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Stabilized Finite Element Methods for Biot's Consolidation Problems Using Equal Order Elements

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Abstract. Using the standard mixed Galerkin methods with equal order elements to solve Biot's consolidation problems, the pressure close to the initial time produces large non-physical oscillations. In this paper, we propose a class of fully discrete stabilized methods using equal order elements to reduce the effects of non-physical oscillations. Optimal error estimates for the approximation of displacements and pressure at every time level are obtained, which are valid even close to the initial time. Numerical experiments illustrate and confirm our theoretical analysis.

AMS subject classifications: 65M60

Key words: Biot's problem, LBB condition, stabilized method, error estimates, numerical experiments, Terzaghi problem.

1 Introduction

The Biot's consolidation model describes the time-dependent interaction between the deformation of an elastic porous material and the fluid flow inside of it. This problem was first proposed by Terzaghi [1], and summarized by Biot [2–4]. Biot's model is widely used in geomechanics, hydrogeology, petrol engineering and biomechanics. This paper focuses on the quasi-static Biot's consolidation model.

Variational principles for Biot's consolidation problem and finite element approximations based on the Galerkin method were presented in [5–9]. Asymptotic behavior of semi-discrete finite element approximations of the Biot's consolidation problem was discussed in [8]. The authors analyzed the standard mixed Galerkin methods for the Biot's consolidation problem in [9]. Long-time stability was proved since they obtained

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an exponential decay of the error in the initial data in time. Error estimates on the LB-B (Ladyzhenskaya-Babuška-Brezzi) stable and LBB unstable spaces combinations were presented. For the LBB stable cases, the error estimates were optimal even close to the initial time. On the other hand, for the unstable cases, especially those with equal-order of interpolation, the lack of stability close to the initial time results from an unstable approximation of the initial condition. Therefore, oscillations of the pressure close to the initial time may happen. Some researches [9, 10] are devoted to working on this issue. Murad and Loula [9] proposed a post-processing technique, which has to use the LBB stable space combinations in the post-processing. Paper [10] presented a penalty stabilized scheme using equal-order linear space, the numerical experiments showed the good stability and convergence of their method, but no further theoretical analysis is given. A fully discrete stabilized discontinuous Galerkin method was proposed in [11], error estimates for the pressure close to the initial time and numerical experiments were not given.

Motivated by the stabilized methods for the Stokes problem [13–19], in this paper we proposed a large class of fully discrete stabilized methods including the method in [10]. By adding a weak consistent term with time derivate of pressure, we obtain additional control of the pressure. Then we establish the error estimates for the velocities and pressure with arbitrary combination of interpolations. The error estimates are optimal even close to the initial time. Numerical experiments illustrate and confirm our theoretical analysis.

An outline of the paper is as follows. In Section 2, we introduce the quasi-static Biot's consolidation model. In Section 3, we propose and analyze the stability of our methods. In Section 4 we give error estimates for our scheme. In Section 5, we give some numerical experiments. In Section 6, we conclude the whole paper.

Throughout this paper, we use C to denote a positive constant independent of Δt and h, not necessarily the same at each occurrence.

2 The quasi-static Biot's consolidation model

Let $\Omega \in \mathbb{R}^d$ $(d=1,2,3)$ be a bounded domain with polygonal or polyhedral boundary $\Gamma=\partial\Omega$. We use $W^{m,p}(\Omega)$, $W_0^{m,p}(\Omega)$ to denote the m-order Sobolev spaces on Ω, and use $\|\cdot\|_{m,p}$, $|\cdot|_{m,p}$ to denote the norm and semi-norm on these spaces, respectively. When p=2, we set $\mathbf{H}^m(\Omega)=W^{m,p}(\Omega)$, $\mathbf{H}_0^m(\Omega)=W_0^{m,p}(\Omega)$ and $\|\cdot\|_m=\|\cdot\|_{m,p}$, $|\cdot|_m=|\cdot|_{m,p}$. Denote the inner product of $\mathbf{H}^m(\Omega)$ by $(\cdot,\cdot)_m$ and $(\cdot,\cdot)=(\cdot,\cdot)_0$. Let X denote a Banach space with the norm $\|\cdot\|_X$. We define

$$L^\infty(0,T;X)=\Big\{v\in X: ess\ \sup_{0\le t\le T}\|v\|_X^2<\infty\Big\}, \tag{2.1a}$$

$$L^2(0,T;X)=\Big\{v\in X: \int_0^T \|v\|_X^2 dt<\infty\Big\}, \tag{2.1b}$$

$$H^1(0,T;X)=\left\{v\in X: \int_0^T \left(\|v\|_X^2+\left\|\frac{dv}{dt}\right\|_X^2\right)dt<\infty\right\}. \tag{2.1c}$$

Vector analogues of the Sobolev spaces along with vector-valued functions are denoted by upper and lower case bold face font, respectively, e.g., $\mathbf{H}_0^1(\Omega)$, $\mathbf{L}^2(\Omega)$ and $\mathbf{u}$. Let $\mathcal{T}_h=\{K\}$ be a quasi-uniform simplex partitioning of Ω, h_K stands for the diameter of K and $h=\max_{K\in\mathcal{T}_h}h_K$.

To simplify of notations, we consider the Biot's consolidation problem for constant coefficient case:

$$\begin{cases} -A\mathbf{u}+\nabla p=\mathbf{f} & \text{in } \Omega\times(0,T], \\ \nabla\cdot D_t\mathbf{u}-Bp=g & \text{in } \Omega\times(0,T], \end{cases} \tag{2.2}$$

where

$$A\mathbf{u}=\mu\Delta\mathbf{u}+(\mu+\lambda)\nabla\cdot\mathbf{u}, \tag{2.3a}$$

$$Bp=\frac{\kappa}{\boldsymbol{\eta}}\Delta p. \tag{2.3b}$$

$\mathbf{f}=\mathbf{f}(x,t)\in\mathbb{R}^d\times[0,T]$ represents a given body force, $g=g(x,t)\in\mathbb{R}\times[0,T]$ represents a forced fluid ex-traction or injection process, $\mathbf{u}=\mathbf{u}(x,t)\in\mathbb{R}^d\times[0,T]$, represents the displacements, $p=p(x,t)\in\mathbb{R}\times[0,T]$ is the pore pressure. The coefficients μ, λ and κ are Lamé constants and the permeability of the porous skeleton, respectively, and η denotes the viscosity of the pore fluid, T denotes a positive constant. D_t denotes time derivative. The initial condition is given by

$$\nabla\cdot\mathbf{u}(x,0)=0 \quad \text{on } \Omega, \tag{2.4}$$

and the boundary condition is given by

$$\mathbf{u}=0, \quad \frac{\kappa}{\eta}\nabla p\cdot\mathbf{n}=g_1 \quad \text{on } \Gamma_1, \tag{2.5}$$

and

$$p=0, \quad (\lambda\nabla\cdot\mathbf{u}I+2\mu\varepsilon(\mathbf{u}))\mathbf{n}=\mathbf{f}_1 \quad \text{on } \Gamma_2, \tag{2.6}$$

where $\Gamma_1\cup\Gamma_2=\Gamma$ with Γ_1 and Γ_2 disjoint subsets of Γ with non-null measure, $\mathbf{f}_1=\mathbf{f}_1(t)\in\mathbb{R}^d\cap\mathbf{L}^2(\Gamma_1)$ and $g_1=g_1(t)\in\mathbb{R}\cap\mathbf{L}^2(\Gamma_2)$, I is a $d\times d$ matrix, $\varepsilon(\mathbf{u})=(\nabla\mathbf{u}+\nabla^{\mathbf{T}}\mathbf{u})/\mathbf{2}$.

In this paper we always assume the solution of (2.2)-(2.6) is smooth enough. We define the forms

$$a(\mathbf{u},\mathbf{v})=2\mu(\varepsilon(\mathbf{u}),\varepsilon(\mathbf{v}))+\lambda(\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{v}), \tag{2.7a}$$

$$b(\mathbf{v},p)=(\nabla\cdot\mathbf{v},p), \tag{2.7b}$$

$$d(p,q)=\left(\frac{\kappa}{\eta}\nabla p,\nabla q\right), \tag{2.7c}$$

$$F(\mathbf{v})=(\mathbf{f},\mathbf{v})+\int_{\Gamma_2}\mathbf{f}_1\mathbf{v}d\Gamma, \tag{2.7d}$$

$$G(q)=(g,q)+\int_{\Gamma_1}g_1 q d\Gamma, \tag{2.7e}$$

and we introduce the displacements and pressure spaces

$$\mathbf{V}=\left\{\mathbf{v}\in\mathbf{H}^1(\Omega):\mathbf{v}=\mathbf{0}\text{ on }\Gamma_1\right\}, \tag{2.8a}$$

$$Q=\left\{q\in H^1(\Omega):q=0\text{ on }\Gamma_2\right\}. \tag{2.8b}$$

From Korn's inequality and the fact that Γ_1 has non-null measure we have:

Lemma 2.1. $\|\varepsilon(\mathbf{v})\|_0$ *is a norm on* $\mathbf{V}$*, and is equivalent to* $\|\mathbf{v}\|_1$*. Then the variational formulation for problem* (2.2) *with the initial condition* (2.4) *and the boundary conditions* (2.5)-(2.6) *consists of the following:*

Find the initial value $(\mathbf{u}^0,p^0)\in\mathbf{V}\times\mathbf{L}^2(\Omega)$ *such that*

$$\begin{cases} a(\mathbf{u}^0,\mathbf{v})-b(\mathbf{v},p^0)=F^0(\mathbf{v}), & \forall\mathbf{v}\in\mathbf{V},\\ b(\mathbf{u}^0,q)=0, & \forall q\in\mathbf{L}^2(\Omega),\end{cases} \tag{2.9}$$

and $(\mathbf{u},p)\in\mathbf{V}\times Q$ *for any* $t\in(0,T]$ *such that*

$$\begin{cases} a(\mathbf{u},\mathbf{v})-b(\mathbf{v},p)=F(\mathbf{v}), & \forall\mathbf{v}\in\mathbf{V},\\ b(D_t\mathbf{u},q)+d(p,q)=G(q), & \forall q\in Q.\end{cases} \tag{2.10}$$

Let $\mathbf{V}_h=\mathbf{V}\times\mathbf{P}_\mathbf{l}(\mathcal{T}_h)$ and $Q_h=Q\times\mathbf{P}_m(\mathcal{T}_h)$, where $l\geq 1$, $m\geq 1$, then we have the following approximation properties.

