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AN H-BASED $\mathbf{A}-\phi$ METHOD WITH A NONMATCHING GRID FOR EDDY CURRENT PROBLEM WITH DISCONTINUOUS COEFFICIENTS *1)

Tong Kang
(*Department of Applied Mathematics, School of Sciences, Beijing Broadcasting Institute, Beijing 100024, China*)

Zheng-peng Wu
(*Department of Applied Mathematics, School of Sciences, Beijing Broadcasting Institute, Beijing 100024, China*)
(*ICMSEC, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100080, China*)

De-hao Yu
(*LSEC, ICMSEC, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100080, China*)

Abstract

In this paper, we investigate the finite element $\mathbf{A}-\phi$ method to approximate the eddy current equations with discontinuous coefficients in general three-dimensional Lipschitz polyhedral eddy current region. Nonmatching finite element meshes on the interface are considered and optimal error estimates are obtained.

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Key words: Eddy current problem, Finite element $\mathbf{A}-\phi$ method, Nonmatching meshes, Error estimates.

1. Introduction

The eddy current model emerges from Maxwell's equations by formally dropping the displacement currents, which amounts to neglecting capacitive effects (space charges) and provides a reasonable approximation in the low frequency range and in the presence of high conductivity. Various formulations of the eddy current problem have been suggested in [1], which differ in their choice of the primary unknown. Instead of finding magnetic and electric fields directly, the $\mathbf{A}-\phi$ approach is to seek vector and scalar potentials. Although this method increases the number of scalar unknowns and equations, this apparent complication is justified by a better way of dealing with the possible discontinuities in process of the numerical approximations.

It is well-known that the $\mathbf{A}-\phi$ method has been applied to the eddy current model extensively in practice, but further theoretical research in this aspect has rarely shown so far. For some recent relative work, we refer readers to [2, 8-12] for eddy current problem. In [4], Ciarlet and Zou first studied both nodal finite element methods and edge finite element methods for Maxwell equations. Chen et al. in [3] also discussed a fully discrete finite element method for Maxwell equations with discontinuous coefficients by introducing Lagrangian multipliers. In

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this paper, we will study eddy current equations with discontinuous coefficients by using the above methods and give their error estimates in the meanwhile.

This paper is organized as follows. In section 2, the eddy current model in eddy current region is first presented. Second, we give its $\mathbf{A}-\phi$ variational form based on an optimal control formulation of the interface problem and study the feather of its solutions in section 3. Finally, the fully-discrete coupled and decoupled $\mathbf{A}-\phi$ schemes are proposed and their optimal error estimates are obtained in section 4 and 5 respectively.

2. Eddy Current Problem

For simplification, this paper is only concerned with the following eddy current equations in the eddy current region (high conductivity) neglecting the effect of outside source current:

$$\mathbf{curl}\,\mathbf{H} = \sigma\mathbf{E}, \quad \text{in } \Omega\times(0,T), \tag{2.1}$$

$$\mathbf{curl}\,\mathbf{E} = -\frac{\partial(\mu\mathbf{H})}{\partial t}, \quad \text{in } \Omega\times(0,T). \tag{2.2}$$

Here $\Omega\subset\mathbb{R}^3$ is a simply-connected Lipschitz polyhedral domain with connected boundary which is occupied by the dielectric material. We assume that the magnetic permeability parameter μ and the conductivity σ of the medium are discontinuous across an interface $\Gamma\subset\Omega$ respectively, where Γ is the boundary of a simply-connected Lipschitz polyhedral domain Ω_1 with $\overline{\Omega}_1\subset\Omega$ and $\Omega_2=\Omega\setminus\overline{\Omega}_1$. Ω_2 should be multiply-connected as Ω_1 is simply-connected and lies strictly in Ω. Without loss of generality we consider only the case with μ and σ being two piecewise constant function in the domain Ω, namely,

$$\mu=\begin{cases}\mu_1 & \text{in } \Omega_1,\\ \mu_2 & \text{in } \Omega_2,\end{cases}\qquad \sigma=\begin{cases}\sigma_1 & \text{in } \Omega_1,\\ \sigma_2 & \text{in } \Omega_2,\end{cases}$$

and μ_i, σ_i $(i=1,2)$ are positive constants. It is known that magnetic and electric fields must satisfy the following jump conditions across the interface Γ:

$$[\mathbf{H}\times\mathbf{n}]=\mathbf{0}, \tag{2.3}$$

$$[\mathbf{E}\times\mathbf{n}]=\mathbf{0}, \tag{2.4}$$

where $\mathbf{n}$ is the unit outward normal to $\partial\Omega_1$. Throughout the paper, the jump of any function A across the interface Γ is defined as

$$[A]:=A_2|_\Gamma - A_1|_\Gamma$$

with $A_i=A|_{\Omega_i}$, $i=1,2$. From (2.1) and (2.4) we can see that,

$$[\frac{1}{\sigma}\,\mathbf{curl}\,\mathbf{H}\times\mathbf{n}]=\mathbf{0}, \quad \text{on } \Gamma\times(0,T). \tag{2.5}$$

We supplement the equation (2.1)-(2.2) with the boundary condition

$$\mathbf{H}\times\mathbf{n}=\mathbf{h}(\mathbf{x},t), \tag{2.6}$$

and the initial condition

$$\mathbf{H}(\mathbf{x},0)=\mathbf{H}_0(\mathbf{x}), \quad \text{in } \Omega, \tag{2.7}$$

with

$$\mathrm{div}(\mu \mathbf{H}_0) = 0.$$

By taking divergence to both hand sides of (2.2), we easily see that,

$$\mathrm{div}(\mu \mathbf{H}) = 0, \quad \text{in } \Omega \times (0, T).$$

From the equation (2.1), we can suggest the introduction of a vector $\mathbf{A}$, defined by

$$\sigma \mathbf{E} = \mathbf{curl}\, \mathbf{A}, \quad \text{in } \Omega_1 \cup \Omega_2 \times (0, T). \tag{2.8}$$

We then have

$$\mathbf{H} = \mathbf{A} + \nabla \phi, \quad \text{in } \Omega_1 \cup \Omega_2 \times (0, T), \tag{2.9}$$

where $\phi(t)$ is an arbitrary scalar function. We assume that $\nabla \phi_1 \times \mathbf{n} = \nabla \phi_2 \times \mathbf{n} = \mathbf{0}$ on Γ with $\phi_i = \phi|_{\Omega_i}$, $i = 1, 2$. So the system (2.1)-(2.2) is taken as the following $\mathbf{A} - \phi$ form:

$$\mu \frac{\partial(\mathbf{A} + \nabla \phi)}{\partial t} + \mathbf{curl}\, (\frac{1}{\sigma} \mathbf{curl}\, \mathbf{A}) = 0, \quad \text{in } \Omega_1 \cup \Omega_2 \times (0, T), \tag{2.10}$$

$$\mathrm{div}(\mu(\mathbf{A} + \nabla \phi)) = 0, \quad \text{in } \Omega_1 \cup \Omega_2 \times (0, T) \tag{2.11}$$

with the following interface and boundary conditions

$$[\mathbf{A} \times \mathbf{n}] = \mathbf{0}, \quad \nabla \phi_1 \times \mathbf{n} = \nabla \phi_2 \times \mathbf{n} = \mathbf{0}, \quad \text{on } \Gamma \times (0, T), \tag{2.12}$$

$$[\frac{1}{\sigma}\, \mathbf{curl}\, \mathbf{A} \times \mathbf{n}] = \mathbf{0}, \quad \text{on } \Gamma \times (0, T), \tag{2.13}$$

$$\mathbf{A} \times \mathbf{n} = \mathbf{0}, \quad \nabla \phi \times \mathbf{n} = \mathbf{h}(\mathbf{x}, t), \quad \text{on } \partial\Omega \times (0, T), \tag{2.14}$$

and the initial conditions

$$\phi(\mathbf{x}, 0) = \phi_0(\mathbf{x}) \quad \text{and} \quad \mathbf{A}(\mathbf{x}, 0) = \mathbf{H}_0(\mathbf{x}) - \nabla \phi_0, \quad \text{in } \Omega. \tag{2.15}$$

where ϕ_0 is a given function with $\nabla \phi_0 \times \mathbf{n} = \mathbf{h}(\mathbf{x}, 0)$ on $\partial\Omega$ and $\nabla \phi_0 \times \mathbf{n} = \mathbf{0}$ on Γ. For the sake of simplicity, we assume that $\mathbf{h}(\mathbf{x}, t) = \mathbf{0}$ in the following theoretical analysis.

