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EIGENVALUES OF THE NEUMANN-POINCARÉ OPERATOR FOR TWO INCLUSIONS WITH CONTACT OF ORDER m: A NUMERICAL STUDY*

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Abstract

In a composite medium that contains close-to-touching inclusions, the pointwise values of the gradient of the voltage potential may blow up as the distance δ between some inclusions tends to 0 and as the conductivity contrast degenerates. In a recent paper [9], we showed that the blow-up rate of the gradient is related to how the eigenvalues of the associated Neumann-Poincaré operator converge to $\pm\frac{1}{2}$ as $\delta \to 0$, and on the regularity of the contact. Here, we consider two connected 2-D inclusions, at a distance $\delta > 0$ from each other. When $\delta = 0$, the contact between the inclusions is of order $m \geq 2$. We numerically determine the asymptotic behavior of the first eigenvalue of the Neumann-Poincaré operator, in terms of δ and m, and we check that we recover the estimates obtained in [10].

Mathematics subject classification: Primary 35J25, 73C40.
Key words: Elliptic equations, Eigenvalues, Numerical approximation.

1. Eigenvalues of the Neumann-Poincaré Operator for two Inclusions

Let $D_1, D_2 \subset \mathbb{R}^2$ be two bounded, smooth inclusions separated by a distance $\delta > 0$. We assume that D_1 and D_2 are translates of two reference touching inclusions

$$D_1 \;=\; D_1^0 + (0, \delta/2), \quad D_2 \;=\; D_2^0 + (0, -\delta/2).$$

We assume that D_1^0 lies in the lower half–plane $x_1 < 0$, D_2^0 in the upper half–plane, and that they meet at the point 0 tangentially to the x_1–axis (see Figure 1.1). We make the following additional assumptions on the geometry:

A1. The inclusions D_1^0 and D_2^0 are strictly convex and only meet at the point 0.

A2. Around the point 0, ∂D_1^0 and ∂D_2^0 are parametrized by 2 curves $(x, \psi_1(x))$ and $(x, -\psi_2(x))$ respectively. The graph of ψ_1 (resp. ψ_2) lies below (resp. above) the x–axis.

A3. The boundary ∂D_i^0 of each inclusion is globally $\mathcal{C}^{1,\alpha}$ for some $0 < \alpha \leq 1$.

A4. The function $\psi_1(x) + \psi_2(x)$ is equivalent to $C|x|^m$ as $x \to 0$, where $m \geq 2$ is a fixed integer and C is a positive constant.

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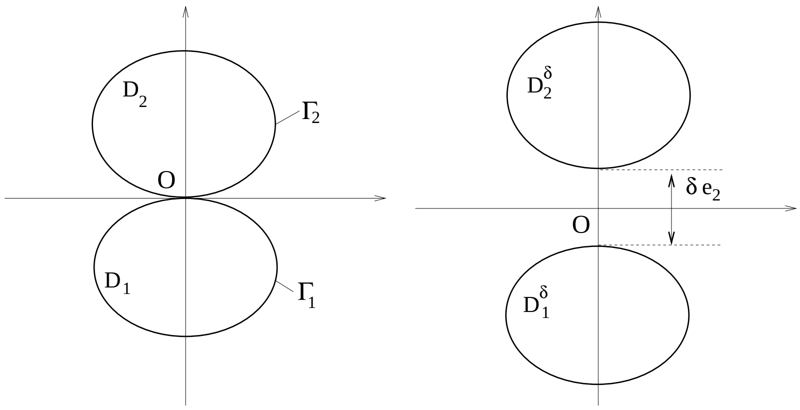

Fig. 1.1. The touching and non–touching configurations.

Let $a(X)$ be a piecewise constant function that takes the value $0 < k \neq 1$ in each inclusion and 1 in $\mathbb{R}^2 \setminus \overline{D_1 \cup D_2}$, that is

$$a(X) = 1 + (k-1)\chi_{D_1 \cup D_2}(X),$$

where $\chi_{D_1 \cup D_2}$ is the characteristic function of $D_1 \cup D_2$. Given a harmonic function H, we denote u the solution to the PDE

$$\begin{cases} \operatorname{div}(a(X)\nabla u(X)) = 0 & \text{in } \mathbb{R}^2 \\ u(X) - H(X) \to 0 & \text{as } |X| \to \infty. \end{cases} \tag{1.1}$$

Since H is harmonic in the whole space the regularity of u at a fixed value k, only depends on the smoothness of the inclusions and of their distribution [15].