Lemma 2.2. *There exists a pair of interpolation operator* $(\mathbf{I}_h,J_h)$ *from* $\mathbf{V}\times Q$ *to* $\mathbf{V}_h\times Q_h$ *such that,*

$$\|\mathbf{u}-\mathbf{I}_h\mathbf{u}\|_0+h|\mathbf{u}-\mathbf{I}_h\mathbf{u}|_1\leq Ch^{l+1}|\mathbf{u}|_{l+1}, \qquad \forall\mathbf{u}\in\mathbf{V}\cap\mathbf{H}^{l+1}(\Omega), \tag{2.11a}$$

$$\|p-J_hp\|_0+h|p-J_hp|_1\leq Ch^{m+1}|p|_{m+1}, \qquad \forall p\in q\cap\mathbf{H}^{m+1}(\Omega). \tag{2.11b}$$

We also introduce the interpolation operator used in our paper, find $(\mathbf{u}_I,p_I)\in\mathbf{V}_h\times Q_h$ such that

$$\begin{cases} a(\mathbf{u}_I,\mathbf{v}_h)-b(\mathbf{v}_h,p_I)=a(\mathbf{u},\mathbf{v}_h)-b(\mathbf{v}_h,p), & \forall\mathbf{v}_h\in\mathbf{V}_h,\\ d(p_I,q_h)=d(p,q_h), & \forall q_h\in Q_h.\end{cases} \tag{2.12}$$

It's obvious problem (2.12) is well-defined. Using the technique in [12, 20], we have the error estimates

$$\begin{aligned}
&\|\mathbf{u}-\mathbf{u}_I\|_0+h|\mathbf{u}-\mathbf{u}_I|_1\\
&\le Ch^{s+1}\left(\|\mathbf{u}\|_{l+1}+\|p\|_{m+1}\right), \qquad \forall\mathbf{u}\in\mathbf{V}\cap\mathbf{H}^{l+1}(\Omega),\quad p\in Q\cap H^{m+1}(\Omega), &(2.13a)\\
&\|p-p_I\|_0+h|p-p_I|_1\le Ch^{m+1}\|p\|_{m+1},\quad \forall p\in Q\cap H^{m+1}(\Omega), &(2.13b)
\end{aligned}$$

where $s=\min(l,m+1)$. Applying D_t and D_{tt} on (2.12) and using the technique in [20] again, we have the error estimates

$$\begin{aligned}
&\|D_t(\mathbf{u}-\mathbf{u}_I)\|_0+h|D_t(\mathbf{u}-\mathbf{u}_I)|_1\le Ch^{s+1}\left(\|D_t\mathbf{u}\|_{l+1}+\|D_tp\|_{m+1}\right),\\
&\quad\forall D_t\mathbf{u}\in\mathbf{V}\cap\mathbf{H}^{l+1}(\Omega),\quad D_tp\in Q\cap H^{m+1}(\Omega), &(2.14a)\\
&\|D_t(p-p_I)\|_0+h|D_t(p-p_I)|_1\le Ch^{m+1}\|D_tp\|_{m+1},\\
&\quad\forall D_tp\in Q\cap H^{m+1}(\Omega), &(2.14b)\\
&\|D_{tt}(\mathbf{u}-\mathbf{u}_I)\|_0+h|D_{tt}(\mathbf{u}-\mathbf{u}_I)|_1\le Ch^{s+1}\left(\|D_{tt}\mathbf{u}\|_{l+1}+\|D_{tt}p\|_{m+1}\right),\\
&\quad\forall D_{tt}\mathbf{u}\in\mathbf{V}\cap\mathbf{H}^{l+1}(\Omega),\quad D_{tt}p\in Q\cap H^{m+1}(\Omega), &(2.14c)\\
&\|D_{tt}(p-p_I)\|_0+h|D_{tt}(p-p_I)|_1\le Ch^{m+1}\|D_{tt}p\|_{m+1},\\
&\quad\forall D_{tt}p\in Q\cap H^{m+1}(\Omega). &(2.14d)
\end{aligned}$$

For simplicity, we define

$$E^n=d_t\mathbf{u}^n-D_t\mathbf{u}^n, \tag{2.15}$$

where $d_t\mathbf{u}^n$ is defined as

$$d_t\mathbf{u}^n:=\frac{\mathbf{u}^n-\mathbf{u}^{n-1}}{\Delta t}. \tag{2.16}$$

We also define

$$\begin{aligned}
e_{\mathbf{u}}&=\boldsymbol{\xi}+\boldsymbol{\eta}=(\mathbf{u}_I-\mathbf{u}_h)+(\mathbf{u}-\mathbf{u}_I), &(2.17a)\\
e_p&=\gamma+\beta=(p_I-p_h)+(p-p_I). &(2.17b)
\end{aligned}$$

The standard Galerkin method using θ- difference for (2.9), (2.10) reads.

Find $\left(\mathbf{u}_h^0,p_h^0\right)\in\mathbf{V}_h\times Q_h$ such that

$$\begin{cases}
a\left(\mathbf{u}_h^0,\mathbf{v}_h\right)-b\left(\mathbf{v}_h,p_h^0\right)=F^0(\mathbf{v}_h), & \forall\mathbf{v}_h\in\mathbf{V}_h,\\
b\left(\mathbf{u}_h^0,q_h\right)=0, & \forall q_h\in Q_h.
\end{cases} \tag{2.18}$$

Find $\left(\mathbf{u}_h^n,p_h^n\right)\in\mathbf{V}_h\times Q_h$ such that

$$\begin{cases}
a\left(\mathbf{u}_h^n,\mathbf{v}_h\right)-b\left(\mathbf{v}_h,p_h^n\right)=F^n(\mathbf{v}_h), & \forall\mathbf{v}_h\in\mathbf{V}_h,\\
b\left(d_t\mathbf{u}_h^n,q_h\right)+\theta d\left(p_h^n,q_h\right)+(1-\theta)d\left(p_h^{n-1},q_h\right)=\theta G^n(q_h)+(1-\theta)G^{n-1}(q_h), & \forall q_h\in Q_h,
\end{cases}$$

where $0\le\theta\le 1$, $1\le n\le N$. When $\theta=1$ we obtain the backwards difference methods.

Find $\left(\mathbf{u}_h^n,p_h^n\right)\in\mathbf{V}_h\times Q_h$ such that

$$\left\{\begin{array}{ll} a\left(\mathbf{u}_h^n,\mathbf{v}_h\right)-b\left(\mathbf{v}_h,p_h^n\right)=F^n\left(\mathbf{v}_h\right), & \forall\mathbf{v}_h\in\mathbf{V}_h,\\ b\left(d_t\mathbf{u}_h^n,q_h\right)+d\left(p_h^n,q_h\right)=G^n\left(q_h\right), & \forall q_h\in Q_h,\end{array}\right. \tag{2.19}$$

where $1\le n\le N$.

We notice that (2.18) is a Stokes problem, it's not well-posed unless $\mathbf{V}_h\times Q_h$ satisfies the discrete LBB condition. If the initial value isn't needed (the initial pressure isn't needed), we don't need to solve (2.18), since we can use $b\left(\mathbf{u}_h^0,q_h\right)=0$ in (2.19), when $n=1$. Otherwise, we propose the following scheme to solve the initial value.

Let the conforming space $\mathbf{V}_{h0}\times Q_{h0}\subset\mathbf{V}\times Q$ satisfy the condition

$$\mathbf{V}_h\times Q_h\subseteq\mathbf{V}_{h0}\times Q_{h0}, \tag{2.20}$$

and there exists a pair of interpolation operator $(\mathbf{I}_h,J_h)$ from $\mathbf{V}\times Q$ to $\mathbf{V}_h\times Q_h$ such that,

$$\|\mathbf{u}-\mathbf{I}_{h0}\mathbf{u}\|_0+h|\mathbf{u}-\mathbf{I}_{h0}\mathbf{u}|_1\le Ch^{l_0+1}\|\mathbf{u}\|_{l_0+1},\qquad \forall\mathbf{u}\in\mathbf{V}\cap\mathbf{H}^{l_0+1}(\Omega), \tag{2.21a}$$

$$\|p-J_{h0}p\|_0+h|p-J_{h0}p|_1\le Ch^{m_0+1}\|p\|_{m_0+1},\qquad \forall p\in Q\cap H^{m_0+1}(\Omega). \tag{2.21b}$$

From (2.20), it's obviously that

$$l\le l_0,\quad m\le m_0. \tag{2.22}$$

We use two steps to solve the initial value.

