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Compact Difference Scheme for Time-Fractional Fourth-Order Equation with First Dirichlet Boundary Condition

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Abstract. The convergence of a compact finite difference scheme for one- and two-dimensional time fractional fourth order equations with the first Dirichlet boundary conditions is studied. In one-dimensional case, a Hermite interpolating polynomial is used to transform the boundary conditions into the homogeneous ones. The Stephenson scheme is employed for the spatial derivatives discretisation. The approximate values of the normal derivative are obtained as a by-product of the method. For periodic problems, the stability of the method and its convergence with the accuracy $\mathcal{O}(\tau^{2-\alpha})+\mathcal{O}(h^4)$ are established, with the similar error estimates for two-dimensional problems. The results of numerical experiments are consistent with the theoretical findings.

AMS subject classifications: 35R11, 65M06, 65M12, 65M15

Key words: Fractional partial differential equation, compact finite difference scheme, fourth-order equation, Stephenson scheme, stability and convergence.

1. Introduction

Let ${}_0^CD_t^\alpha v$, $0<\alpha<1$ be the Caputo fractional derivative of a function $v(x,t)$ [30] defined by

$$ {}_0^CD_t^\alpha v(x,t) := \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\frac{\partial v}{\partial \tau}(x,\tau)d\tau. \tag{1.1}$$

We consider the one-dimensional time fractional fourth-order equation

$$ {}_0^CD_t^\alpha v(x,t)+\frac{\partial^4 v}{\partial x^4}(x,t)=g(x,t), \quad x\in(x_L,x_R), \quad t\in(0,T] \tag{1.2}$$

with the initial and boundary conditions

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$$\begin{aligned} &v(x,0) = v_0(x), \quad x \in [x_L, x_R] \\ &v(x_L,t) = \phi_L(t), \quad \frac{\partial v}{\partial x}(x_L,t) = \psi_L(t), \\ &v(x_R,t) = \phi_R(t), \quad \frac{\partial v}{\partial x}(x_R,t) = \psi_R(t). \end{aligned} \tag{1.3}$$

Considering the above problem in the situation where the time variable t does not appear, we obtain the biharmonic equation, which finds applications in incompressible fluid dynamics and in two-dimensional elasticity theory [2]. Numerical schemes for such problems are developed and well studied. Thus, for multi-space nonlinear parabolic partial differential equations and vibration problems, implicit difference schemes of order two in time and order four in space are, respectively, presented in [27] and [28]. It was already noted that for fourth-order diffusion equation with the second Dirichlet boundary conditions — i.e. if a second order derivative appears in the boundary conditions, the finding of numerical solutions is relatively easy. Thus writing the second order derivative as an auxiliary variable, one can split the original problem into a coupled system of two second-order equations with appropriate boundary conditions. However, the discretisation of the first Dirichlet boundary conditions requires special attention in order to match the global accuracy. As an uncoupled scheme, the Stephenson schemes of second and fourth order have been presented in [34], fourth order accurate schemes in [4] and [12], and a compact discretisation of the biharmonic problem with a fast FFT algorithm in [3].

Traditional partial differential equations contains the derivatives of integer order only. Recently, fractional differential equations attracted substantial attention because of wide applications — cf. [26,29,30]. Thus for anomalous subdiffusion equations, finite difference schemes are considered in Refs. [5,25,41,45]. Moreover, a difference scheme with spectral method [24] and fast finite difference methods [6,37] are applied to space-fractional diffusion equations, to tempered fractional diffusion equations [17], to time fractional equations [22,39] and to multi-term time-fractional diffusion equations [31]. Compact finite difference schemes for subdiffusion equations are proposed and studied in [1,8,14], where the error estimates $\mathcal{O}(\tau + h^4)$, $\mathcal{O}(\tau^{2-\alpha} + h^4)$, $\mathcal{O}(\tau^2 + h^4)$ are, respectively, obtained. For one-dimensional space and time fractional Bloch-Torrey equation, the stability and convergence of a high-order difference scheme have been studied in [44] by the discrete energy method. Various high-order difference schemes for Stokes' first problem are considered for heated generalized second grade fluid with fractional derivatives [21] and for distributed-order time-fractional equations [11]. Galerkin and spectral element methods for fractional equations have been investigated in [23,32,40].