3. The Variational Formulation

For solving the system (2.10)-(2.15), the finite element method with a matching finite element mesh on the interface Γ need impose a serious restriction: both must match with each other on Γ. We are now going to relax this restriction and consider a nonmatching mesh on the interface Γ that allows the triangulations in Ω_1 and Ω_2 to be generated independently. This advantage, however, brings some difficulty to the convergence analysis since the resulting finite element spaces will be nonconforming for $\mathbf{A}$. So we will deal with the constraint $[\mathbf{A} \times \mathbf{n}] = [\frac{1}{\sigma}\, \mathbf{curl} \mathbf{A} \times \mathbf{n}] = \mathbf{0}$ on Γ by a Lagrangian multiplier approach.

First, we introduce some notations that will be used throughout the paper.

Let $L^p(0, T; X)$ denote the set of all strongly measurable functions $u(t, \cdot)$ from $[0, T]$ into the Hilbert space X such that

$$\int_0^T \|u(t)\|_X^p \, dt < \infty, \qquad 1 \le p < \infty.$$

We say $u \in H^s(0,T;X)$ (s is a positive integer number) if and only if u, $\dfrac{\partial u}{\partial t}, \cdots, \dfrac{\partial^s u}{\partial^s t}$ are in $L^2(0,T;X)$. Let

$$H(\mathbf{curl};\Omega) = \{\mathbf{v} \in L^2(\Omega)^3;\ \mathbf{curl}\,\mathbf{v} \in L^2(\Omega)^3\},$$

$$H^\alpha(\mathbf{curl};\Omega) = \{\mathbf{v} \in H^\alpha(\Omega)^3;\ \mathbf{curl}\,\mathbf{v} \in H^\alpha(\Omega)^3\},\quad (\alpha > 0),$$

$$H_0(\mathbf{curl};\Omega) = \{\mathbf{v} \in H(\mathbf{curl};\Omega);\ \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\}$$

with the norms

$$\|\mathbf{v}\|_{\mathbf{curl},\Omega} = \left(\|\mathbf{v}\|_{0,\Omega}^2 + \|\mathbf{curl}\,\mathbf{v}\|_{0,\Omega}^2\right)^{1/2},$$

$$\|\mathbf{v}\|_{\alpha,\mathbf{curl},\Omega} = \left(\|\mathbf{v}\|_{\alpha,\Omega}^2 + \|\mathbf{curl}\,\mathbf{v}\|_{\alpha,\Omega}^2\right)^{1/2}.$$

Here and in what follows, $\|\cdot\|_{0,\Omega}$ denotes the $L^2(\Omega)^3$-norm (or the $L^2(\Omega)$-norm for scalar functions) and for $s > 0$, $\|\cdot\|_{s,\Omega}$ denotes the norm of the Sobolev space $H^s(\Omega)^3$ (or $H^s(\Omega)$ for scalar functions). Similar definitions are adopted for Ω_1 and Ω_2. The constant C will always represent a generic constant independent of the time step and the mesh size.

For the convenience of presentation, let us introduce the following spaces:

$$X_1 = H(\mathbf{curl};\ \Omega_1), \qquad X_2 = \{\mathbf{v} \in H(\mathbf{curl};\ \Omega_2);\ \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\},$$

$$Y_1 = \{\psi \in H^1(\Omega_1);\ \nabla\psi \times \mathbf{n} = 0 \text{ on } \partial\Omega_1\},\ \ Y_2 = \{\psi \in H^1(\Omega_2);\ \nabla\psi \times \mathbf{n} = 0 \text{ on } \partial\Omega_2\}.$$

Set $X = X_1 \times X_2$ and $Y = Y_1 \times Y_2$.

Second, to establish an appropriate variational formulation for the system (2.10)-(2.15), we need use a few important mathematical analysis tools borrowed from [3].

Let

$$T(\Gamma) = \{\mathbf{s} \in H^{-1/2}(\Gamma)^3;\ \exists\, \mathbf{v} \in H_0(\mathbf{curl};\Omega) \text{ such that } \mathbf{v} \times \mathbf{n} = \mathbf{s} \text{ on } \Gamma\}.$$

It is not difficult to see that $T(\Gamma)$ is a Banach space under the norm:

$$\|\mathbf{s}\|_{T(\Gamma)} = \inf\{\|\mathbf{v}\|_{\mathbf{curl},\Omega};\mathbf{v} \in H_0(\mathbf{curl};\Omega) \text{ and } \mathbf{v} \times \mathbf{n} = \mathbf{s} \text{ on } \Gamma\}.$$

For any $\mathbf{s} \in T(\Gamma)$, we define

$$\ll \mathbf{s},\mathbf{w} \gg_{1,\Gamma} = \int_{\Omega_1} \mathbf{v} \cdot \mathbf{curl}\,\mathbf{w}\,dx - \int_{\Omega_1} \mathbf{curl}\,\mathbf{v} \cdot \mathbf{w}\,dx, \qquad \forall\, \mathbf{w} \in X_1, \tag{3.1}$$

$$\ll \mathbf{s},\mathbf{w} \gg_{2,\Gamma} = \int_{\Omega_2} \mathbf{curl}\,\mathbf{v} \cdot \mathbf{w}\,dx - \int_{\Omega_2} \mathbf{v} \cdot \mathbf{curl}\,\mathbf{w}\,dx, \qquad \forall\, \mathbf{w} \in X_2, \tag{3.2}$$

where $\mathbf{v} \in H_0(\mathbf{curl};\Omega)$ such that $\mathbf{v} \times \mathbf{n} = \mathbf{s}$ on Γ. We know that (3.1)-(3.2) are independent of the choice of $\mathbf{v} \in H(\mathbf{curl};\Omega)$ such that $\mathbf{v} \times \mathbf{n} = \mathbf{s}$ on Γ.

Then, we have Lemma 3.1-3.2 which are proved in [3].

Lemma 3.1. *For any* $\mathbf{s} \in T(\Gamma)$*, we have the equality*

$$\|\mathbf{s}\|_{T(\Gamma)} = \sup_{\mathbf{w} \in X_1 \times X_2} \frac{\ll \mathbf{s},\mathbf{w} \gg_{1,\Gamma} - \ll \mathbf{s},\mathbf{w} \gg_{2,\Gamma}}{\|\mathbf{w}\|_{X_1 \times X_2}}.$$

A direct consequence of this lemma is that $T(\Gamma)$ *is indeed a Hilbert space.*

In practice, Lemma 3.1 is rather inconvenient as it uses information from both Ω_1 and Ω_2 to define the norm on $T(\Gamma)$. To overcome the inconvenience, we note that

$$\|\mathbf{s}\|_{1,\Gamma} = \sup_{\mathbf{w}\in X_1} \frac{\ll \mathbf{s},\mathbf{w} \gg_{1,\Gamma}}{\|\mathbf{w}\|_{X_1}}, \qquad \|\mathbf{s}\|_{2,\Gamma} = \sup_{\mathbf{w}\in X_2} \frac{\ll \mathbf{s},\mathbf{w} \gg_{1,\Gamma}}{\|\mathbf{w}\|_{X_2}} \tag{3.3}$$

are also norms of $T(\Gamma)$. So we have

Lemma 3.2. *The norms* $\|\cdot\|_{1,\Gamma}$ *and* $\|\cdot\|_{2,\Gamma}$ *are equivalent to* $\|\cdot\|_{T(\Gamma)}$.