One can express u in terms of layer potentials [1, 22]

$$u(X) = S_1\varphi_1(X) + S_2\varphi_2(X) + H(X), \tag{1.2}$$

where S_i denotes the single layer potential on ∂D_i, defined for $\varphi \in H^{-1/2}(\partial D_i)$ by

$$S_i\varphi(X) = \frac{1}{2\pi}\int_{\partial D_i} \ln|X - Y|\, \varphi(Y)\, d\sigma(Y).$$

Denoting the conductivity contrast by

$$\lambda = \frac{k+1}{2(k-1)} \in \Big(-\infty, -\frac{1}{2}\Big) \cup \Big(\frac{1}{2}, +\infty\Big)$$

and expressing the transmission conditions satisfied by u, one sees that the layer potential $\varphi = (\varphi_1, \varphi_2) \in H^{-1/2}(\partial D_1) \times H^{-1/2}(\partial D_2)$ satisfies the system of integral equations

$$(\lambda I - K_\delta^*) \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \partial_{\nu_1} H_{|\partial D_1} \\ \partial_{\nu_2} H_{|\partial D_2} \end{pmatrix}, \tag{1.3}$$

where $\nu_i(X)$ denotes the outer normal at a point $X \in \partial D_i$. The operator K_δ^* is the Neumann-Poincaré operator for the system of two inclusions

$$K_\delta^* \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} K_1^* & \partial_{\nu_1} S_{2|\partial D_1} \\ \partial_{\nu_2} S_{1|\partial D_2} & K_2^* \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \tag{1.4}$$

where the integral operators K_i^* are defined on $H^{-1/2}(\partial D_i)$ by

$$K_i^*\varphi(X) = \frac{1}{2\pi}\int_{\partial D_i} \frac{(X-Y)\cdot\nu_i(X)}{|X-Y|^2}\,\varphi(Y)\,d\sigma(Y).$$

In such a system of inclusions, for a fixed contrast $|\lambda| > \frac{1}{2}$, the gradient of the potential is bounded pointwise [1, 11, 20] *independently of* δ. This is an important fact from the point of view of material sciences, where one would like to control the 'hot spots' where gradients may become large [12]. The pointwise control of the gradients is also particularly pertinent in the context of solid mechanics. For instance, the constitutive laws of classical models of plasticity or fracture involve pointwise values of the stress tensor. Similar qualitative results hold in this case [19].

However, the gradients may blow up when both $\delta \to 0$ and the material coefficients inside the inclusions degenerate [11]. How the bounds depend on the inter-inclusion distance in the case of perfectly conducting inclusions was studied in [8, 25]. Several works study the blow-up rate of the gradient in terms of both parameter $\delta \to 0$, and $|\lambda| \to \frac{1}{2}$ when the inclusions are discs. In this case, the voltage potential u can be represented by a series, that lends itself to a precise asymptotic analysis [3, 4, 6, 7, 12, 21]. In particular, optimal upper and lower bounds on ∇u were obtained in [4–6].

In a recent work [10], we have used the above integral representation to derive bounds on ∇u, as we had observed that in (1.3) the parameters λ and δ are decoupled since K_δ^* does not depend on λ. Following [17, 18], we showed that K_δ^* has a spectral decomposition in the space of single layer potentials. We showed that its spectrum splits into two families of ordered eigenvalues $\lambda_n^{\delta,\pm}$ which satisfy

$$\lambda_n^{\delta,+} = -\lambda_n^{\delta,-} \quad \text{and} \quad 0 < \lambda_n^{\delta,+} < \frac{1}{2}.$$

Consequently, denoting by $\varphi_n^{\delta,\pm}$ the associated eigenvectors, the solution to (1.3) can be expressed as

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \sum_{n\geq 1} \frac{\left\langle \varphi_n^{\delta,\pm}, \begin{pmatrix} \partial_{\nu_1} H_{|\partial D_1} \\ \partial_{\nu_2} H_{|\partial D_2} \end{pmatrix} \right\rangle}{\lambda - \lambda_n^{\delta,\pm}} \varphi_n^{\delta,\pm}. \tag{1.5}$$

This formula indicates that the singularities of u are triggered by the fact that $\lambda - \lambda_n^{\delta,\pm}$ may become small. Indeed, $\lambda \to \pm\frac{1}{2}$ as k tends to 0 or to $+\infty$, whereas we have shown that $\lambda_n^{\delta,\pm} \to \pm\frac{1}{2}$ as $\delta \to 0$ [10].