The numerical solutions of fractional equations of fourth-order have been also considered — e.g. a compact algorithm for sub-diffusion equations with the first Dirichlet conditions [20]. A new variable was introduced and a high order difference scheme was developed with the convergence order $\mathcal{O}(\tau^{2-\alpha}+h^4)$ in L_2-norm. In addition, a local discontinuous Galerkin method for time-fractional fourth-order differential equations was studied in [16,38], an implicit compact finite difference scheme for the fourth-order fractional diffusion-wave system in [19], and the hyperbolic equation describing the random vibrations of beams in [33,43].

In this work, we apply the fourth order Stephenson scheme to time fractional fourth order parabolic problems with the first Dirichlet boundary conditions. This scheme is naturally suited for Dirichlet boundary conditions and, in addition to the numerical solution $\{\mathbf{U}^n\}$, we can simultaneously obtain the derivative $\{\mathbf{U}_x^n\}$ or partial derivatives $\{\mathbf{U}_x^n\}$ and $\{\mathbf{U}_y^n\}$ while considering one- or two-dimensional problems. The efficiency of the Stephenson scheme in solving biharmonic problems is well-known [13] and this why we use it here. Balancing the time and space errors in high-order methods one can use significantly larger space mesh size than the time step size, which allows to reduce the order of the matrices of the corresponding linear systems at each time step. This paper complements the study [20] of difference schemes for the fourth-order time fractional sub-diffusion equations.

The paper is organised as follows. In Section 2 we transform the boundary conditions into homogeneous ones, approximate the fractional time derivative by $L1$-formula and use the one-dimensional Stephenson operator for the discretisation of spatial derivative. All this leads to a compact finite difference scheme for homogeneous problem (2.2). The stability of this scheme is considered in Section 3, while the convergence is discussed in Section 4. In the case of smooth solutions, we show that the numerical scheme under consideration has $(2-\alpha)$ order of accuracy in time and fourth order accurate in space. Section 5 deals with the extension of the method to two-dimensional problems. Numerical experiments, presented in Section 6 show that computational error can be estimated as $\mathscr{O}(\tau^{2-\alpha})+\mathscr{O}(h^4)$, consistent with the theoretical analysis. Our conclusion is in Section 7.

2. Difference Scheme

Let us start with the transformation of the boundary conditions into homogeneous ones. Recalling the Hermite interpolation, we consider the following basis functions:

$$\begin{aligned}
\alpha_L(x) &= \left[1-\frac{2(x-x_L)}{x_L-x_R}\right]\left(\frac{x-x_R}{x_L-x_R}\right)^2, \\
\alpha_R(x) &= \left[1-\frac{2(x-x_R)}{x_R-x_L}\right]\left(\frac{x-x_L}{x_R-x_L}\right)^2, \\
\beta_L(x) &= (x-x_L)\left(\frac{x-x_R}{x_L-x_R}\right)^2, \\
\beta_R(x) &= (x-x_R)\left(\frac{x-x_L}{x_R-x_L}\right)^2.
\end{aligned}$$

It is easily seen that

$$\begin{aligned}
&\alpha_L(x_L)=1, \quad \frac{d\alpha_L}{dx}(x_L)=0, \quad \alpha_L(x_R)=0, \quad \frac{d\alpha_L}{dx}(x_R)=0, \\
&\alpha_R(x_L)=0, \quad \frac{d\alpha_R}{dx}(x_L)=0, \quad \alpha_R(x_R)=1, \quad \frac{d\alpha_R}{dx}(x_R)=0, \\
&\beta_L(x_L)=0, \quad \frac{d\beta_L}{dx}(x_L)=1, \quad \beta_L(x_R)=0, \quad \frac{d\beta_L}{dx}(x_R)=0,
\end{aligned}$$

$$\beta_R(x_L)=0,\quad \frac{d\beta_R}{dx}(x_L)=0,\quad \beta_R(x_R)=0,\quad \frac{d\beta_R}{dx}(x_R)=1.$$