Taking any $\overline{\mathbf{A}}\in X$, $\nabla\overline{\phi}$ (any $\overline{\phi}\in Y$) in (2.10) as test functions respectively and applying the standard technique of integration by parts lead immediately to the following weak formulations of (2.10)-(2.15):

Problem (I). Find $(\mathbf{A},\ \phi,\ \mathbf{p})\in H^1(0,T;X)\times H^1(0,T;Y)\times L^2(0,T;T(\Gamma))$ such that

$$\begin{aligned}\sum_{i=1}^{2}\Big\{(\mu\frac{\partial(\mathbf{A}+\nabla\phi)}{\partial t},\overline{\mathbf{A}})_{\Omega_i}+(\frac{1}{\sigma}\,\mathbf{curl}\,\mathbf{A},\mathbf{curl}\,\overline{\mathbf{A}})_{\Omega_i}\Big\}&\\ +\ll \mathbf{p},\overline{\mathbf{A}}\gg_{2,\Gamma}-\ll \mathbf{p},\overline{\mathbf{A}}\gg_{1,\Gamma}=0,\quad \forall\overline{\mathbf{A}}\in X,&\end{aligned} \tag{3.4}$$

$$\sum_{i=1}^{2}(\mu\frac{\partial(\mathbf{A}+\nabla\phi)}{\partial t},\nabla\overline{\phi})_{\Omega_i}=0,\quad \forall\overline{\phi}\in Y, \tag{3.5}$$

$$\ll \mathbf{A},\ \overline{\mathbf{p}}\gg_{2,\Gamma}-\ll \mathbf{A},\ \overline{\mathbf{p}}\gg_{1,\Gamma}=0,\quad \forall\,\overline{\mathbf{p}}\in T(\Gamma). \tag{3.6}$$

The system (3.4)-(3.6) is consistent with the finite element discretization on a nonmatching grid on the interface Γ and can be derived based on an optimal control formulation of the interface problem (2.10)-(2.15).

In order to analyze the feature of the solution of Problem (I), we first study the existence and uniqueness of the solution to the following problem:

Problem (II). Find $(\mathbf{H},\ \mathbf{p})\in H^1(0,T;X)\times L^2(0,T;T(\Gamma))$ such that

$$\begin{aligned}\sum_{i=1}^{2}\Big\{(\mu\frac{\partial\mathbf{H}}{\partial t},\overline{\mathbf{H}})_{\Omega_i}+(\frac{1}{\sigma}\,\mathbf{curl}\,\mathbf{H},\mathbf{curl}\,\overline{\mathbf{H}})_{\Omega_i}\Big\}&\\ +\ll \mathbf{p},\overline{\mathbf{H}}\gg_{2,\Gamma}-\ll \mathbf{p},\overline{\mathbf{H}}\gg_{1,\Gamma}=0,\quad \forall\overline{\mathbf{H}}\in X,&\end{aligned} \tag{3.7}$$

$$\ll \mathbf{H},\ \overline{\mathbf{p}}\gg_{2,\Gamma}-\ll \mathbf{H},\ \overline{\mathbf{p}}\gg_{1,\Gamma}=0,\quad \forall\,\overline{\mathbf{p}}\in T(\Gamma). \tag{3.8}$$

Furthermore, we only need to discuss the following stationary variational problem, whose result can be extended to Problem (II) by the standard analytic method.

Problem (III). Given $\mathbf{f}\in L^2(\Omega)^3$, find $(\mathbf{Q},\ \mathbf{p})\in X\times T(\Gamma)$ such that

$$\begin{aligned}\sum_{i=1}^{2}\Big\{(\alpha_i\,\mathbf{curl}\,\mathbf{Q},\mathbf{curl}\,\overline{\mathbf{Q}})_{\Omega_i}+(\beta_i\,\mathbf{Q},\overline{\mathbf{Q}})_{\Omega_i}\Big\}&\\ +\ll \mathbf{p},\overline{\mathbf{Q}}\gg_{2,\Gamma}-\ll \mathbf{p},\overline{\mathbf{Q}}\gg_{1,\Gamma}=\sum_{i=1}^{2}(\mathbf{f},\overline{\mathbf{Q}})_{\Omega_i},\quad \forall\overline{\mathbf{Q}}\in X,&\end{aligned} \tag{3.9}$$

$$\ll \mathbf{Q},\ \overline{\mathbf{p}}\gg_{2,\Gamma}-\ll \mathbf{Q},\ \overline{\mathbf{p}}\gg_{1,\Gamma}=0,\quad \forall\,\overline{\mathbf{p}}\in T(\Gamma). \tag{3.10}$$

Here α_i and β_i are piecewise positive constants in Ω_i for $i=1,2$.

Theorem 3.1. *There exists a unique solution* $(\mathbf{Q},\mathbf{p})\in X\times T(\Gamma)$ *to Problem (III).*

Proof. First, define a bilinear form $a : X \times X \to R$:

$$a(\mathbf{u},\mathbf{v}) = \sum_{i=1}^{2}\Big\{(\alpha_i\,\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\mathbf{v})_{\Omega_i} + (\beta_i\,\mathbf{u},\mathbf{v})_{\Omega_i}\Big\}, \qquad \mathbf{u},\,\mathbf{v}\in X.$$

It is obvious that

$$a(\mathbf{u},\mathbf{u}) \ge a_0\|\mathbf{u}\|_X^2 \tag{3.11}$$

for some constant $a_0 > 0$.

Next, we verify the inf-sup condition: there exists a constant $C > 0$ such that

$$\sup_{\mathbf{B}\in X}\frac{\ll \mathbf{s},\mathbf{B}_2 \gg_{2,\Gamma} - \ll \mathbf{s},\mathbf{B}_1 \gg_{1,\Gamma}}{\|\mathbf{B}\|_X} \ge C\|\mathbf{s}\|_{T(\Gamma)}, \qquad \forall\,\mathbf{s}\in T(\Gamma), \tag{3.12}$$

where $\mathbf{B} = \mathbf{B}_i$ in Ω_i for $i = 1, 2$. Let $\mathbf{B} \in H(\mathbf{curl};\,\Omega_1)$ be the solution of the following problem:

$$(\mathbf{curl}\,\mathbf{B},\mathbf{curl}\,\overline{\mathbf{B}})_{\Omega_1} + (\mathbf{B},\overline{\mathbf{B}})_{\Omega_1} = \ll \mathbf{s},\overline{\mathbf{B}} \gg_{1,\Gamma}, \qquad \forall\,\overline{\mathbf{B}}\in H(\mathbf{curl};\,\Omega_1). \tag{3.13}$$

We define

$$\widetilde{\mathbf{B}} = \begin{cases} -\mathbf{B} & \text{in } \Omega_1,\\ \ \ \mathbf{0} & \text{in } \Omega_2. \end{cases} \tag{3.14}$$

It is obvious that $\widetilde{\mathbf{B}} \in X$ and

$$\|\widetilde{\mathbf{B}}\|_X = \|\mathbf{B}\|_{\mathbf{curl},\Omega_1}. \tag{3.15}$$

Thus, by (3.13), we are able to obtain

$$\ll \mathbf{s},\widetilde{\mathbf{B}}_2 \gg_{2,\Gamma} - \ll \mathbf{s},\widetilde{\mathbf{B}}_1 \gg_{1,\Gamma} = \ll \mathbf{s},\mathbf{B} \gg_{1,\Gamma} = \|\mathbf{B}\|^2_{\mathbf{curl},\Omega_1}$$

which yields, together with Lemma 3.2,

$$\frac{\ll \mathbf{s},\widetilde{\mathbf{B}}_2 \gg_{2,\Gamma} - \ll \mathbf{s},\widetilde{\mathbf{B}}_1 \gg_{1,\Gamma}}{\|\widetilde{\mathbf{B}}\|_X} = \|\mathbf{B}\|_{\mathbf{curl},\Omega_1} = \|\mathbf{s}\|_{1,\Gamma} \ge C\|\mathbf{s}\|_{T(\Gamma)}.$$

From (3.11)-(3.12), we then have finished the proof of the theorem.