We do not know if the expansion (1.5) holds in a pointwise sense, except in the case of discs [9], where we can then directly relate the bounds on ∇u to the asymptotic behavior of the eigenvalues. One of the difficulties is that K_δ^* is not self-adjoint. One can nevertheless symmetrize the operator [17]: The expansion (1.5) holds in the sense of the following inner-product on the space $H^{-1/2}(\partial D_1) \times H^{-1/2}(\partial D_2)$

$$\begin{aligned} <\varphi,\psi>_S &= < -S[\varphi], \psi >_{L^2} \\ &:= -\int_{\partial D_1} S_1[\varphi_1]\psi_1 - \int_{\partial D_2} S_2[\varphi_2]\psi_2, \end{aligned} \tag{1.6}$$

for which K_δ^* becomes a compact self-adjoint operator, which therefore has a spectral decomposition. Moreover, this implies that the eigenvalues of K_δ^* can be obtained via a min-max principle known as the Poincaré variational problem (in the terminology of [17]). It consists in optimizing the ratio

$$J(u) = \frac{\displaystyle\int_{D_1 \cup D_2} |\nabla u|^2}{\displaystyle\int_{\mathbb{R}^2 \setminus \overline{D_1 \cup D_2}} |\nabla u|^2},$$

among all functions $u \in W^{1,2}(\mathbb{R}^2)$ whose restriction to $D = D_1 \cup D_2$ and to $D' = \mathbb{R}^2 \setminus \overline{D_1 \cup D_2}$ is harmonic.

Consider the weighted Sobolev space

$$\mathcal{W}_0^{1,-1}(\mathbb{R}^2) := \left\{ \begin{array}{l} \dfrac{u(X)}{(1+|X|^2)^{1/2}\log(2+|X|^2)} \in L^2(\mathbb{R}^2) \\ \\ \nabla u \in L^2(\mathbb{R}^2),\ u(X) = o(1) \text{ as } |X| \to \infty \end{array} \right\},$$

equipped with the scalar product $\displaystyle\int_{\mathbb{R}^2} \nabla u \cdot \nabla v$ [22]. We have shown in [10] that the spectrum of K_δ^* is related to the sprectrum of the operator T_δ defined for $u \in W_0^{1,-1}(\mathbb{R}^2)$ by

$$\forall\, v \in W_0^{1,-1}(\mathbb{R}^2), \quad \int_{\mathbb{R}^2} \nabla T_\delta u(X) \cdot \nabla v(X) = \int_{D_1 \cup D_2} \nabla u(X) \cdot \nabla v(X).$$

This operator is self adjoint, satisfies $||T_\delta|| \leq 1$. Proposition 4 and Lemmas 1 and 2 in [10] show that its eigenvalues can be grouped in two families $\beta_n^{\delta,+} \subset [0, \frac{1}{2}]$, and $\beta_n^{\delta,-} \subset [\frac{1}{2}, 1]$, which are symmetric with respect to $\frac{1}{2}$. The values $\beta_0^{\delta,-} = 1$ is an eigenvalue of T_δ, with associated eigenspace

$$\mathrm{Ker}\,(I - T_\delta) = \Big\{ v|_{D'} \equiv 0,\ v|_D \in H_0^1(D) \Big\}.$$

Due to the symmetry, $\beta_0^{\delta,+} = 0$ is also an eigenvalue, and its eigenspace is

$$\mathrm{Ker}\,(T_\delta) = \Big\{ v|_{D'} \in \mathcal{W}_0^{1,-1}(D'),\ v|_D \equiv 0 \Big\} \ \cup\ \mathbb{R}\, w_0,$$

where w_0 is defined by

$$\left\{ \begin{array}{lll} \Delta w_0(X) = 0 & & \text{in } D', \\ w_0(X) = C_j & & \text{on } \partial D_j \ \ j = 1, 2, \\ \displaystyle\int_{\partial D_j} \frac{\partial w_0}{\partial \nu} = (-1)^j & & j = 1, 2. \end{array} \right. \tag{1.7}$$

The constants $C_1, C_2 \in \mathbb{R}$ are chosen so that $w_0 \in \mathcal{W}_0^{1,-1}(\mathbb{R}^2)$.