Considering the Hermite interpolating polynomial

$$H(x,t)=\alpha_L(x)\phi_L(t)+\alpha_R(x)\phi_R(t)+\beta_L(x)\psi_L(t)+\beta_R(x)\psi_R(t), \tag{2.1}$$

we observe that the function $u(x,t):=v(x,t)-H(x,t)$ satisfies the boundary value problem

$$\begin{aligned}
&{}_0^C D_t^\alpha u(x,t)+\frac{\partial^4 u}{\partial x^4}(x,t)=g(x,t)-{}_0^C D_t^\alpha H(x,t)\equiv f(x,t),\\
&x\in(x_L,x_R),\quad t\in(0,T],\\
&u(x,0)=v_0(x)-H(x,0)\equiv u_0(x),\quad x\in[x_L,x_R],\\
&u(x_L,t)=\frac{\partial u}{\partial x}(x_L,t)=u(x_R,t)=\frac{\partial u}{\partial x}(x_R,t)=0.
\end{aligned} \tag{2.2}$$

Our next task is to construct the solution of (2.2).

Remark 2.1. Since the Hermite interpolation function $H(x,t)$ is a cubic polynomial with respect to x, one has $(\partial^4 H/\partial x^4)(x,t)=0$. That simplifies the first equation in (2.2).

Let M and N be positive integers. Setting $h:=(x_R-x_L)/M$ and $\tau:=T/N$, we introduce the uniform grid of mesh points (x_i,t_n), where $x_i=x_L+ih$, $i=0,1,\cdots,M$ and $t_n=n\tau$, $n=0,1,\cdots,N$. Then $u(x_i,t_n)$ is the value of the exact solution at the mesh point (x_i,t_n), and U_i^n refer to the solution of the below difference scheme at the same mesh point.

We recall auxiliary results concerning the approximation of fractional derivatives.

Lemma 2.1 (cf. Gao & Sun [14], Sun & Wu [35]). *If $f(t)\in C^2[0,t_k]$ and*

$$\begin{aligned}
\bar{R}f(t_k):=&\frac{1}{\Gamma(1-\alpha)}\int_0^{t_k}\frac{f'(s)}{(t_k-s)^\alpha}ds-\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\\
&\times\Big[a_0 f(t_k)-\sum_{j=1}^{k-1}(a_{k-j-1}-a_{k-j})f(t_i)-a_{k-1}f(t_0)\Big],
\end{aligned}$$

then

$$\big|\bar{R}f(t_k)\big|\le\frac{1}{\Gamma(2-\alpha)}\Big[\frac{1-\alpha}{12}+\frac{2^{2-\alpha}}{2-\alpha}-(1+2^{-\alpha})\Big]\max_{0\le t\le t_k}|f''(t)|\tau^{2-\alpha},$$

where $0<\alpha\le 1$ and $a_j=(j+1)^{1-\alpha}-j^{1-\alpha}$.

Assume that $u(t)\in C^2[0,t_n]$ and consider the term

$$D^\alpha u^n:=\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\Big[a_0u(t_n)-\sum_{l=1}^{n-1}(a_{n-l-1}-a_{n-l})u(t_l)-a_{n-1}u(t_0)\Big] \tag{2.3}$$

with the coefficients $a_l=(l+1)^{1-\alpha}-l^{1-\alpha}$. It follows from Lemma 2.1 that

$$D^\alpha u^n={}_0^C D_t^\alpha u^n+\mathscr{O}(\tau^{2-\alpha})\quad\text{for}\quad 0<\alpha<1. \tag{2.4}$$

