=\frac{80}{999}$ by CMS.

	Δt				
	$\frac{1}{200}$	$\frac{1}{400}$	$\frac{1}{800}$	$\frac{1}{1600}$	$\frac{1}{3200}$
$\|\|e_\phi\|\|_2$	$1.8842e-4$	$8.8443e-5$	$4.2470e-5$	$2.0483e-5$	$9.7384e-6$
Order	-	1.0911	1.0583	1.0520	1.0727
$\|\|e_\phi\|\|_\infty$	$3.4293e-5$	$1.6142e-5$	$7.7622e-6$	$3.7462e-6$	$1.7818e-6$
Order	-	1.0871	1.0563	1.0510	1.0721
$\|\|e_\psi\|\|_2$	$1.0653e-4$	$4.7540e-5$	$2.2188e-5$	$1.0540e-5$	$4.9723e-6$
Order	-	1.1640	1.0994	1.0739	1.0839
$\|\|e_\psi\|\|_\infty$	$1.9359e-5$	$8.6841e-6$	$4.0639e-6$	$1.9331e-6$	$9.1257e-7$
Order	-	1.1565	1.0955	1.0719	1.0829

Table 2: Temporal convergent rate of ϕ and ψ with $\Delta x = \frac{80}{999}$ by CMP.

	Δt				
	$\frac{1}{200}$	$\frac{1}{400}$	$\frac{1}{800}$	$\frac{1}{1600}$	$\frac{1}{3200}$
$\|\|e_\phi\|\|_2$	$1.8848e-4$	$8.8462e-5$	$4.2477e-5$	$2.0486e-5$	$9.7398e-6$
Order	-	1.0584	1.0804	1.0521	1.0727
$\|\|e_\phi\|\|_\infty$	$3.4289e-5$	$1.6139e-5$	$7.7606e-6$	$3.7454e-6$	$1.7814e-6$
Order	-	1.0872	1.0563	1.0510	1.0721
$\|\|e_\psi\|\|_2$	$1.0659e-4$	$4.7556e-5$	$2.2193e-5$	$1.0542e-5$	$4.9730e-6$
Order	-	1.1644	1.0995	1.0739	1.0840
$\|\|e_\psi\|\|_\infty$	$1.9358e-5$	$8.6830e-6$	$4.0632e-6$	$1.9327e-6$	$9.1230e-7$
Order	-	1.1567	1.0956	1.0720	1.0830

Fig. 1 and Fig. 2 display the collision of two solitons with different damping coefficients a, b and nonlinear coupling parameter γ by using CMS algorithm and CMP algorithm, respectively. As can be seen from the left graphs of Fig. 1 and Fig. 2, the propagation of two solitary over the time interval [0,50] is travelling from left and right as required and presenting the good preservation of the phase space structure with $\gamma=1$. The phenomenon indicates that the collision is elastic. But from the right graphs of Fig. 1 and Fig. 2, we can see that the collision takes place at $t=18$, and then the two solitary seems to be oscillated with $\gamma=2$. The phenomenon indicates the collision is inelastic.



(a) (b) (c)

(d) (e) (f)

Figure 1: Collision of two solitons of CMS algorithm with different damping coefficients a, b and coupling parameter γ.