All the other eigenvalues $\beta_n^{\delta,+}$ are given by the following min-max principle

$$\begin{aligned} \beta_n^{\delta,+} &= \min_{\substack{u \in W_0^{1,-1}(\mathbb{R}^2) \\ u \perp w_0, w_1^{\delta,+}, \cdots, w_n^{\delta,+}}} \frac{\int_D |\nabla u(X)|^2\, dX}{\int_{\mathbb{R}^2} |\nabla u(X)|^2\, dX} \\ &= \max_{\substack{F_n \subset W_0^{1,-1}(\mathbb{R}^2) \\ \dim(F_n) = n+1}} \ \min_{u \in F_n} \frac{\int_D |\nabla u(X)|^2\, dX}{\int_{\mathbb{R}^2} |\nabla u(X)|^2\, dX}. \end{aligned}$$

The eigenvalues of T_δ are related to the $\lambda_n^{\delta,\pm}$'s by

$$\beta_n^{\delta,\pm} = \frac{1}{2} - \lambda_n^{\delta,\pm}.$$

The min-max characterization of T_δ allows to derive an asymptotic expansion of the eigenvalues of the Neumann-Poincaré operator (see [10], Theorem 1) as $\delta \to 0$.

Theorem 1.1. *For two close to touching inclusions with contact of order m, the eigenvalues of the Neumann-Poincaré operator K_δ^* split in two families $(\lambda_n^\pm)_{n\geq 1}$, with*

$$\begin{cases} \lambda_n^+ \sim \frac{1}{2} - c_n^+ \delta^{\frac{m-1}{m}} + o(\delta^{\frac{m-1}{m}}), \\ \lambda_n^- \sim -\frac{1}{2} + c_n^- \delta^{\frac{m-1}{m}} + o(\delta^{\frac{m-1}{m}}), \end{cases} \tag{1.8}$$

where $(c_n^\pm)_{n\geq 1}$ are increasing sequences of positive numbers, that only depend on the shapes of the inclusions, and that satisfy $c_n^\pm \sim n$ as $n \to \infty$.

In this work, we consider a numerical approximation of the spectral problem for T_δ so as to give a numerical validation of the rates of convergence of $\lambda_1^{\delta,+}$ as $\delta \to 0$. The first eigenvalue $\lambda_1^{\delta,+}$ is of importance in applications since it is related to the spectral radius of the operator K_δ^*, and gives the rate of convergence of Neumann series that appears in solving the integral equation (1.3) [24].

In Section 2, we show that the asymptotic behavior of the eigenvalues of T_δ can be estimated by the eigenvalues of an operator of similar type, but defined on a ball B_R that contains the inclusions. In fact, by considering the auxiliary spectral problem in a large ball B_R, we reduce the computation to a bounded domain.

In Section 3, we explain how we discretized the latter spectral problem, by choosing a basis of functions which are harmonic polynomials on each inclusion, extended as harmonic functions in $B_R \setminus \overline{D_1 \cup D_2}$. Finally, numerical results for $\beta_1^{\delta,+}$ with different contact orders m are presented in Section 4.

2. Comparison of T_δ with an Operator Defined on a Bounded Domain

Let $R > 2$ be large enough, so that $D_1 \cup D_2 \subset B_{R/2}$ when $\delta < \delta_0$. It follows from the Riesz Theorem that for any $u \in H_0^1(B_R)$, there exists a unique $B_\delta u \in H_0^1(B_R)$ such that

$$\forall\, v \in H_0^1(B_R), \quad \int_{B_R} \nabla B_\delta u(X) \cdot \nabla v(X) = \int_{D_1 \cup D_2} \nabla u(X) \cdot \nabla v(X).$$

The operator B_δ maps $H_0^1(B_R)$ into itself, and it is easily seen to satisfy $||B_\delta|| \leq 1$. The argument in [10] concerning T_δ shows that B_δ is self adjoint and of Fredholm type, thus has a spectral decomposition. Let $b_n^{\delta,\pm}$ denote its eigenvalues.