Step 1 Find $\left(\mathbf{u}_{h0}^0,p_{h0}^0\right)\in\mathbf{V}_{h0}\times Q_{h0}$ such that

$$\left\{\begin{array}{ll} a\left(\mathbf{u}_{h0}^0,\mathbf{v}_h\right)-b\left(\mathbf{v}_h,p_{h0}^0\right)=F^0\left(\mathbf{v}_h\right), & \forall\mathbf{v}_h\in\mathbf{V}_{h0},\\ b\left(\mathbf{u}_{h0}^0,q_h\right)+\alpha M_{h0}\left(p_{h0}^0,q_h\right)=0, & \forall q_h\in Q_{h0},\end{array}\right. \tag{2.23}$$

where $M_{h0}(p_h,q_h)$ is a LBB stabilized term and $\alpha=1$. If $\mathbf{V}_{h0}\times Q_{h0}$ satisfies the LBB condition, $\alpha=0$. Then we have the error estimates,

$$\left|\mathbf{u}^0-\mathbf{u}_{h0}^0\right|_1+\left\|p^0-p_{h0}^0\right\|_0\le Ch^{s_0}\left(\left\|\mathbf{u}^0\right\|_{l_0+1}+\left\|p^0\right\|_{m_0+1}\right), \tag{2.24}$$

where

$$s_0=\min(l_0,m_0+1).$$

Step 2 Find $\left(\mathbf{u}_{h0}^0,p_{h0}^0\right)\in\mathbf{V}_h\times Q_h$ such that

$$\left\{\begin{array}{ll} a\left(\mathbf{u}_h^0,\mathbf{v}_h\right)-b\left(\mathbf{v}_h,p_h^0\right)=a\left(\mathbf{u}_{h0}^0,\mathbf{v}_h\right)-b\left(\mathbf{v}_h,p_{h0}^0\right), & \forall\mathbf{v}_h\in\mathbf{V}_h,\\ d\left(p_h^0,q_h\right)=d\left(p_{h0}^0,q_h\right), & \forall q_h\in Q_h.\end{array}\right. \tag{2.25}$$

Then we have the error estimates,

$$\left|\mathbf{u}^0-\mathbf{u}_h^0\right|_1\le Ch^s\left(\left\|\mathbf{u}^0\right\|_{l+1}+\left\|p^0\right\|_{m+1}\right), \tag{2.26a}$$

$$\left\|p^0-p_h^0\right\|_0\le Ch^{s_0}\left(\left\|\mathbf{u}^0\right\|_{l_0+1}+\left\|p^0\right\|_{m_0+1}\right)+Ch^{m+1}\left\|p^0\right\|_{m+1}. \tag{2.26b}$$

3 New stabilized methods

3.1 The stabilized scheme

In this section, we will give a unified form of our new stabilized methods. We invoke the: let the symmetrical bilinear functional $M_h(p,q)$ satisfy the hypothesis:

$$M_h(p,q)\leq M_h(p,p)^{1/2}M_h(q,q)^{1/2}, \qquad \forall p,q\in\mathbf{V}, \tag{3.1a}$$

$$M_h(p_h,p_h)\leq\|p_h\|^2, \qquad \forall p_h\in\mathbf{V}_h, \tag{3.1b}$$

$$M_h(p,p)\leq Ch^{2r}\|p\|_r^2, \qquad \forall p\in\mathbf{H}^r(\Omega), \tag{3.1c}$$

where r is a fixed integer satisfying $0\leq r\leq m$. And there exists an interpolation operator $\mathbf{j}_h:\mathbf{H}_0^1(\Omega)\cap\mathbf{V}\rightarrow\mathbf{H}_0^1(\Omega)\cap\mathbf{V}_h$ for any $\mathbf{v}\in\mathbf{H}_0^1(\Omega)\cap\mathbf{V}$ and $q_h\in Q_h$, there holds

$$|(\nabla\cdot(\mathbf{j}_h\mathbf{v}-\mathbf{v}),q_h)|\leq Ch^{-1}M_h(q_h,q_h)^{1/2}(\|\mathbf{j}_h\mathbf{v}-\mathbf{v}\|_0+h|\mathbf{j}_h\mathbf{v}-\mathbf{v}|_1), \tag{3.2a}$$

$$\|\mathbf{v}-\mathbf{j}_h\mathbf{v}\|_0+h|\mathbf{v}-\mathbf{j}_h\mathbf{v}|_1\leq C\mathbf{h}^{m+1}\|\mathbf{v}\|_{m+1}, \quad \mathbf{v}\in\mathbf{V}\cap\mathbf{H}_0^1(\Omega)\cap\mathbf{H}^{m+1}(\Omega). \tag{3.2b}$$

The stabilized method for the Biot's consolidation problems reads: seek $(\mathbf{u}_h,p_h)\in\mathbf{V}_h\times Q_h$ such that

$$\begin{cases} a(\mathbf{u}_h,\mathbf{v}_h)-b(\mathbf{v}_h,p_h)=F(\mathbf{v}_h), & \forall\mathbf{v}_h\in\mathbf{V}_h,\\ b(D_t\mathbf{u}_h,q_h)+d(p_h,q_h)+M_h(D_tp_h,q_h)=G(q_h), & \forall q_h\in Q_h.\end{cases}$$

The θ- difference scheme of last scheme reads:

Seek $\left(\mathbf{u}_h^n,p_h^n\right)\in\mathbf{V}_h\times Q_h$ such that

$$\begin{cases} a\left(\mathbf{u}_h^n,\mathbf{v}_h\right)-b\left(\mathbf{v}_h,p_h^n\right)=F^n(\mathbf{v}_h), & \forall\mathbf{v}_h\in\mathbf{V}_h,\\ b\left(d_t\mathbf{u}_h^n,q_h\right)+\theta d\left(p_h^n,q_h\right)+\theta M_h(d_tp_h^n,q_h) & \\ \quad+(1-\theta)d\left(p_h^{n-1},q_h\right)+(1-\theta)M_h(d_tp_h^{n-1},q_h) & \\ =\theta G^n(q_h)+(1-\theta)G^{n-1}(q_h), & \forall q_h\in Q_h,\end{cases}$$

where $0\leq\theta\leq1$, $1\leq n\leq N$. When $\theta=0$ and $\theta=\frac{1}{2}$, the last scheme is forward difference and Crank-Nicolson difference scheme, respectively. In this paper we only give an analysis of the backwards difference scheme, i.e., $\theta=1$ in last scheme: seek $\left(\mathbf{u}_h^n,p_h^n\right)\in\mathbf{V}_h\times Q_h$ such that

$$\begin{cases} a\left(\mathbf{u}_h^n,\mathbf{v}_h\right)-b\left(\mathbf{v}_h,p_h^n\right)=F^n(\mathbf{v}_h), & \forall\mathbf{v}_h\in\mathbf{V}_h,\\ b\left(d_t\mathbf{u}_h^n,q_h\right)+d\left(p_h^n,q_h\right)+M_h(d_tp_h^n,q_h)=G^n(q_h), & \forall q_h\in Q_h.\end{cases} \tag{3.3}$$

Subtracting (3.3) from (2.10) we get the error estimates equation,

$$\begin{cases} a(\boldsymbol{\xi}^n,\mathbf{v}_h)-b(\mathbf{v}_h,\gamma^n)=0, & \forall\mathbf{v}_h\in\mathbf{V}_h,\\ b(d_t\boldsymbol{\xi}^n,q_h)+d(\gamma^n,q_h)+M_h(d_t\gamma^n,q_h) & \\ =b(E^n,q_h)-b(d_t\boldsymbol{\eta}^n,q_h)-M_h(d_t\beta^n,q_h)+M_h(d_tp^n,q_h), & \forall q_h\in Q_h.\end{cases} \tag{3.4}$$

Applying d_t on the first equation in (3.5) we get

$$a(d_t\boldsymbol{\xi}^n,\mathbf{v}_h)-b(\mathbf{v}_h,d_t\gamma^n)=0,\quad \forall \mathbf{v}_h\in\mathbf{V}_h,\quad n\geq 1. \tag{3.5}$$

Lemma 3.1. *There exists a constant C, for any $p_h\in Q_h$ such that*

$$\sup_{\mathbf{v}_h\in\mathbf{V}_h,\mathbf{v}_h\neq 0}\frac{(\nabla\cdot\mathbf{v}_h,p_h)}{\|\mathbf{v}_h\|_1}+CM_h(p_h,p_h)^{1/2}\geq C\|p_h\|_0. \tag{3.6}$$

Proof. We define

$$C_0=\frac{1}{|\Omega|}\int_\Omega p_h dx. \tag{3.7}$$

So for any $p_h\in Q_h$, there exists $\mathbf{v}\in\mathbf{H}_0^1(\Omega)\subset\mathbf{V}$, such that

$$\nabla\cdot\mathbf{v}=p_h-C_0,\quad \|\mathbf{v}_h\|_1\leq C\|p_h-C_0\|_0. \tag{3.8}$$