Let $\Delta_x u_i^n$ and $\delta_x^2 u_i^n$ be the spatial difference operators defined on the set of mesh functions u_i^n by

$$\Delta_x u_i^n = \frac{1}{2h}\left(u_{i+1}^n - u_{i-1}^n\right), \quad \delta_x^2 u_i^n = \frac{1}{h^2}\left(u_{i-1}^n - 2u_i^n + u_{i+1}^n\right).$$

At the interior points x_i, $1 \le i \le M-1$ the difference scheme is constructed following the ideas of [12]. Consider an operator $\tilde{\delta}_x^4$, approximating the partial derivative $\partial^4/\partial x^4$. Let V_i^n be an approximation of $(\partial u/\partial x)(x_i, t_n)$. We define the operator $\tilde{\delta}_x^4$ by

$$\tilde{\delta}_x^4 U_i^n = \frac{12}{h^2}\left(\Delta_x V_i^n - \delta_x^2 U_i^n\right). \tag{2.5}$$

This approximation of the fourth-order derivative of u^n at x_i is called the Stephenson's scheme [34]. Using (2.5) and Taylor series expansions, we write

$$\tilde{\delta}_x^4 u_i^n = \frac{\partial^4 u}{\partial x^4}\bigg|_i^n + \mathscr{O}(h^4).$$

We now consider an implicit compact finite difference scheme for the problem (2.2) — viz. **Scheme I**. Find $\{U_i^n\}$, $0 \le i \le M,\ 0 \le n \le N$ satisfying

$$\begin{aligned}
&D^\alpha U_i^n + \tilde{\delta}_x^4 U_i^n = f_i^n, && 1 \le i \le M-1, \quad 1 \le n \le N,\\
&\frac{1}{6}V_{i-1}^n + \frac{2}{3}V_i^n + \frac{1}{6}V_{i+1}^n = \Delta_x U_i^n, && 1 \le i \le M-1, \quad 1 \le n \le N,\\
&U_i^0 = u_0(x_i), && 1 \le i \le M-1,\\
&U_0^n = U_M^n = V_0^n = V_M^n = 0.
\end{aligned} \tag{2.6}$$

Using the notation $\mathbf{U}^n := (U_1^n, U_2^n, \cdots, U_{M-1}^n)^T$ and $\mathbf{V}^n := (V_1^n, V_2^n, \cdots, V_{M-1}^n)^T$, we note that (2.6) is a coupled system — i.e. the terms $\mathbf{U}^n$ and $\mathbf{V}^n$ have to be simultaneously determined. However, we can eliminate the unknown $\mathbf{V}^n$, derive $\mathbf{U}^n$ and find $\mathbf{V}^n$ afterwards.

Let $\mathbf{I}$ be the identity matrix and $\mathbf{K}$, $\mathbf{P}$, $\mathbf{T}$, $\mathbf{S}$ be defined by

$$\mathbf{K} := \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}_{(M-1)\times(M-1)},$$

$$\mathbf{P} = \begin{pmatrix} 4 & 1 & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ & & & 1 & 4 \end{pmatrix}_{(M-1)\times(M-1)},$$

$$\mathbf{T} := 6\mathbf{I} - \mathbf{P}, \quad \mathbf{S} := \frac{6}{h^4}(3\mathbf{K}\mathbf{P}^{-1}\mathbf{K} + 2\mathbf{T}).$$

The second equation in (2.6) can be written as

$$\mathbf{PV}^n = \frac{3}{h}\mathbf{KU}^n, \tag{2.7}$$

and the operator $\tilde{\delta}_x^4$ has the following matrix representation:

$$\mathbf{Su} = \frac{6}{h^4}\left[3\mathbf{KP}^{-1}\mathbf{K} + 2\mathbf{T}\right]\mathbf{u}.$$