Thus, from the result of Theorem 3.1, we conclude that the solution ($\mathbf{H}$, $\mathbf{p}$) of Problem (II) is existing and unique. Meanwhile, by using an appropriate application of the Green's formula, the Lagrange multiplier $\mathbf{p}$ in Problem (II) satisfies the following relation:

$$\mathbf{p} = \frac{1}{\sigma}\,\mathbf{curl}\,\mathbf{H}\times\mathbf{n}, \qquad \text{in } \ T(\Gamma)\times(0,T).$$

For the solution $\mathbf{H}$ of Problem (II) and a given $\phi \in Y$, let $\mathbf{A} = \mathbf{H} - \nabla\phi$. Especially, if we append the divergence-free property of $\mathbf{A}$, $\mathbf{A}$ is unique. Taking $\overline{\mathbf{H}} = \overline{\mathbf{A}}$ and $\overline{\mathbf{H}} = \nabla\overline{\phi}$ for any $\overline{\mathbf{A}} \in X$ and $\overline{\phi} \in Y$ respectively in (3.7)-(3.8), we conclude that $\mathbf{A}$, ϕ, $\mathbf{p}$ satisfy Problem (I); that is:

Theorem 3.2. *The solution* ($\mathbf{A}$, ϕ, $\mathbf{p}$) *of Problem (I) is existing, but only* $\mathbf{p}$ *is unique. Furthermore, the Lagrange multiplier* $\mathbf{p}$ *in Problem (I) satisfies:*

$$\mathbf{p} = \frac{1}{\sigma}\,\mathbf{curl}\,\mathbf{A}\times\mathbf{n}, \qquad \text{in } \ T(\Gamma)\times(0,T). \tag{3.16}$$

4. A Fully-discrete Coupled $\mathbf{A}-\phi$ Scheme with a Nonmatching Grid for Eddy Current Problem

In this section we propose a finite element method for solving Problem (I), which allows a nonmatching finite element grid on the interface Γ.

Let $\mathcal{T}^{h_1}$ and $\mathcal{T}^{h_2}$ be a shape regular triangulation of Ω_1 and Ω_2 respectively. They induce naturally two finite element triangulations Γ_{h_1} and Γ_{h_2} on the interface Γ. Let Γ_{h_0} be an another shape regular triangulation over Γ. Note that Γ_{h_i}, $i=0,1,2$, are allowed to be different from each other. However, we make the following reasonable assumption:

(H1) Each triangle in Γ_{h_1} and Γ_{h_2} must be contained in some triangles of Γ_{h_0}.

We introduce the Nédélec $H(\mathbf{curl},\Omega_i)$-conforming edge element space defined by

$$X_{h_i}=\{\mathbf{v}_h\in X_i;\, \mathbf{v}_h=\mathbf{a}_K+\mathbf{b}_K\times\mathbf{x}\ \text{ on } K,\ \forall K\in\mathcal{T}^{h_i}\},\quad i=1,2$$

where $\mathbf{a}_K$ and $\mathbf{b}_K$ are two constant vectors. It is known that any function $\mathbf{v}_h\in X_{h_i}$ is uniquely determined by the degrees of freedom in the set $M_E(\mathbf{v})$ of the moments on each element $K\in\Gamma_{h_i}$, which is given by

$$M_E(\mathbf{v})=\left\{\int_e \mathbf{v}\cdot\tau\, ds;\ e \text{ is an edge of } K\right\}.$$

Here τ is the unit vector along the edge. For $i=1,2$ and any $\mathbf{v}\in H^s(\Omega_i)^3$ with $\mathbf{curl}\,\mathbf{v}\in L^p(\Omega_i)^3$, where $s>1/2$ and $p>2$, we can define an interpolation $\pi_h\mathbf{v}\in X_{h_i}$, and $\pi_h\mathbf{v}$ has the same degrees of freedom (defined by $M_E(\mathbf{v})$) as $\mathbf{v}$ on each K in Γ_{h_i}.

Let $T_{h_0}(\Gamma)$ be the Nédélec $T(\Gamma)$-conforming edge element space defined by

$$T_{h_0}(\Gamma)=\{\mathbf{s}_h\in T(\Gamma);\, \mathbf{s}_h=(\alpha_\tau+\beta_\tau\times\mathbf{n})\times\mathbf{n},\ \text{ on any } \tau\in\Gamma_{h_0},\ \alpha_\tau,\,\beta_\tau\in R^3\}.$$

We also define the following finite element spaces

$$Y_{h_i}=\{\varphi_h\in Y_i;\, \varphi_h|_K\in\mathcal{P}_1,\,\forall K\in\mathcal{T}^{h_i}\},\quad i=1,2,$$

where $\mathcal{P}_1$ is the space of linear polynomials. Let Π_h be the standard interpolating operator. Now set

$$X_h=X_{h_1}\times X_{h_2},\qquad Y_h=Y_{h_1}\times Y_{h_2}.$$

We will assume the inf-sup condition:

(H2) There exists a constant $C^*>0$ independent of h_0, h_1, h_2 such that

$$\sup_{\mathbf{w}_{h_i}\in X_{h_i}}\frac{\ll \mathbf{s}_h,\mathbf{w}_{h_i}\gg_{i,\Gamma}}{\|\mathbf{w}_{h_i}\|_{\mathbf{curl},\Omega_i}}\ge C^*\|\mathbf{s}_h\|_{T(\Gamma)},\quad \forall\mathbf{s}_h\in T_{h_0}(\Gamma),\quad i=1 \text{ or } 2. \tag{4.1}$$

The assumption (H2) indicate that the mesh Γ_{h_0} should be coarse enough compared with the meshes $\mathcal{T}^{h_1}$ or $\mathcal{T}^{h_2}$ in order to stabilize the effect of the introduced Lagrangian multiplier. In subsection 4.4 of [3], by using a general compactness argument, (4.1) is verified to be valid at least when the mesh h_1 or h_2 is suitably small compared with h_0.

Let us divide the time interval $(0,T)$ into M equally-spaced subintervals by using nodal points

$$0=t_0<t_1<\cdots<t_M=T$$

with $t_n=n\tau$ and $\tau=T/M$, and denote n-th subinterval by $I^n=(t_{n-1},t_n]$. For a continuous mapping $u:\ [0,T]\to L^2(\Omega)$ or $L^2(\Omega)^3$, we define $u^n(\cdot)=u(\cdot,t_n)$ for $1\le n\le M$.

Now we are in a position to introduce the discrete version of Problem (I).

Problem (VI). For $n = 0, 1, \cdots, M-1$, find $(\mathbf{A}_h^{n+1},\ \phi_h^{n+1},\ \mathbf{p}_h^{n+1}) \in X_h \times Y_h \times T_{h_0}(\Gamma)$ such that

$$\mathbf{A}_h^0 = \pi_h \mathbf{A}_0, \quad \phi_h^0 = \Pi_h \phi_0 \tag{4.2}$$

and

$$\begin{aligned}\sum_{i=1}^{2} \Big\{ (\mu \frac{\mathbf{A}_h^{n+1} - \mathbf{A}_h^n}{\tau}, \overline{\mathbf{A}}_h)_{\Omega_i} + (\frac{1}{\sigma}\, \mathbf{curl}\, \mathbf{A}_h^{n+1}, \mathbf{curl}\, \overline{\mathbf{A}}_h)_{\Omega_i} + (\mu \nabla \frac{\phi_h^{n+1} - \phi_h^n}{\tau}, \overline{\mathbf{A}}_h)_{\Omega_i} \Big\} \\ + \ll \mathbf{p}_h^{n+1}, \overline{\mathbf{A}}_h \gg_{2,\Gamma} - \ll \mathbf{p}_h^{n+1}, \overline{\mathbf{A}}_h \gg_{1,\Gamma} = 0, \quad \forall \overline{\mathbf{A}}_h \in X_h,\end{aligned} \tag{4.3}$$

$$\sum_{i=1}^{2} \Big\{ (\mu \frac{\mathbf{A}_h^{n+1} - \mathbf{A}_h^n}{\tau}, \nabla \overline{\phi}_h)_{\Omega_i} + (\mu \nabla \frac{\phi_h^{n+1} - \phi_h^n}{\tau}, \nabla \overline{\phi}_h)_{\Omega_i} \Big\} = 0, \quad \forall \overline{\phi}_h \in Y_h, \tag{4.4}$$

$$\ll \mathbf{A}_h^{n+1},\ \overline{\mathbf{p}}_h \gg_{2,\Gamma} - \ll \mathbf{A}_h^{n+1},\ \overline{\mathbf{p}}_h \gg_{1,\Gamma} = 0, \quad \forall\, \overline{\mathbf{p}}_h \in T_{h_0}(\Gamma). \tag{4.5}$$