Then we have

$$\begin{aligned}\|p_h-C_0\|_0^2&=(\nabla\cdot\mathbf{v},p_h-C_0)=(\nabla\cdot\mathbf{v},p_h)\\&=(\nabla\cdot(\mathbf{v}-\mathbf{j}_h\mathbf{v}),p_h)+(\nabla\cdot\mathbf{j}_h\mathbf{v},p_h)\\&\leq Ch^{-1}M_h(p_h,p_h)^{1/2}(\|\mathbf{j}_h\mathbf{v}-\mathbf{v}\|_0+h|\mathbf{v}-\mathbf{j_h}\mathbf{v}|_1)+(\nabla\cdot\mathbf{j}_h\mathbf{v},p_h)\\&\leq CM_h(p_h,p_h)^{1/2}\|p_h-C_0\|_0+(\nabla\cdot\mathbf{j}_h\mathbf{v},p_h),\end{aligned} \tag{3.9}$$

this implies (3.6). ☐

Lemma 3.2. *Under the definitions from* (2.15) *to* (2.17b)*, we have the estimates*

$$\|\nabla\cdot E^n\|_0^2\leq C\Delta t\|\nabla\cdot D_{tt}\mathbf{u}\|^2_{\mathbf{L}^2(t_{n-1},t_n;\mathbf{L}^2(\Omega))}, \tag{3.10a}$$

$$\|\nabla\cdot d_t\eta^n\|_0^2\leq C\frac{h^{2s}}{\Delta t}\left(\|D_t\mathbf{u}\|^2_{\mathbf{L}^2\left(t_{n-1},t_n;\mathbf{H}^{l+1}(\Omega)\right)}+\|D_tp\|^2_{\mathbf{L}^2(t_{n-1},t_n;\mathbf{H}^{m+1}(\Omega))}\right). \tag{3.10b}$$

Theorem 3.1. *Let $\left(\mathbf{u}_h^n,p_h^n\right)\in\mathbf{V}_h\times Q_h$ and $(\mathbf{u},p)\in\mathbf{V}\times Q$ be the solution to* (3.3) *and* (2.10)*, respectively. Then, we have the error estimates*

$$\begin{aligned}&\max_{1\leq n\leq N}\left(2\mu\|\varepsilon(\mathbf{u}^n-\mathbf{u}_h^n)\|_0^2+\lambda\|\nabla\cdot(\mathbf{u}^n-\mathbf{u}_h^n)\|_0^2\right)\\&\quad+\max_{1\leq n\leq N}\left(\frac{\kappa}{\boldsymbol{\eta}}\|\nabla(p^n-p_h^n)\|_0^2+M_h(p^n-p_h^n,p^n-p_h^n)\right)\\&\quad+\Delta t\sum_{i=1}^N\left(2\mu\left\|\varepsilon\left(d_t\left(\mathbf{u}^i-\mathbf{u}_h^i\right)\right)\right\|_0^2+\lambda\left\|\nabla\cdot d_t\left(\mathbf{u}^i-\mathbf{u}_h^i\right)\right\|_0^2\right)\\&\quad+\Delta t\sum_{i=1}^N\left(\frac{\kappa}{\boldsymbol{\eta}}\left\|\nabla\left(p^i-p_h^i\right)\right\|_0^2+M_h\left(d_t\left(p^i-p_h^i\right),d_t\left(p^i-p_h^i\right)\right)\right)\\&\leq C_T\left(h^{2m}+h^{2l}+h^{2r}+\Delta t^2\right)+Ch^{2s_0-2}\left(\left\|\mathbf{u}^0\right\|^2_{l_0+1}+\left\|p^0\right\|^2_{m_0+1}\right),\end{aligned} \tag{3.11}$$

where C_T is represented by

$$\begin{aligned}C_T=&C\Big(\|\nabla\cdot D_{tt}\mathbf{u}\|^2_{\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))}+\|D_t\mathbf{u}\|^2_{\mathbf{L}^2\left(0,T;\mathbf{H}^{l+1}(\Omega)\right)}+\|D_tp\|^2_{\mathbf{L}^2(0,T;\mathbf{H}^{m+1}(\Omega))}\Big)\\&+C\Big(\|D_t\mathbf{u}\|^2_{\mathbf{L}^\infty\left(0,T;\mathbf{H}^{l+1}(\Omega)\right)}+\|D_tp\|^2_{\mathbf{L}^\infty(0,T;\mathbf{H}^{m+1}(\Omega))}\Big).\end{aligned}\tag{3.12}$$

Proof. Testing (3.4) with $\mathbf{v}_h=d_t\boldsymbol{\xi}^n$, $q_h=\gamma^n+d_t\gamma^n$, we get

$$\begin{aligned}&a(\boldsymbol{\xi}^n,d_t\xi^n)+d(\gamma^n,d_t\gamma^n)+M_h(d_t\gamma^n,\gamma^n)+b(d_t\boldsymbol{\xi}^n,d_t\gamma^n)\\&\quad+d(\gamma^n,\gamma^n)+M_h(d_t\gamma^n,d_t\gamma^n)\\=&b(E^n,\gamma^n)-b(d_t\boldsymbol{\eta}^n,\gamma^n)+b(E^n,d_t\gamma^n)-b(d_t\boldsymbol{\eta}^n,d_t\gamma^n)-M_h(d_t\beta^n,\gamma^n)\\&\quad+M_h(d_tp^n,\gamma^n)-M_h(d_t\beta^n,d_t\gamma^n)+M_h(d_tp^n,d_t\gamma^n).\end{aligned}\tag{3.13}$$

Testing (3.5) with $\mathbf{v}_h=d_t\boldsymbol{\xi}^n$ we get

$$a(d_t\xi^n,d_t\boldsymbol{\xi}^n)-b(d_t\boldsymbol{\xi}^n,d_t\gamma^n)=0.\tag{3.14}$$

Add (3.13), (3.14) together we get

$$\begin{aligned}&a(\boldsymbol{\xi}^n,d_t\boldsymbol{\xi}^n)+d(\gamma^n,d_t\gamma^n)+M_h(d_t\gamma^n,\gamma^n)+a(d_t\boldsymbol{\xi}^n,d_t\boldsymbol{\xi}^n)\\&\quad+d(\gamma^n,\gamma^n)+M_h(d_t\gamma^n,d_t\gamma^n)\\=&b(E^n,\gamma^n)-b(d_t\boldsymbol{\eta}^n,\gamma^n)+b(E^n,d_t\gamma^n)-b(d_t\boldsymbol{\eta}^n,d_t\gamma^n)-M_h(d_t\beta^n,\gamma^n)\\&\quad+M_h(d_tp^n,\gamma^n)-M_h(d_t\beta^n,d_t\gamma^n)+M_h(d_tp^n,d_t\gamma^n).\end{aligned}\tag{3.15}$$

Then we have

$$\begin{aligned}&\Delta t d_t\left(a(\boldsymbol{\xi}^n,\boldsymbol{\xi}^n)+d(\gamma^n,\gamma^n)+M_h(\gamma^n,\gamma^n)\right)\\&\quad+2\Delta t\left(a(d_t\boldsymbol{\xi}^n,d_t\boldsymbol{\xi}^n)+d(\gamma^n,\gamma^n)+M_h(d_t\gamma^n,d_t\gamma^n)\right)\\\le&C\Delta t\left(\|\nabla\cdot E^n\|_0+\|\nabla\cdot d_t\boldsymbol{\eta}^n\|_0\right)\left(\|\gamma^n\|_0+\|d_t\gamma^n\|\right)\\&\quad+C\Delta t\Big(M_h(d_t\beta^n,d_t\beta^n)^{1/2}+M_h(d_tp^n,d_tp^n)^{1/2}\Big)\left(\|\gamma^n\|_0+\|d_t\gamma^n\|\right).\end{aligned}\tag{3.16}$$

From Lemma 3.1 and (3.5), we get

$$\begin{aligned}\|d_t\gamma^n\|\le&C\sup_{\mathbf{v}\in\mathbf{V}_h,v\neq0}\frac{b(\mathbf{v},d_t\gamma^n)}{\|\mathbf{v}\|_1}+CM_h(d_t\gamma^n,d_t\gamma^n)^{1/2}\\=&C\sup_{\mathbf{v}\in\mathbf{V}_h,\mathbf{v}\neq0}\frac{a(d_t\boldsymbol{\xi}^n,\mathbf{v})}{\|\mathbf{v}\|_1}+CM_h(d_t\gamma^n,d_t\gamma^n)^{1/2}\\\le&Ca(d_t\boldsymbol{\xi}^n,d_t\boldsymbol{\xi}^n)^{1/2}+CM_h(d_t\gamma^n,d_t\gamma^n)^{1/2}.\end{aligned}\tag{3.17}$$