Theorem 2.1. *Let $n \geq 1$. There exists a constant C independent of δ and n such that*

$$\frac{1}{C}\, b_n^{\delta,+} \;\leq\; \beta_n^{\delta,+} \;\leq\; C b_n^{\delta,+}. \tag{2.1}$$

Proof. Let $f \in H^{1/2}(\partial D)$ and let $u_f \in W_0^{1,-1}(\mathbb{R}^2)$ and $v_f \in H_0^1(B_R)$ denote the functions which are harmonic in $\mathbb{R}^2 \setminus D$ and in $B_R \setminus D$ respectively, which are also harmonic in D, and which satisfy $u_f = v_f = f$ on ∂D. We will show that there exists a constant $C > 0$ independent of δ and n such that for all $f \in H^{1/2}(\partial D) \setminus \{0\}$,

$$\frac{1}{C}\frac{\displaystyle\int_D |\nabla v_f|^2}{\displaystyle\int_{B_R} |\nabla v_f|^2} \leq \frac{\displaystyle\int_D |\nabla u_f|^2}{\displaystyle\int_{\mathbb{R}^2} |\nabla u_f|^2} \leq C\frac{\displaystyle\int_D |\nabla v_f|^2}{\displaystyle\int_{B_R} |\nabla v_f|^2}. \tag{2.2}$$

The statement of the theorem follows then from the min-max principle for the operators T_δ and B_δ.

To prove (2.2), we first note that since u_f and v_f are harmonic in D and coincide on ∂D, $u_f \equiv v_f$ on ∂D, so that

$$\int_D |\nabla u_f|^2 = \int_D |\nabla v_f|^2. \tag{2.3}$$

Since the extension of v_f by 0 outside of B_R is a function of $W_0^{1,-1}(\mathbb{R}^2)$, we see that

$$\int_{\mathbb{R}^2} |\nabla u|^2 \leq \min_{w \in W_0^{1,-1}(\mathbb{R}^2)} \int_{\mathbb{R}^2} |\nabla w|^2 \leq \int_{B_R} |\nabla v|^2,$$

which together with (2.3) proves the right-hand inequality in (2.2).

To prove the other inequality, let χ denote a smooth cut-off function, such that $\chi \equiv 1$ in $B_{R/2}$ and $\chi \equiv 0$ outside B_R. We may also assume that $||\chi||_{W^{1,\infty}} \leq 1$. The function $\tilde{u}_f = \chi u_f$ lies in $H_0^1(B_R)$, and there is a constant C that only depends on R such that

$$\int_{B_R \setminus \overline{D}} |\nabla \tilde{u}_f|^2 \leq C \int_{\mathbb{R}^2 \setminus \overline{D}} |\nabla u_f|^2.$$

Since $\tilde{u}_f = u_f = v_f$ on ∂D, it follows from the Dirichlet principle that

$$\int_{B_R \setminus \overline{D}} |\nabla v_f|^2 \leq \int_{B_R \setminus \overline{D}} |\nabla \tilde{u}_f|^2,$$

which combined with (2.3) yields the desired inequality. □

3. Discretization

In the sequel, we estimate numerically the rate of convergence to 0 of the first non-degenerate eigenvalue $b_1^{\delta,+}$, from which, using Theorem 1.1, we will infer the behavior of $\beta_1^{\delta,+}$. To this end, we use the min-max principle to approximate $b_1^{\delta,+}$ by

$$b_{1,N}^{\delta,+} = \min_{u \in V_N} \frac{\int_D |\nabla u(X)|^2 dX}{\int_{B_R} |\nabla u(X)|^2 dX}, \tag{3.1}$$

where V_N is a finite dimensional subspace of $H_0^1(B_R)$. We construct approximation spaces V_N in the following fashion Let $X_1 = (x_1 + iy_1) \in D_1, X_2 = (x_2 + iy_2) \in D_2$ and $n \in \mathbb{N}$. Define $\phi_{n,1}^{\pm}, \phi_{n,2}^{\pm} : \mathbb{R}^2 \longrightarrow \mathbb{C}$ by $\phi_{n,1}(z) = (z - X_1)^n, \phi_{n,2}(z) = (z - X_2)^n$, where $z = x + iy$. Let $w_m, m \geq 1$ be the $H_0^1(D)$ functions which are harmonic in $B_R \setminus \overline{D}$ and such that

$$\begin{cases} w_{4n-3} = Re(\phi_{n,1}) & \text{in } D_1 \\ w_{4n-3} = 0 & \text{in } D_2, \end{cases} \qquad \begin{cases} w_{4n-2} = Im(\phi_{n,1}) & \text{in } D_1 \\ w_{4n-2} = 0 & \text{in } D_2, \end{cases}$$

$$\begin{cases} w_{4n-1} = 0 & \text{in } D_1 \\ w_{4n-1} = Re(\phi_{n,2}) & \text{in } D_2, \end{cases} \qquad \begin{cases} w_{4n} = 0 & \text{in } D_1 \\ w_{4n} = Im(\phi_{n,2}) & \text{in } D_2. \end{cases}$$