We then have the following result:

Theorem 4.1. *Under the assumptions* (H1)-(H2) *and* $\mathbf{A}_h^{n+1} \times \mathbf{n} = \mathbf{0}$ *on* $\partial\Omega$*, the solution* $(\mathbf{A}_h^{n+1},\ \phi_h^{n+1},\ \mathbf{p}_h^{n+1}) \in X_h \times Y_h \times T_{h_0}(\Gamma)$ *of Problem (VI) is existing.*

Proof. We first introduce the following discrete version of Problem (III) and prove that it has a unique solution.

Problem (V). Find $(\mathbf{Q}_h,\ \mathbf{p}_h) \in X_h \times T_{h_0}(\Gamma)$ such that

$$\begin{aligned}\sum_{i=1}^{2} \Big\{ (\alpha_i\, \mathbf{curl}\, \mathbf{Q}_h, \mathbf{curl}\, \overline{\mathbf{Q}}_h)_{\Omega_i} + (\beta_i\, \mathbf{Q}_h, \overline{\mathbf{Q}}_h)_{\Omega_i} \Big\} \\ + \ll \mathbf{p}_h, \overline{\mathbf{Q}}_h \gg_{2,\Gamma} - \ll \mathbf{p}_h, \overline{\mathbf{Q}}_h \gg_{1,\Gamma} = \sum_{i=1}^{2} (\mathbf{f}, \overline{\mathbf{Q}}_h)_{\Omega_i},\ \ \forall \overline{\mathbf{Q}}_h \in X_h.\end{aligned} \tag{4.6}$$

$$\ll \mathbf{Q}_h,\ \overline{\mathbf{p}}_h \gg_{2,\Gamma} - \ll \mathbf{Q}_h,\ \overline{\mathbf{p}}_h \gg_{1,\Gamma} = 0, \quad \forall\, \overline{\mathbf{p}}_h \in Q_{h_0}(\Gamma). \tag{4.7}$$

Furthermore, we only need verify the inf-sup condition: for any $\mathbf{s}_h \in T_{h_0}(\Gamma)$, there exists a constant $C > 0$ such that

$$\sup_{\mathbf{B}_h \in X_h} \frac{\ll \mathbf{s}_h, \mathbf{B}_{h_2} \gg_{2,\Gamma} - \ll \mathbf{s}_h, \mathbf{B}_{h_1} \gg_{1,\Gamma}}{\|\mathbf{B}_h\|_{1,\Omega}} \geq C \|\mathbf{s}_h\|_{T(\Gamma)}, \tag{4.8}$$

where $\mathbf{B}_h = \mathbf{B}_{h_i}$ in Ω_i for $i = 1, 2$. Without loss of generality we assume that (4.1) is valid for $i = 1$. Thus there exists a $\mathbf{w}_{h_1} \in X_{h_1}$ such that

$$\frac{\ll \mathbf{s}_h, \mathbf{w}_{h_1} \gg_{1,\Gamma}}{\|\mathbf{w}_{h_1}\|_{\mathbf{curl},\Omega_1}} \geq C^* \|\mathbf{s}_h\|_{T(\Gamma)}. \tag{4.9}$$

Let $\mathbf{B}_{h_1} \in X_{h_1}$ be the solution of the following problem:

$$(\mathbf{curl}\, \mathbf{B}_{h_1}, \mathbf{curl}\, \overline{\mathbf{B}}_{h_1})_{\Omega_1} + (\mathbf{B}_{h_1}, \overline{\mathbf{B}}_{h_1})_{\Omega_1} = \ll \mathbf{s}_h, \overline{\mathbf{B}}_{h_1} \gg_{1,\Gamma}, \quad \forall\, \overline{\mathbf{B}}_{h_1} \in X_{h_1}. \tag{4.10}$$

Taking $\overline{\mathbf{B}}_{h_1} = \mathbf{B}_{h_1}$ and $\overline{\mathbf{B}}_{h_1} = \mathbf{w}_{h_1}$ as test functions respectively and using (4.9), we obtain

$$C^* \|\mathbf{s}_h\|_{T(\Gamma)} \leq \|\mathbf{B}_{h_1}\|_{\mathbf{curl},\Omega_1} \leq \|\mathbf{s}_h\|_{1,\Gamma} \leq C \|\mathbf{s}_h\|_{T(\Gamma)}. \tag{4.11}$$

We define

$$\widetilde{\mathbf{B}}_h = \begin{cases} -\mathbf{B}_{h_1} & \text{in } \Omega_1, \\ \mathbf{0} & \text{in } \Omega_2. \end{cases}$$

Then,

$$\|\widetilde{\mathbf{B}}_h\|_X = \|\mathbf{B}_{h_1}\|_{\mathbf{curl},\Omega_1}. \tag{4.12}$$

Thus, by (4.11), we are able to obtain

$$\ll \mathbf{s}_h, \widetilde{\mathbf{B}}_{h_2} \gg_{2,\Gamma} - \ll \mathbf{s}_h, \widetilde{\mathbf{B}}_{h_1} \gg_{1,\Gamma} = \ll \mathbf{s}_h, \mathbf{B}_{h_1} \gg_{1,\Gamma} = \|\mathbf{B}_{h_1}\|^2_{\mathbf{curl},\Omega_1}$$

which yields,

$$\frac{\ll \mathbf{s}_h, \widetilde{\mathbf{B}}_{h_2} \gg_{2,\Gamma} - \ll \mathbf{s}_h, \widetilde{\mathbf{B}}_{h_1} \gg_{1,\Gamma}}{\|\widetilde{\mathbf{B}}_h\|_X} = \frac{\ll \mathbf{s}_h, \mathbf{B}_{h_1} \gg_{1,\Gamma}}{\|\mathbf{B}_{h_1}\|_{\mathbf{curl},\Omega_1}} = \|\mathbf{B}_{h_1}\|_{\mathbf{curl},\Omega_1} \geq C\|\mathbf{s}_h\|_{T(\Gamma)}.$$

Therefore, Problem (V) has a unique solution $(\mathbf{Q}_h, \mathbf{p}_h)$.

From the definition of the edge element space, $\nabla\varphi_h \in X_h$ for any $\varphi_h \in Y_h$. Thus, for the solution $\mathbf{Q}_h$ of Problem (V) and a given $\phi_h^{n+1} \in Y_h$, $\mathbf{A}_h^{n+1} = \mathbf{Q}_h - \nabla\phi_h^{n+1} \in X_h$. Noting that $\operatorname{div}\mathbf{A}_h^{n+1} = 0$, we have that $\mathbf{A}_h^{n+1}$ is unique; while ϕ_h^{n+1} depending on its boundary condition is not unique. $\alpha = \dfrac{1}{\sigma}$, $\beta = \dfrac{\mu}{\tau}$ and $\mathbf{f} = \dfrac{\mu}{\tau}(\mathbf{A}_h^n + \nabla\phi_h^n)$. Taking $\overline{\mathbf{Q}}_h = \overline{\mathbf{A}}_h$ and $\overline{\mathbf{Q}}_h = \nabla\overline{\phi}_h$ for any $\overline{\mathbf{A}}_h \in X_h$ and $\overline{\phi}_h \in Y_h$ in (4.6)-(4.7) respectively and noting that

$$\mathbf{p}_h = \frac{1}{\sigma}\,\mathbf{curl}\,\mathbf{Q}_h \times \mathbf{n} = \mathbf{p}_h^{n+1}, \qquad \text{in } T(\Gamma),$$

we see that $(\mathbf{A}_h^{n+1}, \phi_h^{n+1}, \mathbf{p}_h^{n+1})$ satisfies Problem (VI). Thus, we finish the proof of Theorem 4.1.