Combining (3.16), (3.17), and Lemma 3.2 we get

$$\begin{aligned}&\Delta t d_t\left(a(\boldsymbol{\xi}^n,\boldsymbol{\xi}^n)+d(\gamma^n,\gamma^n)+M_h(\gamma^n,\gamma^n)\right)\\&\quad+\Delta t\left(a(d_t\boldsymbol{\xi}^n,d_t\boldsymbol{\xi}^n)+d(\gamma^n,\gamma^n)+M_h(d_t\gamma^n,d_t\gamma^n)\right)\\\le& Ch^{2s}\left(\|D_t\mathbf{u}\|^2_{\mathbf{L}^2\left(t_{n-1},t_n;\mathbf{H}^{l+1}(\Omega)\right)}+\|D_tp\|^2_{\mathbf{L}^2(t_{n-1},t_n;\mathbf{H}^{m+1}(\Omega))}\right)\\&\quad+C\Delta t^2\|\nabla\cdot D_{tt}\mathbf{u}\|^2_{\mathbf{L}^2(t_{n-1},t_n;\mathbf{L}^2(\Omega))}+Ch^{2r}\|D_tp\|^2_{\mathbf{L}^2(t_{n-1},t_n;\mathbf{H}^{r+1}(\Omega))}.\end{aligned}\tag{3.18}$$

Adding last equation form 1 to n, we get

$$\begin{aligned}&\left(2\mu\|\varepsilon(\boldsymbol{\xi}^n)\|_0^2+\lambda\|\nabla\cdot\boldsymbol{\xi}^n\|_0^2+\frac{\kappa}{\boldsymbol{\eta}}\|\nabla\gamma^n\|_0^2+M_h(\gamma^n,\gamma^n)\right)\\&\quad+\Delta t\sum_{i=1}^n\left(2\mu\left\|\varepsilon\left(d_t\boldsymbol{\xi}^i\right)\right\|_0^2+\lambda\left\|\nabla\cdot d_t\boldsymbol{\xi}^i\right\|_0^2\right)\\&\quad+\Delta t\sum_{i=1}^n\left(\frac{\kappa}{\boldsymbol{\eta}}\left\|\nabla\gamma^i\right\|_0^2+M_h(d_t\gamma^n,d_t\gamma^n)\right)\\\le& C_T\left(h^{2s}+\Delta t^2+h^{2r}\right)+Ch^{2s_0-2}\left(\left\|\mathbf{u}^0\right\|^2_{l_0+1}+\left\|p^0\right\|^2_{m_0+1}\right).\end{aligned}\tag{3.19}$$

We use triangle inequality to get our final result. □

3.2 The form of the stabilization term

In this section we will give the concrete form of the stabilized term $M_h(p,q)$, satisfying the assumptions (3.1a)-(3.2b).

3.2.1 Penalty stabilized method

The penalty stabilized term proposed in [10] is $C_0h^2(\nabla d_tp_h^n,\nabla q_h)$, where the numerical analysis is not given. This method is include in the class of our stabilized method, since we can define

$$M_h(p_h,q_h)=C_0h^2(\nabla p_h,\nabla q_h),\tag{3.20}$$

then we have (3.1a), (3.1c) with $r=1$. Take $\mathbf{j}_h\mathbf{u}=\mathbf{u}_I$, we also have (3.2a), (3.2b). As for (3.1b), we have

$$M_h(p_h,p_h)=C_0h^2\|\nabla(p_h-C_1)\|_0^2\le C\|p_h-C_1\|_0^2,\tag{3.21}$$

from the arbitrariness of C_1, we have (3.1b).

So the actual stabilized term in (3.7) is

$$M_h(d_tp_h^n,q_h)=C_0h^2(\nabla d_tp_h^n,\nabla q_h).$$

3.2.2 Pressure stabilized method

When $l\geq m\geq 1$, we can define

$$M_h(p_h,q_h)=C_0\big((id-\boldsymbol{\pi}_{m-1})p_h,(id-\boldsymbol{\pi}_{m-1})q_h\big), \tag{3.22}$$

or

$$M_h(p_h,q_h)=C_0h^2\big(\nabla(id-\boldsymbol{\pi}_{m-1})p_h,\nabla(id-\boldsymbol{\pi}_{m-1})q_h\big). \tag{3.23}$$

When $m=1$, (3.24) is the same as (3.20), and is equivalent to (3.22) since

$$Ch\|\nabla p_h\|_0=Ch\|\nabla(id-\boldsymbol{\pi}_0)p_h\|_0\leq\|(id-\boldsymbol{\pi}_0)p_h\|_0\leq Ch\|\nabla p_h\|_0. \tag{3.24}$$

When $l\geq m\geq 2$, we only have to verify (3.2a), (3.2b). Obviously, $\mathbf{V}_h\times\mathbf{P}_{m-1}(\mathcal{T}_h)$ satisfying the discrete LBB condition, so we have Fortin interplant $\boldsymbol{\pi}_{\mathcal{F}}$, for any $\mathbf{v}\in\mathbf{H}_0^1(\Omega)\cap\mathbf{V}$, such that

$$\begin{aligned}
&\boldsymbol{\pi}_{\mathcal{F}}\mathbf{v}\in\mathbf{H}_0^1(\Omega)\cap\mathbf{V}_h, \quad b(q_{h0},v)=b(q_{h0},\boldsymbol{\pi}_{\mathcal{F}}\mathbf{v}), \qquad \forall q_{h0}\in P_{m-1}(\mathcal{T}_h), && \text{(3.25a)}\\
&\|\mathbf{v}-\boldsymbol{\pi}_{\mathcal{F}}\mathbf{v}\|_1\leq Ch^{k-1}\|\mathbf{v}\|_k, \quad 1\leq k\leq m+1, \qquad \mathbf{v}\in\mathbf{H}^{m+1}(\Omega)\cap\mathbf{H}_0^1(\Omega)\cap\mathbf{V}. && \text{(3.25b)}
\end{aligned}$$

So we have

$$\begin{aligned}
|(\nabla\cdot(\boldsymbol{\pi}_{\mathcal{F}}\mathbf{u}-\mathbf{u}),q_h)|&=|(\nabla\cdot(\boldsymbol{\pi}_{\mathcal{F}}\mathbf{u}-\mathbf{u}),(id-\boldsymbol{\pi}_{m-1})q_h)|\\
&\leq C\|\nabla(\boldsymbol{\pi}_{\mathcal{F}}\mathbf{u}-\mathbf{u})\|_0\|(id-\boldsymbol{\pi}_{m-1})q_h\|_0.
\end{aligned} \tag{3.26}$$

Taking $r=m$ and $\mathbf{j}_h=\boldsymbol{\pi}_{\mathcal{F}}$ we have (3.2a), (3.2b). The actual stabilized term in (3.7) is

$$M_h(d_tp_h^n,q_h)=C_0\big((id-\boldsymbol{\pi}_{m-1})d_tp_h^n,(id-\boldsymbol{\pi}_{m-1})q_h\big).$$

Or

$$M_h(d_tp_h^n,q_h)=C_0h^2\big(\nabla(id-\boldsymbol{\pi}_{m-1})d_tp_h^n,\nabla(id-\boldsymbol{\pi}_{m-1})q_h\big).$$

3.2.3 Orthogonal projection and local projection stabilized method

The orthogonal projection method is

$$M_h(p_h,q_h)=Ch^2\big((id-\boldsymbol{\pi}_h)\nabla p_h,(id-\boldsymbol{\pi}_h)\nabla q_h\big). \tag{3.27}$$

Where $\boldsymbol{\pi}_h$ is the $\mathbf{L}^2$-projection onto $\mathbf{H}_0^1(\Omega)\cap\mathbf{V}_h$. For any $\mathbf{u}\in\mathbf{H}_0^1(\Omega)\cap\mathbf{V}$, we have

$$\begin{aligned}
&|(\nabla\cdot(\boldsymbol{\pi}_h\mathbf{u}-\mathbf{u}),q_h)|=|(\boldsymbol{\pi}_h\mathbf{u}-\mathbf{u},\nabla q_h)|\\
=&|(\boldsymbol{\pi}_h\mathbf{u}-\mathbf{u},(id-\boldsymbol{\pi}_h)\nabla q_h)|\leq C\|(id-\boldsymbol{\pi}_h)\nabla q_h\|_0\|\boldsymbol{\pi}_h\mathbf{u}-\mathbf{u}\|_0.
\end{aligned} \tag{3.28}$$

Taking $r=m$ and $\mathbf{j}_h=\boldsymbol{\pi}_h$, we have verified (3.2a), (3.2b). For the local projection stabilized method we refer to [15], in where exists a interplant $\mathbf{j}_h$ such that $\mathbf{j}_h\mathbf{u}-\mathbf{u}$ is orthogonal to the projection space. This is similar to the orthogonal projection method. So the actual stabilized term in (3.7) is

$$M_h(d_t p_h^n, q_h) = Ch^2\left((id-\boldsymbol{\pi}_h)\nabla d_t p_h^n, (id-\boldsymbol{\pi}_h)\nabla q_h\right),$$

where $s_0 = \min(l_0, m_0+1)$.