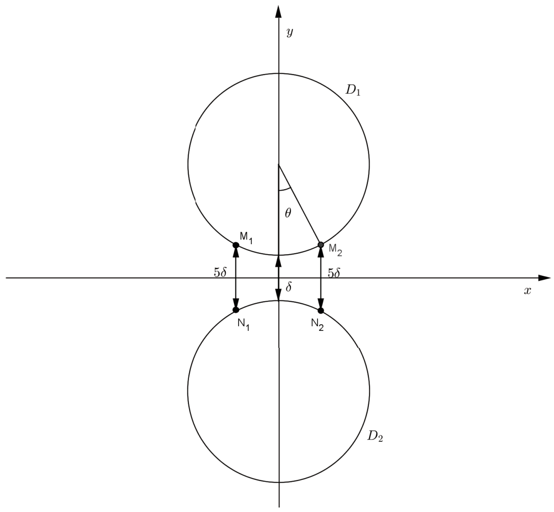

Fig. 3.1. Mesh refinement zone.

We consider a conformal triangulation $\mathcal{T}$ of B_R, which is refined in the neck between the 2 inclusions. The width of the refined zone is chosen so that its thickness is equal to 5δ at its extremities (see for instance Figures 3.1–3.3) for the case of two discs. Let $\hat{w}_m, m \geq 1$ denote the H^1 projection of w_m on the space of functions which are piecewise linear on $\mathcal{T}$. We define V_N as the vector space generated by the functions $\hat{w}_m, m \leq 4N$.

We note that the functions $w_m, m \geq 1$ are linearly independent. Together with the functions $w_{0,1}, w_{0,2}$ in $H_0^1(B_R)$ defined by $\Delta w_{0,i} = 0$ in $B_R \setminus \overline{D}$, and

$$\begin{cases} w_{0,1} = 1 & \text{in } D_1 \\ w_{0,1} = 0 & \text{in } D_2, \end{cases} \qquad \begin{cases} w_{0,2} = 0 & \text{in } D_1 \\ w_{0,2} = 1 & \text{in } D_2, \end{cases}$$

they from a basis of $H_0^1(B_R)$. We also note that the functions $w_{0,i}$ are the eigenfunctions of B_δ associated to the degenerate mode $b_0 = 0$. To compute the eigenvalues $b_{1,N}^{\delta,+}$, we form the

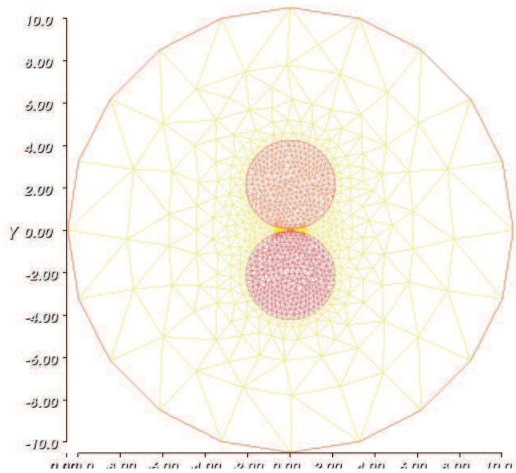

Fig. 3.2. Mesh for $\delta = \frac{1}{16}$.

matrices A and B with entries

$$A_{i,j} = \int_{D_1 \cup D_2} \nabla \hat{w}_i \cdot \nabla \hat{w}_j, \qquad B_{i,j} = \int_{B_R} \nabla \hat{w}_i \cdot \nabla \hat{w}_j,$$

and then compute the generalized eigenvalues of the system $AU = \lambda BU$. We have used the software Freefem++ [14] to compute the vectors $\hat{w}_m$, and Scilab [23] to solve the above matrix eigenvalue problem.