Writing

$$\mathbf{A} = \mathbf{S} + \frac{1}{\tau^\alpha\Gamma(2-\alpha)}\mathbf{I},$$

we represent Scheme I in the form

$$\begin{aligned}
\mathbf{AU}^1 &= \frac{1}{\tau^\alpha\Gamma(2-\alpha)}\mathbf{U}^0 + \mathbf{F}^1,\\
\mathbf{AU}^n &= \frac{1}{\tau^\alpha\Gamma(2-\alpha)}\sum_{l=1}^{n-1}(a_{n-l-1} - a_{n-l})\mathbf{U}^l + \frac{1}{\tau^\alpha\Gamma(2-\alpha)}a_{n-1}\mathbf{U}^0 + \mathbf{F}^n,\\
\mathbf{U}^0 &= (u_0(x_1), u_0(x_2), \cdots, u_0(x_{M-1}))^T,
\end{aligned} \tag{2.8}$$

where

$$\mathbf{F}^n = (f(x_1, t^n), f(x_2, t^n), \cdots, f(x_{M-1}, t^n))^T.$$

We point out that (2.8) contains the unknown $\{\mathbf{U}^n\}$ only, and $\mathbf{V}^n$ can be determined from (2.7) as soon as $\mathbf{U}^n$ have been found.

Remark 2.2. The matrix A contains the factors $1/h^4$ and $1/\tau^\alpha$, which increase the rounding errors in the case of small h and τ. Therefore, in numerical experiments we multiply the corresponding equations by τ^α to reduce the errors.

Theorem 2.1. *Scheme I has a unique solution.*

Proof. The coefficient matrix $\mathbf{A}$ in (2.8) is strictly diagonally dominant and hence invertible. Therefore, the solution of our compact finite difference scheme (2.6) exists and is unique. □

3. Stability Analysis

For vectors $\mathbf{v} = (v_0, v_1, \cdots, v_M)^T, \mathbf{w} = (w_0, w_1, \cdots, w_M)^T$, we define the inner product and norms by

$$(\mathbf{v}, \mathbf{w}) = h\sum_{j=0}^{M} v_j w_j, \quad \|\mathbf{v}\|_2 = (\mathbf{v}, \mathbf{v})^{1/2}, \quad \|\mathbf{v}\|_\infty = \max_{0\le j\le M} |v_j|.$$

In what follows, we will drop subscript 2 in the discrete L_2-norm and write it simply as $\|\cdot\|$.

Lemma 3.1 (cf. Refs. [3, 12])**.** *The symmetric positive definite operator $\tilde{\delta}_x^4$ can be written in the form*

$$\mathbf{S} = \frac{6}{h^4}\mathbf{P}^{-1}\mathbf{T}^2 + \frac{36}{h^4}\left(v_1 v_1^T + v_2 v_2^T\right),$$

where

$$v_1 = (\alpha-\beta)^{1/2}\mathbf{P}^{-1}\left[\frac{\sqrt{2}}{2}e_1 - \frac{\sqrt{2}}{2}e_M\right],$$
$$v_2 = (\alpha+\beta)^{1/2}\mathbf{P}^{-1}\left[\frac{\sqrt{2}}{2}e_1 + \frac{\sqrt{2}}{2}e_M\right]$$

with the constants

$$\alpha = 2\left(2 - e_1^T\mathbf{P}^{-1}e_1\right), \quad \beta = 2e_{M-1}^T\mathbf{P}^{-1}e_1$$

and the vectors $e_1 = (1, 0, \cdots, 0)^T, e_{M-1} = (0, \cdots, 0, 1)^T$.

Lemma 3.2 (cf. Thomas [36])**.** *The matrix*

$$\mathbf{T_r}(a,b,c) = \begin{pmatrix} b & c & & & \\ a & b & c & & \\ & \ddots & \ddots & \ddots & \\ & & a & b & c \\ & & & a & b \end{pmatrix}_{(M-1)\times(M-1)}$$

has the eigenvalues

$$\lambda_j = b + 2c\left(\frac{a}{c}\right)^{1/2}\cos\frac{j\pi}{M}$$

with the corresponding eigenvectors $\mathbf{u}_j = [u_1, \cdots, u_k, \cdots, u_{M-1}]^{\mathsf{T}}$*,* $j = 1, \cdots, M-1$*, where*

$$u_k = 2\left(\frac{a}{c}\right)^{k/2}\sin\frac{kj\pi}{M}, \quad k = 1, \cdots, M-1,$$

and T *denotes the transposition operation.*

If the tridiagonal matrix $\mathbf{T_r}(a,b,c)$ is symmetric — i.e. if $a = c$, then it is orthogonally diagonalisable and has the inverse diagonalisable by the same orthogonal matrix. Since $\mathbf{P}^{-1}$ and $\mathbf{T}^2$ are simultaneously diagonalisable, they commute [18]. Lemmas 3.1 and 3.2 can be used to study the properties of the matrix $\mathbf{S}$.