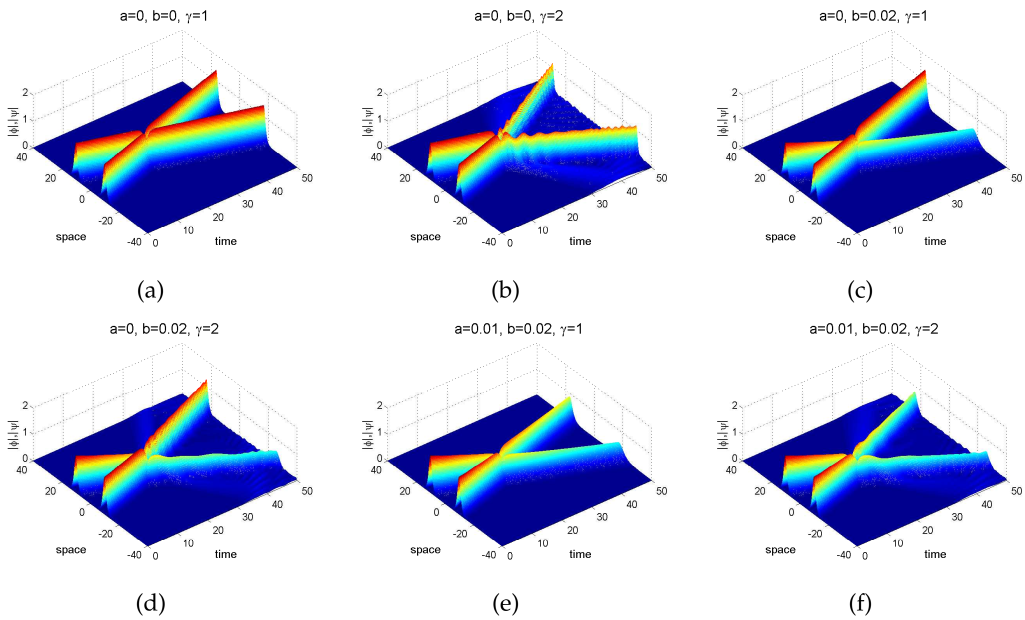


Figure 2: Collision of two solitons of CMP algorithm with different damping coefficients a, b and coupling parameter γ.

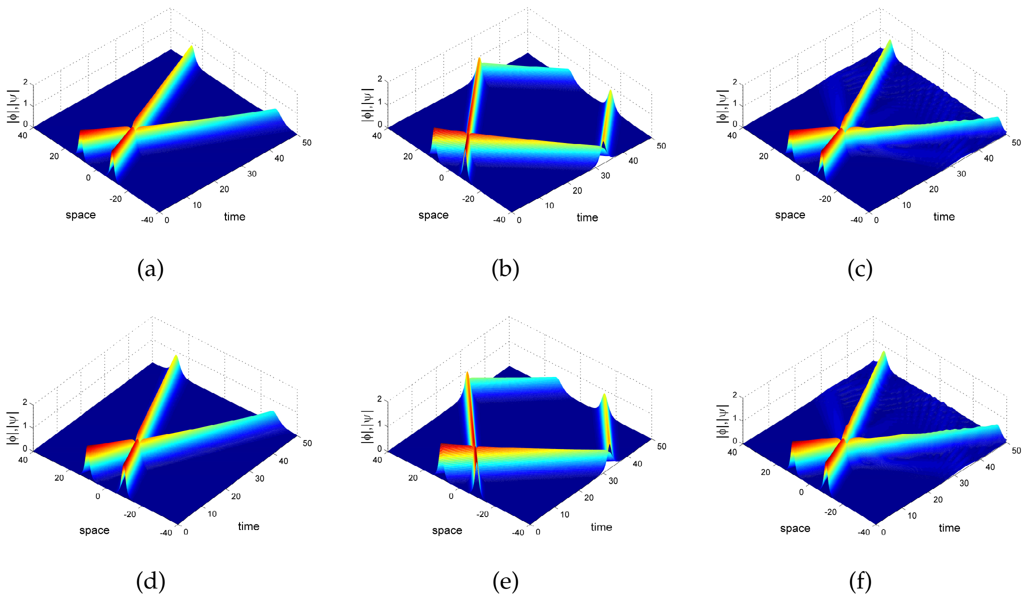
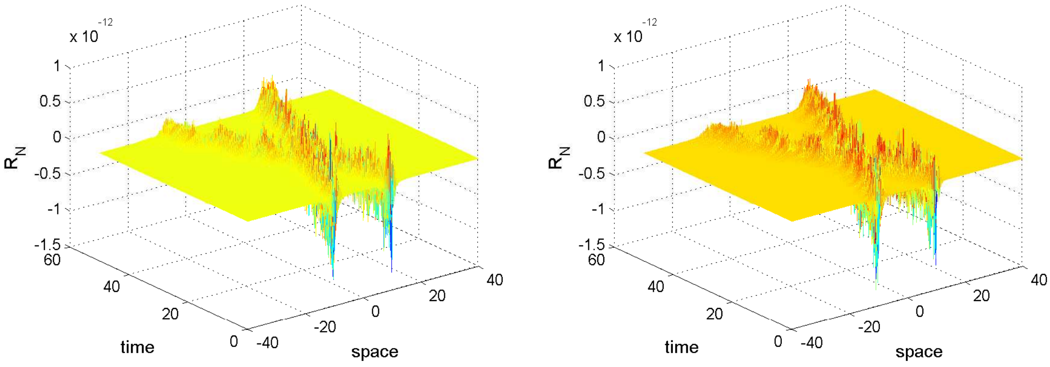
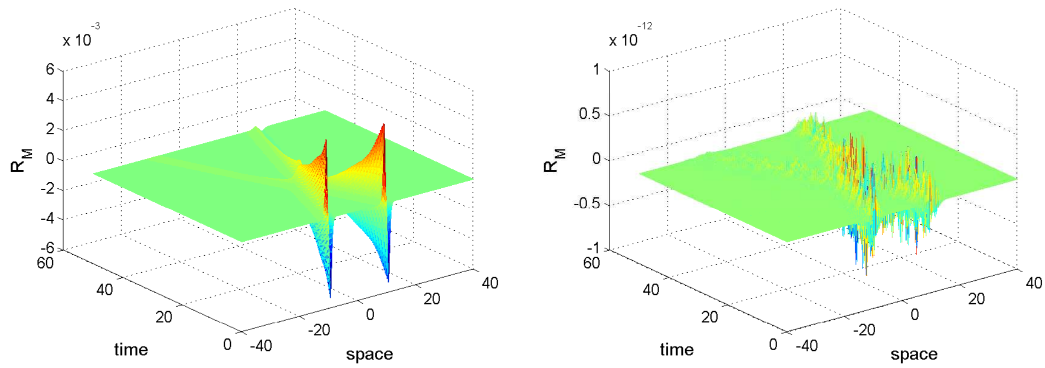