Find $\left(\mathbf{u}_{h0}^0, p_{h0}^0\right) \in \mathbf{V}_h \times Q_h$ such that

$$\begin{cases} a\left(\mathbf{u}_h^0, \mathbf{v}_h\right) - b\left(\mathbf{v}_h, p_h^0\right) = a\left(\mathbf{u}_{h0}^0, \mathbf{v}_h\right) - b\left(\mathbf{v}_h, p_{h0}^0\right), & \forall \mathbf{v}_h \in \mathbf{V}_h, \\ d\left(p_h^0, q_h\right) = d\left(p_{h0}^0, q_h\right), & \forall q_h \in Q_h. \end{cases} \tag{3.29}$$

Then we have the error estimates,

$$\left|\mathbf{u}^0 - \mathbf{u}_h^0\right|_1 \le Ch^s\left(\left\|\mathbf{u}^0\right\|_{l+1} + \left\|p^0\right\|_{m+1}\right), \tag{3.30a}$$

$$\left\|p^0 - p_h^0\right\|_0 \le Ch^{s_0}\left(\left\|\mathbf{u}^0\right\|_{l_0+1} + \left\|p^0\right\|_{m_0+1}\right) + Ch^{m+1}\left\|p^0\right\|_{m+1}. \tag{3.30b}$$

4 Numerical experiments

This section includes two numerical examples to verify and illustrate the convergence and performance of our stabilized methods.

4.1 Convergence verification

To simplify of notaions, we set $d=2$, $\Omega=(0,1)\times(0,1)$, $T=1$, $\mu=\lambda=\kappa=\eta=1$, and the border condition is $\mathbf{u}=\mathbf{0}$, $p=0$ on Γ. The true solution of (2.2) is

$$\begin{cases} u_1 = 10x^2(1-x)^2 y(1-y)(1-2y)\exp(-t), \\ u_2 = -10x(1-x)(1-2x)y^2(1-y)^2\exp(-2t), \\ p = 10x^2(1-x)^2 y(1-y)(1-2y)\exp(-3t). \end{cases} \tag{4.1}$$

The computational mesh is a regular triangulation with $2\times 2\times n$ $(n=1/h)$ triangles, and the stabilized method (3.3) with the cases

$$\text{M1} \qquad \begin{cases} \mathbf{V}_{h0}\times Q_{h0} = (\mathbf{V}\cap\mathbf{P}_2(\mathcal{T}_h))\times(V\cap P_1(\mathcal{T}_h)), \\ \mathbf{V}_h\times Q_h = (\mathbf{V}\cap\mathbf{P}_1(\mathcal{T}_h))\times(V\cap P_1(\mathcal{T}_h)), \\ \Delta t = h, \quad M_h(p_h, q_h) = h^2(\nabla p_h, \nabla q_h), \end{cases} \tag{4.2a}$$

$$\text{M2} \qquad \begin{cases} \mathbf{V}_{h0}\times Q_{h0} = (\mathbf{V}\cap\mathbf{P}_2(\mathcal{T}_h))\times(V\cap P_1(\mathcal{T}_h)), \\ \mathbf{V}_h\times Q_h = (\mathbf{V}\cap\mathbf{P}_1(\mathcal{T}_h))\times(V\cap P_1(\mathcal{T}_h)), \\ \Delta t = h, \quad M_h(p_h, q_h) = ((id-\boldsymbol{\pi}_0)p_h, (id-\boldsymbol{\pi}_0)q_h), \end{cases} \tag{4.2b}$$

$$\text{M3} \qquad \begin{cases} \mathbf{V}_{h0}\times Q_{h0}=(\mathbf{V}\cap\mathbf{P}_2(\mathcal{T}_h))\times(Q\cap P_1(\mathcal{T}_h)), \\ \mathbf{V}_h\times Q_h=(\mathbf{V}\cap\mathbf{P}_1(\mathcal{T}_h))\times(V\cap P_1(\mathcal{T}_h)), \\ \Delta t=h, \quad M_h(p_h,q_h)=h^2((id-\boldsymbol{\pi}_h)\nabla p_h,(id-\boldsymbol{\pi}_h)\nabla q_h), \end{cases} \tag{4.2c}$$

$$\text{M4} \qquad \begin{cases} \mathbf{V}_{h0}\times Q_{h0}=(\mathbf{V}\cap\mathbf{P}_3(\mathcal{T}_h))\times(Q\cap P_2(\mathcal{T}_h)), \\ \mathbf{V}_h\times Q_h=(\mathbf{V}\cap\mathbf{P}_2(\mathcal{T}_h))\times(V\cap P_2(\mathcal{T}_h)), \\ \Delta t=h^2, \quad M_h(p_h,q_h)=h^2((id-\boldsymbol{\pi}_h)\nabla p_h,(id-\boldsymbol{\pi}_h)\nabla q_h). \end{cases} \tag{4.2d}$$

Tables 1-4 give the errors of these methods, where $r_{H(\mathbf{u})}$, $r_{L(p)}$, $r_{\max(\mathbf{u})}$ and $r_{\max(p)}$ represented the convergence rate of $E_{H(u)}$, $E_{H(p)}$, $E_{\max(\mathbf{u})}$ and $E_{\max(p)}$, and

$$E_{H(\mathbf{u})}=\Delta t\left(\sum_{n=1}^{N}|\mathbf{u}^n-\mathbf{u}_h^n|_1^2\right)^{\frac{1}{2}}, \qquad E_{H(p)}=\Delta t\left(\sum_{n=1}^{N}|p^n-p_h^n|_1^2\right)^{\frac{1}{2}}, \tag{4.3a}$$

$$E_{\max(\mathbf{u})}=\max_{1\le n\le N}|\mathbf{u}^n-\mathbf{u}_h^n|_1, \qquad E_{\max(p)}=\max_{1\le n\le N}|p^n-p_h^n|_1. \tag{4.3b}$$

From the tables we can see the convergence rates of our stabilized methods are exactly as we predicted in Section 3.

Table 1: Errors of M1.

$\frac{1}{h}$	$E_{H(\mathbf{u})}$	$E_{H(p)}$	$E_{\max(\mathbf{u})}$	$E_{\max(p)}$	$r_{H(\mathbf{u})}$	$r_{L(p)}$	$r_{\max(\mathbf{u})}$	$r_{\max(p)}$
8	0.05258960	0.02320830	0.0820553	0.0478414				
16	0.02776750	0.01297680	0.0453947	0.0290779	0.92	0.84	0.85	0.72
32	0.01419200	0.00683919	0.0237734	0.0160180	0.97	0.92	0.93	0.86
64	0.00717135	0.00350619	0.0121638	0.0084005	0.98	0.96	0.97	0.93

Table 2: Errors of M2.

$\frac{1}{h}$	$E_{H(\mathbf{u})}$	$E_{H(p)}$	$E_{\max(\mathbf{u})}$	$E_{\max(p)}$	$r_{H(\mathbf{u})}$	$r_{L(p)}$	$r_{\max(u)}$	$r_{\max(p)}$
8	0.05259010	0.02283020	0.0820569	0.0469630				
16	0.02776760	0.01293830	0.0453949	0.0289757	0.92	0.82	0.85	0.70
32	0.01419200	0.00683482	0.0237734	0.0160054	0.97	0.92	0.93	0.86
64	0.00717135	0.00350566	0.0121638	0.0083989	0.98	0.96	0.97	0.93

Table 3: Errors of M3.

$\frac{1}{h}$	$E_{H(\mathbf{u})}$	$E_{H(p)}$	$E_{\max(\mathbf{u})}$	$E_{\max(p)}$	$r_{H(\mathbf{u})}$	$r_{L(p)}$	$r_{\max(\mathbf{u})}$	$r_{\max(p)}$
8	0.05258990	0.02290340	0.0820562	0.0471696				
16	0.02776750	0.01294160	0.0453949	0.0289886	0.92	0.82	0.85	0.70
32	0.01419200	0.00683498	0.0237734	0.0160062	0.97	0.92	0.93	0.86
64	0.00717135	0.00350567	0.0121638	0.0083989	0.98	0.96	0.97	0.93

Table 4: Errors of M4.

$\frac{1}{h}$	$E_{H(\mathbf{u})}$	$E_{H(p)}$	$E_{\max(\mathbf{u})}$	$E_{\max(p)}$	$r_{H(\mathbf{u})}$	$r_{L(p)}$	$r_{\max(\mathbf{u})}$	$r_{\max(p)}$
4	0.02685460	0.01352580	0.0439391	0.0310246				
8	0.00745967	0.00373234	0.0126604	0.0094883	1.85	1.86	1.80	1.71
16	0.00190719	0.00095230	0.0032665	0.0024278	1.97	1.97	1.95	1.97
32	0.00047886	0.00023862	0.0008220	0.0005997	1.99	2.00	1.99	2.02

4.2 The performances close to initial time

In this section we show the performances close to initial time using M1-M4 and the standard Galerkin method (2.19) with the cases

$$G1: \mathbf{V}_h \times Q_h = (\mathbf{V} \cap \mathbf{P}_1(\mathcal{T}_h)) \times (V \cap P_1(\mathcal{T}_h)), \tag{4.4a}$$
$$G2: \mathbf{V}_h \times Q_h = (\mathbf{V} \cap \mathbf{P}_2(\mathcal{T}_h)) \times (V \cap P_2(\mathcal{T}_h)), \tag{4.4b}$$
$$G3: \mathbf{V}_h \times Q_h = (\mathbf{V} \cap \mathbf{P}_2(\mathcal{T}_h)) \times (V \cap P_1(\mathcal{T}_h)), \tag{4.4c}$$

in which we do not have to solve the initial problems. We let $\Delta t = 10^{-6}$. The error of pressure when $t \leq 10^{-4}$ is presented in Fig. 1.

In Fig. 2, we can see that the error of pressure is pretty big when t is towards 0 using equal-order elements; our stabilized method and the standard Galerkin method using LBB stabile element have very good numerical performances.