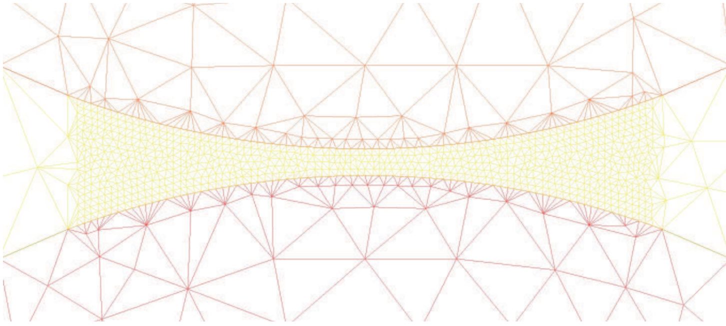

Fig. 3.3. Mesh refinement near the contact point.

4. Numerical Results

We deduce from Theorems 1.1 and 2.1 that

$$\log b_{1,N}^{\delta,+} \sim \log c_1^+ + \frac{m-1}{m} \log \delta$$

as δ tends to 0. In this section, we draw the graph of $\log b_{1,N}^{\delta,+}$ as a function of $\log \delta$, and determine numerically its slope $\frac{m-1}{m}$. We first study the case where the inclusions are two discs, and then we perturb the inclusions to have a contact point with higher order.

4.1. The case of 2 discs

We start with the case of two discs $D_1 = B_r(0, r + \frac{\delta}{2})$ and $D_2 = B_r(0, r - \frac{\delta}{2})$ with $r = 2$. Here, X_1 and X_2 in the construction of V_N, are chosen to be the centers of the discs D_1 and D_2.

Since the contact of order two, i.e.,

$$\psi_1(x) + \psi_2(x) \sim C|x|^2 \quad \text{as } x \to 0,$$

the theoretical slope is $\frac{1}{2}$. Taking $N = 39$, the graph of $\log b_{1,N}^{\delta,+}$ tends to the line with equation $t = -0.7934156 + 0.4307516s$ (see for instance Figure 4.1). The equation of the line is computed using the least squares method.

The dimension of the space V_N is $4N + 2$. Hence, we expect that the numerical slope will tend to the theoretical one when N becomes larger. Table 4.1 and Figure 4.1 give how does the numerical slope behave as a function of N, and shows a good agreement with the theoretical predictions.

Table 4.1: Numerical slope as a function of N.

Values of N	equation of the line approximation
9	$t = -1.09526 + 0.2486835s$
19	$t = -0.9099896 + 0.3700286s$
29	$t = -0.8575362 + 0.4045268s$
39	$t = -0.7934156 + 0.4307516s$

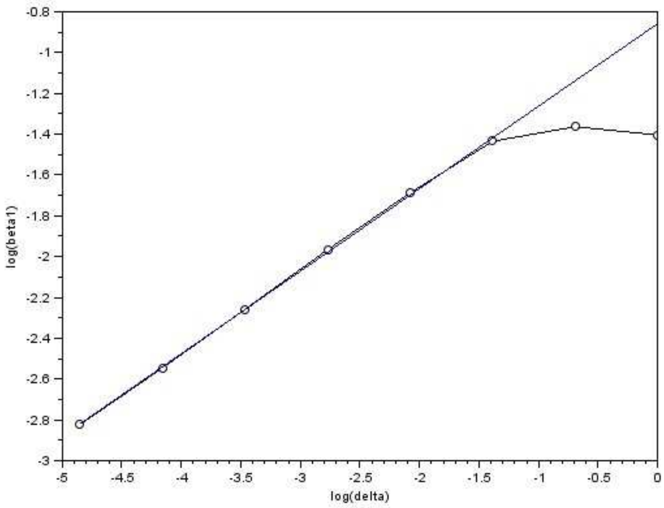

Fig. 4.1. $\log b_{1,N}^{\delta,+}$ as function of $\log \delta$.

4.2. Contact of order m

Now, we consider shapes with different contact orders, i.e.,

$$\psi_1(x) + \psi_2(x) \sim C|x|^m.$$

Let D_1 and D_2 be the perturbed half discs defined by (see Figure 4.3)

$$D_1 = \Big\{ -1 \leq x \leq 1, |x|^m + \delta \leq y \leq 1 + \delta \Big\} \cup \Big\{ x^2 + (y - 1 - \delta)^2 \leq 1, y \geq 1 + \delta \Big\},$$
$$D_2 = \Big\{ -1 \leq x \leq 1, -|x|^m - \delta \geq y \geq -1 - \delta \Big\} \cup \Big\{ x^2 + (y + 1 + \delta)^2 \leq 1, y \leq -1 - \delta \Big\}.$$

The points X_1 and X_2 in the construction of the space V_N, are the centers of the perturbed discs. Table 4.2 provides the numerical results for δ between $\frac{1}{2}$ and $\frac{1}{2^7}$, and $N = 39$.