Figure 3: Collision of two solitons by CMS algorithm (the top three) and CMP algorithm (the bottom three) with different parameter δ, γ. (a) $\delta=0.2$, $\gamma=1$; (b) $\delta=1$, $\gamma=1$; (c) $\delta=0.2$, $\gamma=2$; (d) $\delta=0.2$, $\gamma=1$; (e) $\delta=1$, $\gamma=1$; (f) $\delta=0.2$, $\gamma=2$.



(a) (b)

Figure 4: The residual of conformal charge R_N of CMS (left) and CMP (right) with $a=0.01$, $b=0.02$, $\gamma=1$.



(a) (b)

Figure 5: The residual of conformal momentum R_M of CMS (left) and CMP (right) with $a=0.01$, $b=0.02$, $\gamma=1$.

Fig. 3 displays the interaction of two solitons with different parameters δ, γ by using CMS algorithm and CMP algorithm. It is clear that the interaction is elastic in case of $\gamma=1$, and the parameter δ affects propagation velocity of solitons. In this numerical test, we fix $a=0.01$, and $b=0.02$.

Fig. 4 shows the residuals of CCCL by using CMS algorithm (left) and CMP algorithm (right) with $a=0.01$, $b=0.02$, $\gamma=1$, and the errors of them are $\mathcal{O}(10^{-12})$, which indicates the discrete CCCL is preserved exactly.

Fig. 5 presents the residuals of CMCL by using CMS algorithm (left) and CMP algorithm (right) with $a=0.01$, $b=0.02$, $\gamma=1$, and the errors of them are $\mathcal{O}(10^{-3})$, $\mathcal{O}(10^{-12})$, respectively. From the figure, it is clear that the CMP algorithm conserves the CMCL exactly, but the CMS algorithm does not.