Now we can state the following theorem on the relevant error estimate.

Theorem 4.2. *Under the condition of Theorem 4.1, let* $(\mathbf{A}^{n+1}, \phi^{n+1}, \mathbf{p}^{n+1})$ *and* $(\mathbf{A}_h^{n+1}, \phi_h^{n+1}, \mathbf{p}_h^{n+1})$ *be the solutions of Problem (I) and Problem (VI) at time* $t = t_{n+1}$ *respectively. Supposing that for some* $\alpha > 1/2$,

$$\mathbf{A} \in H^2(0,T; H^\alpha(\mathbf{curl};\Omega_1) \times H^\alpha(\mathbf{curl};\Omega_2)), \qquad \phi \in H^2(0,T; Y \cap H^{1+\alpha}(\Omega_1) \times H^{1+\alpha}(\Omega_2)).$$

Then, we have the following error estimate:

$$\max_{0\leq n\leq M-1}\left(\sum_{i=1}^{2}\|(\mathbf{A}_h^{n+1} + \nabla\phi_h^{n+1}) - (\mathbf{A}^{n+1} + \nabla\phi^{n+1})\|^2_{0,\Omega_i}\right) \leq C\tau^2 + \sum_{i=1}^{2} C_i h_i^{2\alpha} + C_0 h_0^{2\alpha}.$$

Proof. For the convenience of presentation, let $b : X \times T(\Gamma) \to R$ the bilinear form as follows:

$$b\,(\mathbf{Q}, \mathbf{p}) = \ll \mathbf{Q}, \mathbf{p} \gg_{2,\Gamma} - \ll \mathbf{Q}, \mathbf{p} \gg_{1,\Gamma}, \quad \forall\,(\mathbf{Q}, \mathbf{p}) \in X \times T(\Gamma).$$

Define the elliptic projection operator P_h: $X \times T(\Gamma) \to X_h \times T_{h_0}(\Gamma)$. For any $(\mathbf{Q}, \mathbf{p}) \in X \times T(\Gamma)$, we have

$$\begin{aligned} &\sum_{i=1}^{2}\left\{(P_h\mathbf{Q} - \mathbf{Q}, \overline{\mathbf{Q}}_h)_{\Omega_i} + (\frac{1}{\sigma}\,\mathbf{curl}\,(P_h\mathbf{Q} - \mathbf{Q}), \mathbf{curl}\,\overline{\mathbf{Q}}_h)_{\Omega_i}\right\} \\ &\qquad + b\,(\overline{\mathbf{Q}}_h, P_h\mathbf{p} - \mathbf{p}) = 0, \quad \forall\,\overline{\mathbf{Q}}_h \in X_h, \end{aligned} \tag{4.13}$$

$$b\,(P_h\mathbf{Q}-\mathbf{Q},\overline{\mathbf{p}}_h)=0,\quad \forall\,\overline{\mathbf{p}}_h\in T_{h_0}(\Gamma). \tag{4.14}$$

Let $\overline{\mathbf{Q}}_h=\nabla\psi_h\in X_h$ for any $\psi_h\in Y_h$. Then, for any $\mathbf{Q}\in X$, we have by (3.16) and the definition of the spaces Y_h and $T_{h_0}(\Gamma)$,

$$\sum_{i=1}^{2}(P_h\mathbf{Q}-\mathbf{Q},\nabla\psi_h)_{\Omega_i}=0. \tag{4.15}$$

We use the backward difference at $t=t_{n+1}$ for (3.4)-(3.5) and obtain

$$\begin{aligned}&\sum_{i=1}^{2}\left\{(\mu\frac{\mathbf{A}^{n+1}-\mathbf{A}^n}{\tau},\overline{\mathbf{A}})_{\Omega_i}+(\frac{1}{\sigma}\mathbf{curl}\,\mathbf{A}^{n+1},\mathbf{curl}\,\overline{\mathbf{A}})_{\Omega_i}+(\mu\nabla\frac{\phi^{n+1}-\phi^n}{\tau},\overline{\mathbf{A}})_{\Omega_i}\right\}\\&\quad+b\,(\overline{\mathbf{A}},\mathbf{p}^{n+1})=-\sum_{i=1}^{2}\left\{(\mu\mathbf{R}_1^{n+1},\overline{\mathbf{A}})_{\Omega_i}+(\mu\mathbf{R}_2^{n+1},\overline{\mathbf{A}})_{\Omega_i}\right\},\ \ \forall\overline{\mathbf{A}}\in X,\end{aligned} \tag{4.16}$$

$$\begin{aligned}&\sum_{i=1}^{2}\left\{(\mu\frac{\mathbf{A}^{n+1}-\mathbf{A}^n}{\tau},\nabla\overline{\phi})_{\Omega_i}+(\mu\nabla\frac{\phi^{n+1}-\phi^n}{\tau},\nabla\overline{\phi})_{\Omega_i}\right\}\\&\quad=-\sum_{i=1}^{2}\left\{(\mu\mathbf{R}_1^{n+1},\nabla\overline{\phi})_{\Omega_i}+(\mu\mathbf{R}_2^{n+1},\nabla\overline{\phi})_{\Omega_i}\right\},\qquad\forall\overline{\phi}\in Y,\end{aligned} \tag{4.17}$$

$$b\,(\mathbf{A}^{n+1},\overline{\mathbf{p}})=0,\quad \forall\,\overline{\mathbf{p}}\in T(\Gamma), \tag{4.18}$$

where

$$\mathbf{R}_1^{n+1}=\left(\frac{\partial\mathbf{A}}{\partial t}\right)_{t_{n+1}}-\frac{\mathbf{A}^{n+1}-\mathbf{A}^n}{\tau},\qquad \|\mathbf{R}_1^{n+1}\|_{0;\Omega_i}=\mathrm{O}(\tau), \tag{4.19}$$

$$\mathbf{R}_2^{n+1}=\left(\frac{\partial\nabla\phi}{\partial t}\right)_{t_{n+1}}-\nabla\frac{\phi^{n+1}-\phi^n}{\tau},\qquad \|\mathbf{R}_2^{n+1}\|_{0;\Omega_i}=\mathrm{O}(\tau). \tag{4.20}$$