Lemma 3.3. *The matrix* $\mathbf{S}$ *is symmetric positive definite and there is a positive constant* c_0 *such that for any vector* $\mathbf{v} \neq \mathbf{0}^T$ *the inequality*

$$(\mathbf{S}\mathbf{v}, \mathbf{v}) \geq \frac{c_0}{h^4}\|\mathbf{v}\|^2 \tag{3.1}$$

holds.

Proof. Since $(v_1 v_1^T + v_2 v_2^T)$ is a nonnegative definite matrix, it follows from Lemma 3.1 that

$$(\mathbf{Sv}, \mathbf{v}) \geq \left(\frac{6}{h^4} \mathbf{P}^{-1} \mathbf{T}^2 \mathbf{v}, \mathbf{v} \right).$$

Since $\mathbf{P}$ is a symmetric matrix, its eigenvectors $\{\varphi_i\}_{i=1}^{M-1}$ form an orthonormal matrix. By Lemma 3.2, matrices $\mathbf{T}$ and $\mathbf{P}^{-1}$ have the same set of eigenvectors. The corresponding eigenvalues, denoted by $\lambda_i(T)$ and $\lambda_i(P^{-1})$ for $i = 1, \cdots, M-1$, are positive. Since the eigenvectors $\{\varphi_i\}_{i=1}^{M-1}$ constitute an orthonormal basis in $\mathbb{R}^{M-1}$, any vector $\mathbf{v} \in \mathbb{R}^{M-1}$ can be represented in the form $\mathbf{v} = \sum\limits_{i=1}^{M-1} \tilde{v}_i \varphi_i$. Then

$$\|\mathbf{v}\|^2 = (\mathbf{v}, \mathbf{v}) = \left(\sum_{i=1}^{M-1} \tilde{v}_i \varphi_i, \sum_{j=1}^{M-1} \tilde{v}_j \varphi_j \right) = \sum_{i=1}^{M-1} (\tilde{v}_i \varphi_i, \tilde{v}_i \varphi_i)$$

and

$$\begin{aligned}
(\mathbf{P}^{-1}\mathbf{T}^2\mathbf{v}, \mathbf{v}) &= \left(\sum_{i=1}^{M-1} \lambda_i(P^{-1}) \lambda_i^2(T) \tilde{v}_i \varphi_i, \sum_{j=1}^{M-1} \tilde{v}_j \varphi_j \right) \\
&= \sum_{i=1}^{M-1} \left(\lambda_i(P^{-1}) \lambda_i^2(T) \tilde{v}_i \varphi_i, \tilde{v}_i \varphi_i \right) \\
&\geq \min_{1 \leq i \leq M-1} \left\{ \lambda_i(P^{-1}) \lambda_i^2(T) \right\} \sum_{i=1}^{M-1} (\tilde{v}_i \varphi_i, \tilde{v}_i \varphi_i) \\
&= \min_{1 \leq i \leq M-1} \left\{ \lambda_i(P^{-1}) \lambda_i^2(T) \right\} \|\mathbf{v}\|^2.
\end{aligned}$$

Therefore, there is a constant $c_0 > 0$ independent of h, such that

$$(\mathbf{Sv}, \mathbf{v}) \geq \frac{c_0}{h^4} \|\mathbf{v}\|^2,$$

and the proof is completed. $\square$

Let a_l be the coefficients in the Eq. (2.3). Then we have

Lemma 3.4 (cf. Refs. [7,42]). *The sequence $\{a_l\}$ is monotonically decreasing, tends to 0 as n tends to ∞, and satisfies the inequality*

$$(1-\alpha)(l+1)^{-\alpha} < a_l < (1-\alpha) l^{-\alpha}.$$

Using Lemmas 3.3 and 3.4, we can prove the stability of Scheme I.