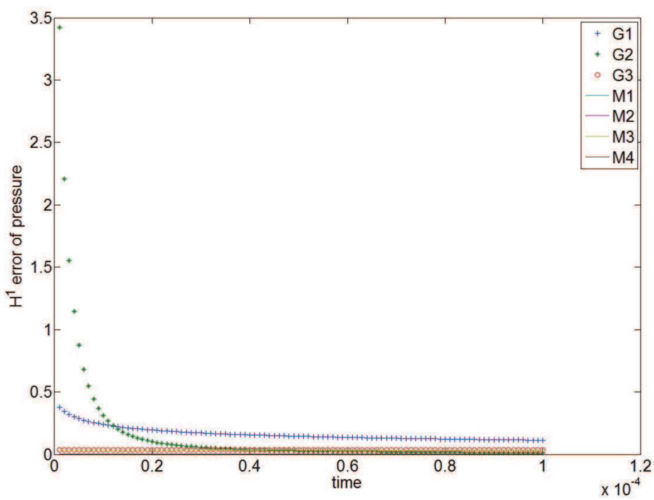

Figure 1: Comparison of different methods.

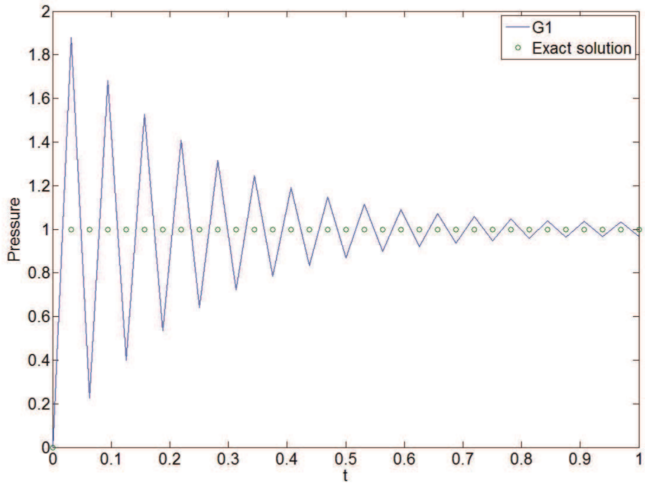

Figure 2: The pressure solved by G1 and the exact pressure.

4.3 Terzaghi problem

We compare the performance of standard Galerkin method and our stabilized method. Consider the 1-d Terzaghi problem:

$$\begin{cases} -\dfrac{\partial^2 \mathbf{u}}{\partial x^2}+\dfrac{\partial p}{\partial x}=0, & (x,t)\in(0,1)\times(0,T], \\ \dfrac{\partial}{\partial t}\left(\dfrac{\partial \mathbf{u}}{\partial x}\right)-\dfrac{\partial \mathbf{u}}{\partial x}=0, & (x,t)\in(0,1)\times(0,T], \end{cases} \tag{4.5}$$

the initial and border condition are

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial x}(0,t)=-1, \quad p(0,t)=0, & t\in[0,T], \\ \mathbf{u}(1,t)=0, \quad \dfrac{\partial p}{\partial x}(1,t)=0, & t\in[0,T], \\ \dfrac{\partial \mathbf{u}}{\partial x}(x,0)=0, & x\in[0,1]. \end{cases} \tag{4.6}$$

The true solution of the equation is

$$\begin{cases} \mathbf{u}(x,t)=(1-x)-\displaystyle\sum_{i=0}^{\infty}\frac{2\cos(m_i x)}{m_i^2}\exp(-m_i^2 t), \\ p(x,t)=\displaystyle\sum_{i=0}^{\infty}\frac{2\sin(m_i x)}{m_i}\exp(-m_i^2 t), \end{cases} \tag{4.7}$$

where

$$m_i=\frac{2n+1}{2}\pi. \tag{4.8}$$

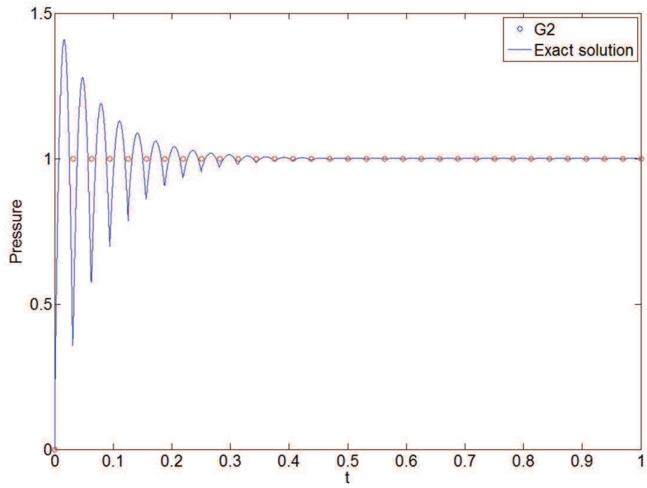

Figure 3: The pressure solved by G2 and the exact pressure.

in which π is the circumference ratio. In our computation, we use 32 elements and $\Delta t = T = 10^{-6}$. Compare the standard Galerkin method G1-G3 and the stabilized method (3.3) with the cases

$$S1:\begin{cases} \mathbf{V}_h\times Q_h=\mathbf{V}_{h0}\times Q_{h0}=(\mathbf{V}\cap\mathbf{P}_1(\mathcal{T}_h))\times(V\cap P_1(\mathcal{T}_h)),\\ M_h(p_h,q_h)=M_{h0}(p_h,q_h)=h^2\left(\dfrac{\partial p_h}{\partial x},\dfrac{\partial q_h}{\partial x}\right),\end{cases} \tag{4.9a}$$

$$S2:\begin{cases} \mathbf{V}_h\times Q_h=\mathbf{V}_{h0}\times Q_{h0}=(\mathbf{V}\cap\mathbf{P}_1(\mathcal{T}_h))\times(V\cap P_1(\mathcal{T}_h)),\\ M_h(p_h,q_h)=M_{h0}(p_h,q_h)=\delta((id-\pi_0)p_h,(id-\pi_0)q_h),\end{cases} \tag{4.9b}$$

$$S3:\begin{cases} \mathbf{V}_h\times Q_h=\mathbf{V}_{h0}\times Q_{h0}=(\mathbf{V}\cap\mathbf{P}_1(\mathcal{T}_h))\times(V\cap P_1(\mathcal{T}_h)),\\ M_h(p_h,q_h)=M_{h0}(p_h,q_h)=h^2\left((id-\pi)\dfrac{\partial p_h}{\partial x},(id-\pi)\dfrac{\partial q_h}{\partial x}\right),\end{cases} \tag{4.9c}$$

$$S4:\begin{cases} \mathbf{V}_h\times Q_h=\mathbf{V}_{h0}\times Q_{h0}=(\mathbf{V}\cap\mathbf{P}_2(\mathcal{T}_h))\times(V\cap P_2(\mathcal{T}_h)),\\ M_h(p_h,q_h)=M_{h0}(p_h,q_h)=((id-\pi_1)p_h,(id-\pi_1)q_h).\end{cases} \tag{4.9d}$$

The figures are presented in Fig. 2-Fig. 8.

From the figures we can see: using the equal-order elements to solve the Biot's problem, the pressure produces severe concussion; using the LBB stable elements the concussion is minimized; our stabilized methods using equal-order elements overcome the non-physical concussion pretty well.

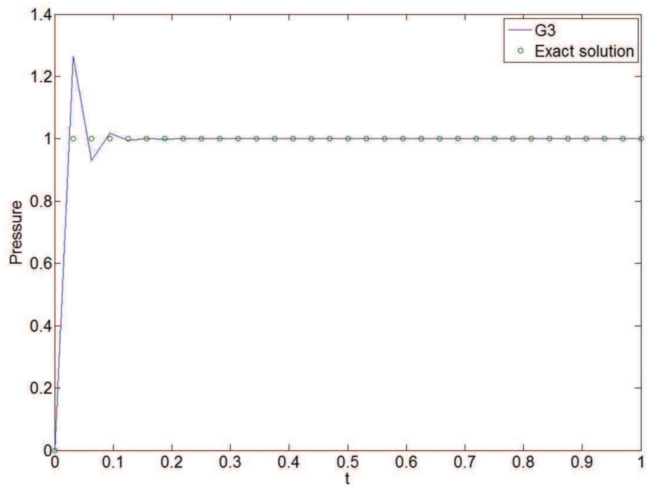

Figure 4: The pressure solved by G3 and the exact pressure.

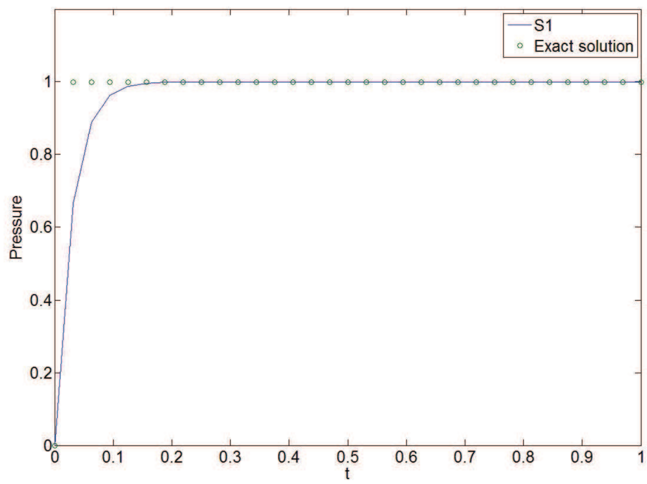

Figure 5: The pressure solved by S1 and the exact pressure.