Let $\overline{\mathbf{A}}=\overline{\mathbf{A}}_h$, $\overline{\phi}=\overline{\phi}_h$ and $\overline{\mathbf{p}}=\overline{\mathbf{p}}_h$. Subtracting (4.16)-(4.18) from (4.3)-(4.5), we have

$$\begin{aligned}&\sum_{i=1}^{2}\left\{(\mu\frac{(\mathbf{A}_h^{n+1}-\mathbf{A}^{n+1})-(\mathbf{A}_h^n-\mathbf{A}^n)}{\tau},\overline{\mathbf{A}}_h)_{\Omega_i}+(\frac{1}{\sigma}\mathbf{curl}\,(\mathbf{A}_h^{n+1}-\mathbf{A}^{n+1}),\mathbf{curl}\,\overline{\mathbf{A}}_h)_{\Omega_i}\right.\\&\qquad\left.+(\mu\nabla\frac{(\phi_h^{n+1}-\phi^{n+1})-(\phi_h^n-\phi^n)}{\tau},\overline{\mathbf{A}}_h)_{\Omega_i}\right\}+b\,(\overline{\mathbf{A}}_h,\mathbf{p}_h^{n+1}-\mathbf{p}^{n+1})\\&\quad=\sum_{i=1}^{2}\left\{(\mu\mathbf{R}_1^{n+1},\overline{\mathbf{A}}_h)_{\Omega_i}+(\mu\mathbf{R}_2^{n+1},\overline{\mathbf{A}}_h)_{\Omega_i}\right\},\qquad\forall\overline{\mathbf{A}}_h\in X_h,\end{aligned} \tag{4.21}$$

$$\begin{aligned}&\sum_{i=1}^{2}\left\{(\mu\frac{(\mathbf{A}_h^{n+1}-\mathbf{A}^{n+1})-(\mathbf{A}_h^n-\mathbf{A}^n)}{\tau},\nabla\overline{\phi}_h)_{\Omega_i}\right.\\&\qquad\left.+(\mu\nabla\frac{(\phi_h^{n+1}-\phi^{n+1})-(\phi_h^n-\phi^n)}{\tau},\nabla\overline{\phi}_h)_{\Omega_i}\right\}\\&\quad=\sum_{i=1}^{2}\left\{(\mu\mathbf{R}_1^{n+1},\nabla\overline{\phi}_h)_{\Omega_i}+(\mu\mathbf{R}_2^{n+1},\nabla\overline{\phi}_h)_{\Omega_i}\right\},\qquad\forall\overline{\phi}_h\in Y_h,\end{aligned} \tag{4.22}$$

$$b\,(\mathbf{A}_h^{n+1}-\mathbf{A}^{n+1},\overline{\mathbf{p}}_h)=0,\quad \forall\,\overline{\mathbf{p}}_h\in T_{h_0}(\Gamma). \tag{4.23}$$

Set $\mathbf{\Theta}_h^{n+1}=\mathbf{A}_h^{n+1}-P_h\mathbf{A}^{n+1}$ and $\eta_h^{n+1}=\phi_h^{n+1}-\Pi_h\phi^{n+1}$. Taking $\overline{\mathbf{A}}_h=\mathbf{\Theta}_h^{n+1}$ and $\overline{\phi}_h=\eta_h^{n+1}$ in (4.21)-(4.22), together with the definition of the projection operator P_h, we come to

$$\begin{aligned}
&\sum_{i=1}^{2}\Big\{(\mu\frac{\mathbf{\Theta}_h^{n+1}-\mathbf{\Theta}_h^{n}}{\tau},\mathbf{\Theta}_h^{n+1})_{\Omega_i}+(\frac{1}{\sigma}\mathbf{curl}\,\mathbf{\Theta}_h^{n+1},\mathbf{curl}\,\mathbf{\Theta}_h^{n+1})_{\Omega_i}\\
&\quad+(\mu\nabla\frac{\eta_h^{n+1}-\eta_h^{n}}{\tau},\mathbf{\Theta}_h^{n+1})_{\Omega_i}\Big\}\\
&\quad=\sum_{i=1}^{2}\Big\{(\mu\frac{(\mathbf{A}^{n+1}-P_h\mathbf{A}^{n+1})-(\mathbf{A}^{n}-P_h\mathbf{A}^{n})}{\tau},\mathbf{\Theta}_h^{n+1})_{\Omega_i}\\
&\qquad+(\mathbf{A}^{n+1}-P_h\mathbf{A}^{n+1},\mathbf{\Theta}_h^{n+1})_{\Omega_i}\\
&\qquad+(\mu\nabla\frac{(\phi^{n+1}-\Pi_h\phi^{n+1})-(\phi^{n}-\Pi_h\phi^{n})}{\tau},\mathbf{\Theta}_h^{n+1})_{\Omega_i}\\
&\qquad+(\mu\mathbf{R}_1^{n+1},\mathbf{\Theta}_h^{n+1})_{\Omega_i}+(\mu\mathbf{R}_2^{n+1},\mathbf{\Theta}_h^{n+1})_{\Omega_i}\Big\},
\end{aligned}\tag{4.24}$$

$$\begin{aligned}
&\sum_{i=1}^{2}\Big\{(\mu\frac{\mathbf{\Theta}_h^{n+1}-\mathbf{\Theta}_h^{n}}{\tau},\nabla\eta_h^{n+1})_{\Omega_i}+(\mu\nabla\frac{\eta_h^{n+1}-\eta_h^{n}}{\tau},\nabla\eta_h^{n+1})_{\Omega_i}\Big\}\\
&\quad=\sum_{i=1}^{2}\Big\{(\mu\frac{(\mathbf{A}^{n+1}-P_h\mathbf{A}^{n+1})-(\mathbf{A}^{n}-P_h\mathbf{A}^{n})}{\tau},\nabla\eta_h^{n+1})_{\Omega_i}\\
&\qquad+(\mu\nabla\frac{(\phi^{n+1}-\Pi_h\phi^{n+1})-(\phi^{n}-\Pi_h\phi^{n})}{\tau},\nabla\eta_h^{n+1})_{\Omega_i}\\
&\qquad+(\mu\mathbf{R}_1^{n+1},\nabla\eta_h^{n+1})_{\Omega_i}+(\mu\mathbf{R}_2^{n+1},\nabla\eta_h^{n+1})_{\Omega_i}\Big\}.
\end{aligned}\tag{4.25}$$

Now adding up (4.24) and (4.25), multiplying both sides by τ and using (4.15) and $a(a-b)\geqslant a^2/2-b^2/2$, for any real numbers a and b, we have

$$\begin{aligned}
&\sum_{i=1}^{2}\Big\{\frac{1}{2}\|\sqrt{\mu}(\mathbf{\Theta}_h^{n+1}+\nabla\eta_h^{n+1})\|_{0;\Omega_i}^2-\frac{1}{2}\|\sqrt{\mu}(\mathbf{\Theta}_h^{n}+\nabla\eta_h^{n})\|_{0;\Omega_i}^2+\tau\|\frac{1}{\sqrt{\sigma}}\mathbf{curl}\,\mathbf{\Theta}_h^{n+1}\|_{0;\Omega_i}^2\Big\}\\
&\quad\le\sum_{i=1}^{2}\Big\{\tau(\mu\frac{(\mathbf{A}^{n+1}-P_h\mathbf{A}^{n+1})-(\mathbf{A}^{n}-P_h\mathbf{A}^{n})}{\tau},\mathbf{\Theta}_h^{n+1}+\nabla\eta_h^{n+1})_{\Omega_i}\\
&\qquad+\tau(\mathbf{A}^{n+1}-P_h\mathbf{A}^{n+1},\mathbf{\Theta}_h^{n+1}+\nabla\eta_h^{n+1})_{\Omega_i}\\
&\qquad+\tau(\mu\nabla\frac{(\phi^{n+1}-\Pi_h\phi^{n+1})-(\phi^{n}-\Pi_h\phi^{n})}{\tau},\mathbf{\Theta}_h^{n+1}+\nabla\eta_h^{n+1})_{\Omega_i}\\
&\qquad+\tau(\mu\mathbf{R}_1^{n+1},\mathbf{\Theta}_h^{n+1}+\nabla\eta_h^{n+1})_{\Omega_i}+\tau(\mu\mathbf{R}_2^{n+1},\mathbf{\Theta}_h^{n+1}+\nabla\eta_h^{n+1})_{\Omega_i}\Big\}.
\end{aligned}\tag{4.26}$$

On the other hand, for the projection operator P_h, we have

$$\begin{aligned}
&\sum_{i=1}^{2}\|\mathbf{A}^{n+1}-P_h\mathbf{A}^{n+1}\|_{\mathbf{curl},\Omega_i}+\|\mathbf{p}^{n+1}-P_h\mathbf{p}^{n+1}\|_{T(\Gamma)}\\
&\le C\inf_{\mathbf{Q}_h\in X_h}\|\mathbf{A}^{n+1}-\mathbf{Q}_h\|_{\mathbf{curl},\Omega_i}+C\inf_{\mathbf{q}_h\in T_{h_0}(\Gamma)}\|\mathbf{q}_h-\mathbf{p}^{n+1}\|_{T(\Gamma)}.
\end{aligned}\tag{4.27}$$