Table 4.2: Numerical results for δ with different values of m.

m	Equation of the line	Theoretical slope	Error
$m = 2$	$t = -0.7934156 + 0.4307516s$	$\frac{1}{2} = 0.5$	0.0692484
$m = 6$	$t = -0.1401772 + 0.8003479s$	$\frac{5}{6} \simeq 0.83$	0.03298543
$m = 9$	$t = -0.2357561 + 0.8508496s$	$\frac{8}{9} \simeq 0.89$	0.03803929

We remark that the computed slopes are in a good agreement with the expected theoretical values.

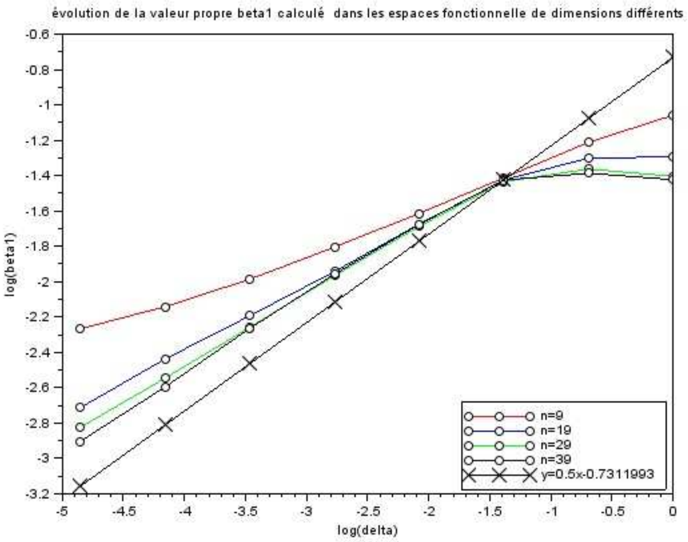

Fig. 4.2. The effect of the dimension of V_N on the values of $b_{1,N}^{\delta,+}$.

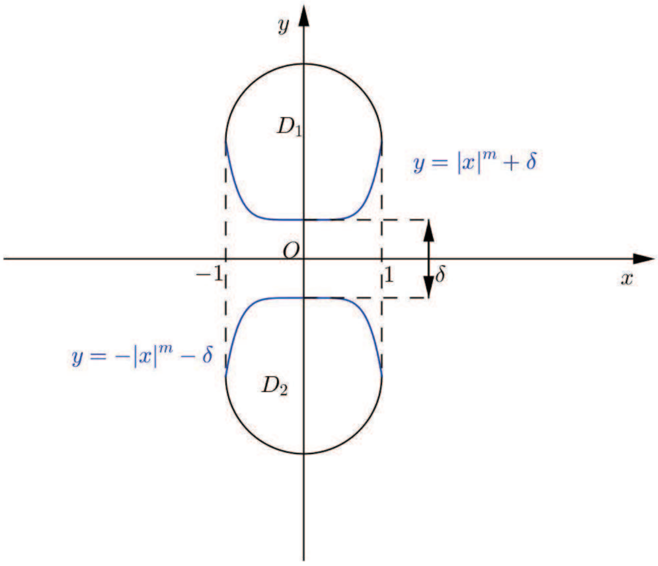

Fig. 4.3. Domains D_1 and D_2

5. Conclusion

We have studied the behavior of the eigenvalues of the Neumann-Poincaré operator for two close-to-touching inclusions in dimension two. We have validated numerically the rates of convergence derived in [10]. We continue to study the asymptotic behavior of the spectrum of the Neumann-Poincaré integral operator for two close-to-touching inclusions in dimension three. We also plan to extend the results of [9] to general geometries in dimension two. In dimension three the sizes of the matrices A and B become too large and this may complicate the computation of the generalized eigenvalues. In another line of research, we propose to use an integral equation approach combined with an asymptotic approximation of the kernels of the off-diagonal operators in the system (1.4) around the contact point. We think that this approach is more appropriate to dimension three and larger. We will report related results in future works.

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