4.4 Flow through a cylinder

We consider the problem on Fig. 9, on the outer border $u=0$, $p=0$, on the inner border $(\lambda\nabla\cdot uI+2\mu\varepsilon(u))n=n$, $p=1$ and $f=0$, $g=0$ and

$$\lambda=\frac{\nu E}{(1+\nu)(1-2\nu)},\quad \mu=\frac{E}{2(1+\nu)}, \tag{4.10a}$$

$$\kappa=10^{-7},\quad \eta=10^{-4},\quad E=3\times10^{4},\quad \nu=0.2,\quad T=\Delta t=10^{-6}. \tag{4.10b}$$

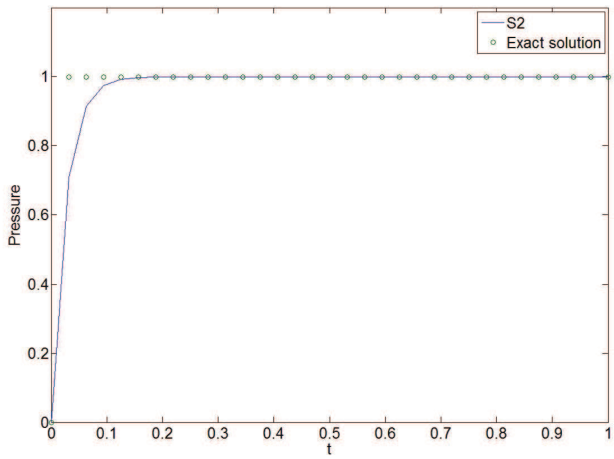

Figure 6: The pressure solved by G3 and the exact pressure.

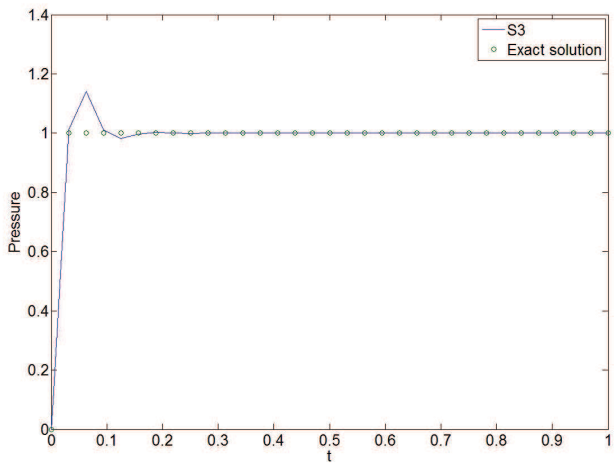

Figure 7: The pressure solved by S3 and the exact pressure.

We plot the pressure at $\Delta t = 10^{-6}$ with G1-G3 and S1-S4 in Figs. 10-16. From the figures we can see: the Galerkin methods using LBB stable elements reduces the oscillations the Galerkin methods with LBB unstable elements, but there are still oscillations on the outer border; our stabilized methods eliminate the oscillations and perform well.

5 Conclusions

This paper gives a new class of weak consistent stabilized methods for the Biot's consolidation problems. We prove the stability and the convergence of our method. The

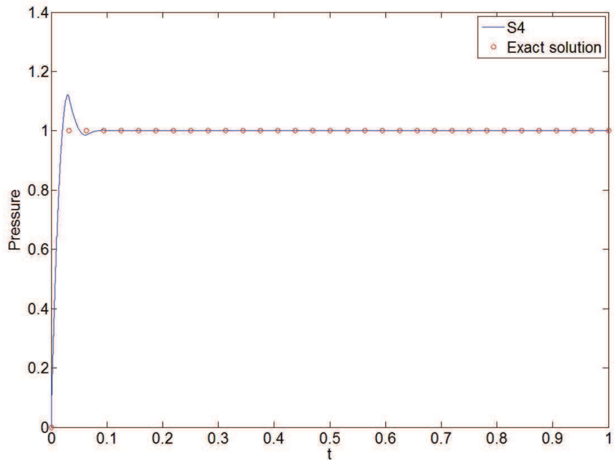

Figure 8: The pressure solved by S4 and the exact pressure.

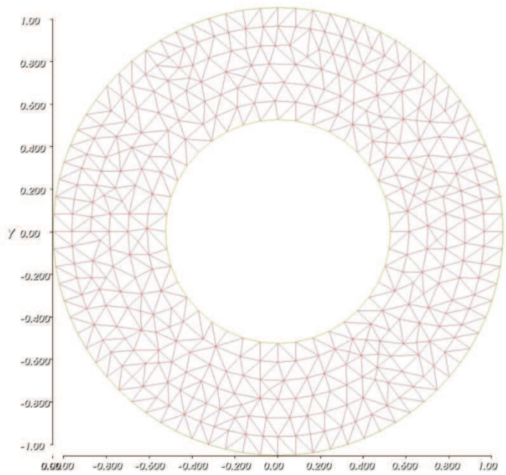

Figure 9: Domain and mesh.

numerical experiments illustrate and confirm the theoretical analysis. Nonconforming stabilized methods for the Biot's soil consolidation problems will be a new topic in forthcoming research.

Acknowledgements

The computations in Section 4.4 were done by FreeFem++ [21]. This research was supported by the Natural Science Foundation of China (No. 11271273).

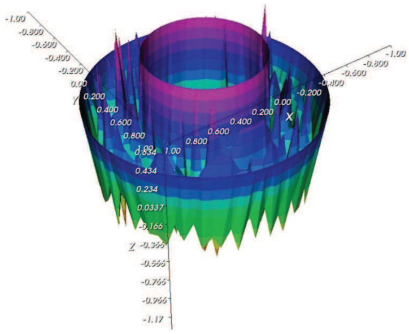

Figure 10: The pressure solved by G1.

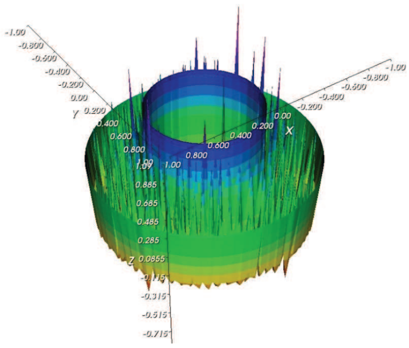

Figure 11: The pressure solved by G2.

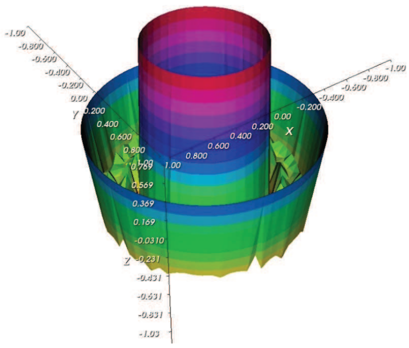

Figure 12: The pressure solved by G3.

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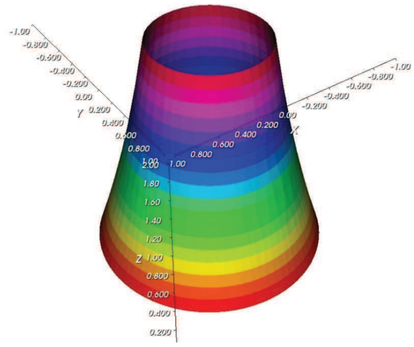

Figure 13: The pressure solved by S1.

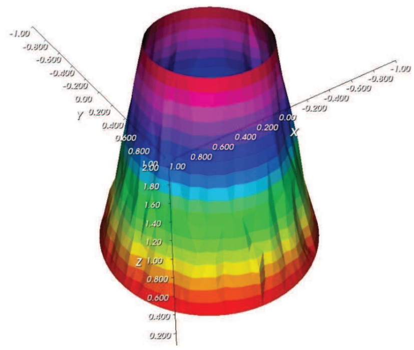

Figure 14: The pressure solved by S2.

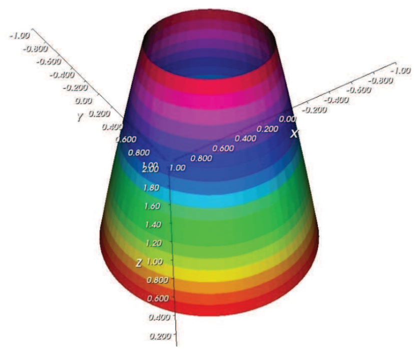

Figure 15: The pressure solved by S3.

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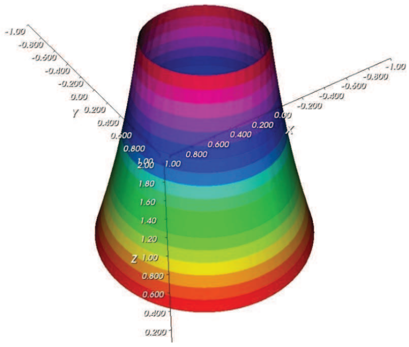

Figure 16: The pressure solved by S4.

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