Now using the edge element interpolation estimate in [5], we get

$$\inf_{\mathbf{Q}_h\in X_{h_i}}\|\mathbf{A}^{n+1}-\mathbf{Q}_h\|_{\mathbf{curl},\Omega_i}\le Ch_i^{\alpha}\|\mathbf{A}^{n+1}\|_{\alpha,\mathbf{curl},\Omega_i}.\tag{4.28}$$

Next we introduce a triangulation $\mathcal{T}^{h_0}$ in Ω_1 whose restriction on Γ coincides with Γ_{h_0} and let X_{h_0} be the Nédélec $H(\mathbf{curl},\Omega_1)$-conforming edge element over the mesh $\mathcal{T}^{h_0}$. Then from the definition of $T_{h_0}(\Gamma)$ we know that

$$T_{h_0}(\Gamma)=\{\mathbf{v}_h\times\mathbf{n};\ \mathbf{v}_h\in X_{h_0}\}.$$

Now using the fact that $\mathbf{p}^{n+1}=\dfrac{1}{\sigma}\mathbf{curl}\,\mathbf{A}^{n+1}\times\mathbf{n}:=\mathbf{T}^{n+1}\times\mathbf{n}$, we can easily show that, by (2.10), $\mathbf{T}^{n+1}\in H^{\alpha}(\mathbf{curl},\Omega_1)$ and

$$\|\mathbf{T}^{n+1}\|_{\alpha,\mathbf{curl},\Omega_1}\le C(\|\mathbf{A}^{n+1}\|_{\alpha,\mathbf{curl},\Omega_1}+\|\phi_t^{n+1}\|_{1+\alpha,\Omega_1}+\|\mathbf{A}_t^{n+1}\|_{\alpha,\Omega_1}).$$

Thus we have by Lemma 3.2 and the standard edge element error estimate in [5] that

$$\inf_{\mathbf{q}_h\in T_{h_0}(\Gamma)}\|\mathbf{q}_h-\mathbf{p}^{n+1}\|_{T(\Gamma)}\le C\inf_{\mathbf{v}_h\in X_{h_0}}\|\mathbf{v}_h-\mathbf{T}^{n+1}\|_{\mathbf{curl},\Omega_1}\le C\,h_0^{\alpha}\|\mathbf{T}^{n+1}\|_{\alpha,\mathbf{curl},\Omega_1}. \tag{4.29}$$

From (4.28)-(4.29), we derive

$$\sum_{i=1}^{2}\|\mathbf{A}^{n+1}-P_h\mathbf{A}^{n+1}\|_{\mathbf{curl},\Omega_i}\le\sum_{i=1}^{2}C_ih_i^{\alpha}+C_0h_0^{\alpha}. \tag{4.30}$$

Similar to the proof of Theorem 3.2 in [11], using the discrete Gronwall's inequality, the finite element interpolation element estimate and (4.30) to (4.26), we easily complete the proof of the theorem with the help of the triangle inequality. We omit the details.

5. A Fully-discrete Decoupled $\mathbf{A}-\phi$ Scheme with a Nonmatching Grid for Eddy Current Problem

To avoid increasing the number of freedoms and equations by solving Problem (VI) directly, we present a new decoupled $\mathbf{A}-\phi$ scheme in this part.

First, we need to extend $\mathbf{A}$ and ϕ with some regularity from the time interval $[0,T]$ to the interval $[-\tau,T]$. Let

$$\mathbf{A}^{-1}=\mathbf{0}\quad\text{and}\quad\phi^{-1}=0.$$

Then, the decoupled $\mathbf{A}-\phi$ scheme is:

$$\mathbf{A}_h^0=\pi_h\mathbf{A}_0,\quad\phi_h^0=\Pi_h\phi_0,\quad\phi_h^{-1}=0 \tag{5.1}$$

and for $n=0,1,\cdots,M-1$, find $(\mathbf{A}_h^{n+1},\ \mathbf{p}_h^{n+1})\in X_h\times T_{h_0}(\Gamma)$ such that

$$\begin{aligned}\sum_{i=1}^{2}\Big\{(\mu\frac{\mathbf{A}_h^{n+1}-\mathbf{A}_h^{n}}{\tau},\overline{\mathbf{A}}_h)_{\Omega_i}+(\frac{1}{\sigma}\,\mathbf{curl}\,\mathbf{A}_h^{n+1},\mathbf{curl}\,\overline{\mathbf{A}}_h)_{\Omega_i}\Big\}+\ll\mathbf{p}_h^{n+1},\overline{\mathbf{A}}_h\gg_{2,\Gamma}\\ -\ll\mathbf{p}_h^{n+1},\overline{\mathbf{A}}_h\gg_{1,\Gamma}=-\sum_{i=1}^{2}(\mu\nabla\frac{\phi_h^{n}-\phi_h^{n-1}}{\tau},\overline{\mathbf{A}}_h)_{\Omega_i},\quad\forall\overline{\mathbf{A}}_h\in X_h,\end{aligned} \tag{5.2}$$

$$\ll\mathbf{A}_h^{n+1},\ \overline{\mathbf{p}}_h\gg_{2,\Gamma}-\ll\mathbf{A}_h^{n+1},\ \overline{\mathbf{p}}_h\gg_{1,\Gamma}=0,\quad\forall\,\overline{\mathbf{p}}_h\in T_{h_0}(\Gamma), \tag{5.3}$$

and find $\phi_h^{n+1} \in Y_h$ such that

$$\sum_{i=1}^{2}(\mu\nabla\frac{\phi_h^{n+1}-\phi_h^n}{\tau}, \nabla\overline{\phi}_h)_{\Omega_i} = -\sum_{i=1}^{2}(\mu\frac{\mathbf{A}_h^{n+1}-\mathbf{A}_h^n}{\tau}, \nabla\overline{\phi}_h)_{\Omega_i}, \quad \forall\overline{\phi}_h \in Y_h. \tag{5.4}$$

From the discussion to Problem (V) in Theorem 4.1 we know that under the same assumptions of Theorem 4.1, the system (5.2)-(5.3) has a unique solution $(\mathbf{A}_h^{n+1}, \mathbf{p}_h^{n+1})$ at each time step. Moreover, by introducing the elliptic projection operator (referring to the proof of Theorem 4.2) and imitating the proof of Theorem 3.3 in [11], we have

Theorem 5.1. *Under the same assumptions of Theorem 4.1, let $(\mathbf{A}^{n+1}, \phi^{n+1}, \mathbf{p}^{n+1})$ and $(\mathbf{A}_h^{n+1}, \phi_h^{n+1}, \mathbf{p}_h^{n+1})$ be the solutions of Problem (I) and the decoupled $\mathbf{A}-\phi$ approximation (5.1)-(5.4) at time $t = t_{n+1}$ respectively. Supposing that for some $\alpha > 1/2$,*

$$\mathbf{A} \in H^3(0,T; H^\alpha(\mathbf{curl};\Omega_1) \times H^\alpha(\mathbf{curl};\Omega_2)), \qquad \phi \in H^2(0,T; Y \cap H^{1+\alpha}(\Omega_1) \times H^{1+\alpha}(\Omega_2)).$$

Then, we have the following error estimate:

$$\max_{0\le n\le M-1}\left(\sum_{i=1}^{2}\|(\mathbf{A}_h^{n+1}+\nabla\phi_h^{n+1}) - (\mathbf{A}^{n+1}+\nabla\phi^{n+1})\|_{0,\Omega_i}^2\right) \le C\tau^2 + \sum_{i=1}^{2} C_i h_i^{2\alpha} + C_0 h_0^{2\alpha}.$$

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