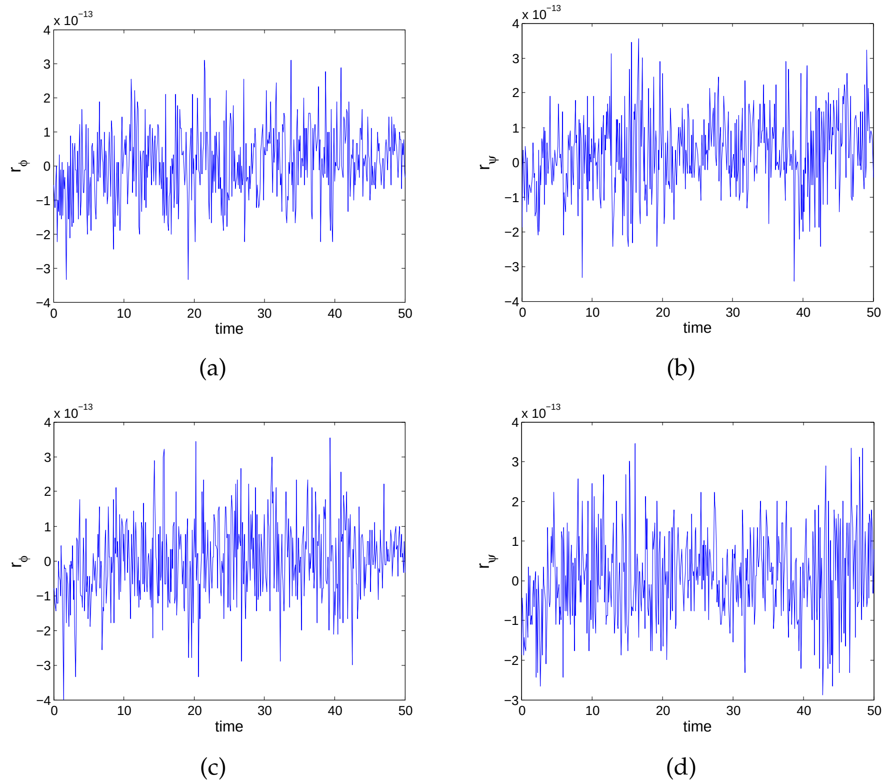


Figure 6: The error of dissipation rate of charge with $a=0.01$, $b=0.02$, and $\gamma=1$: the top two are the error of CMS algorithm; the bottom two are the error of CMP algorithm.

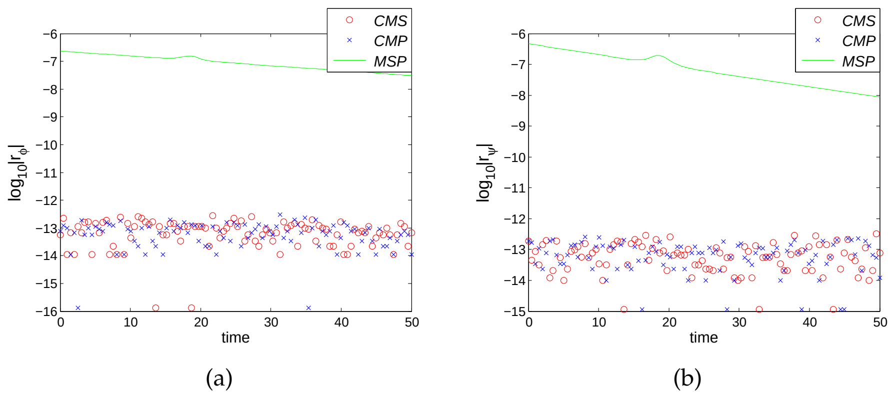


Figure 7: Comparison of the standard MSP scheme with conformal algorithms (CMS and CMP) in dissipation rate of charge for the case $a=0.01$, $b=0.02$, and $\gamma=1$.

Fig. 6 exhibits the ability of CMS algorithm and CMP algorithm in preserving the dissipation rate of charge. The errors of the two algorithms within the roundoff error of machine are $\mathcal{O}(10^{-13})$, which indicate the dissipation rate of charge preserved exactly.

In Fig. 7, we make a comparison of the CMS algorithm and the CMP algorithm with the standard MSP scheme for CDNLS system (1.1). As can be seen in Fig. 7, the conformal algorithm is significantly more accurate in capturing the damped charge than the standard MSP scheme, i.e., the error in the dissipation rate of charge with the conformal algorithms are $\mathcal{O}(10^{-13})$, whereas the standard MSP scheme is $\mathcal{O}(10^{-7})$.

5 Conclusions

In this paper, we derive the conformal multi-symplectic conservation law, the conformal charge conservation law and the conformal momentum conservation law, describing the multi-symplectic structure and the dissipation of charge and momentum, for the CDNLS system. All these conservation laws are local. That is to say, they are independent of the boundary conditions. Based on multi-symplectic Preissman scheme, we develop a conformal multi-symplectic algorithm and a conformal momentum-preserving algorithm, which is proven to admit the discrete conformal multi-symplectic conservation law and the discrete conformal momentum conservation law, respectively. Meanwhile, they also conserve the conformal charge conservation law. With periodic boundary conditions, these algorithms preserve the dissipation rate of charge exactly. Numerical experiments of the two solitons collision are conducted to check the performance of the proposed algorithms and verify the theoretical analysis. Compared with the standard multi-symplectic Preissman scheme, we find that standard method applied to CDNLS system destroys the dissipation rate while the proposed algorithms conserve exactly. Therefore, the proposed conformal algorithms are more advantageous for long time simulation than standard multi-symplectic method for CDNLS system.

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