Theorem 3.1. *Let $\{\mathbf{U}^n\}(n = 1, \cdots, N)$ be the solutions of Scheme I for periodic problem* (1.2)-(1.3)*, then*

$$\|\mathbf{U}^n\| \leq \|\mathbf{U}^0\| + \frac{T^\alpha}{1-\alpha} \Gamma(2-\alpha) \max_{1 \leq l \leq n} \|\mathbf{F}^l\|. \tag{3.2}$$

Proof. Considering the inner products of $\mathbf{U}^1$ and the first equation in (2.8) and also $\mathbf{U}^n$ and the second equation in (2.8) and recalling that $a_0 = 1$, we obtain

$$(\mathbf{A}\mathbf{U}^n, \mathbf{U}^n) = \frac{1}{\tau^\alpha \Gamma(2-\alpha)} \sum_{l=1}^{n-1} (a_{n-l-1} - a_{n-l})(\mathbf{U}^l, \mathbf{U}^n) + \frac{1}{\tau^\alpha \Gamma(2-\alpha)} a_{n-1}(\mathbf{U}^0, \mathbf{U}^n) + (\mathbf{F}^n, \mathbf{U}^n), \quad 1 \le n \le N. \tag{3.3}$$

It follows from (3.1) that

$$(\mathbf{A}\mathbf{U}^n, \mathbf{U}^n) = \frac{6}{h^4}\left((3\mathbf{K}\mathbf{P}^{-1}\mathbf{K} + 2\mathbf{T})\mathbf{U}^n, \mathbf{U}^n\right) + \frac{1}{\tau^\alpha \Gamma(2-\alpha)}(\mathbf{U}^n, \mathbf{U}^n) \ge \left(\frac{c_0}{h^4} + \frac{1}{\tau^\alpha \Gamma(2-\alpha)}\right)\|\mathbf{U}^n\|^2. \tag{3.4}$$

Therefore,

$$|(\mathbf{U}^l, \mathbf{U}^n)| \le \frac{1}{2}\left(\|\mathbf{U}^l\|^2 + \|\mathbf{U}^n\|^2\right), \quad 0 \le l \le n-1, \tag{3.5}$$

$$|(\mathbf{F}^n, \mathbf{U}^n)| \le \frac{a_{n-1}}{2\tau^\alpha \Gamma(2-\alpha)}\|\mathbf{U}^n\|^2 + \frac{\tau^\alpha \Gamma(2-\alpha)}{2a_{n-1}}\|\mathbf{F}^n\|^2. \tag{3.6}$$

Using the inequalities (3.4)-(3.6) in the Eq. (3.3), we write

$$\left(\frac{c_0}{h^4} + \frac{1}{2\tau^\alpha \Gamma(2-\alpha)}\right)\|\mathbf{U}^n\|^2 \le \frac{1}{2\tau^\alpha \Gamma(2-\alpha)} \sum_{l=1}^{n-1}(a_{n-l-1} - a_{n-l})\|\mathbf{U}^l\|^2 + \frac{1}{2\tau^\alpha \Gamma(2-\alpha)} a_{n-1}\|\mathbf{U}^0\|^2 + \frac{\tau^\alpha \Gamma(2-\alpha)}{2a_{n-1}}\|\mathbf{F}^n\|^2. \tag{3.7}$$

If in the left-hand side of (3.7) we replace the positive term c_0/h^4 by zero, it does not influence the inequality sign and (3.7) yields

$$\|\mathbf{U}^n\|^2 \le \sum_{l=1}^{n-1}(a_{n-l-1} - a_{n-l})\|\mathbf{U}^l\|^2 + a_{n-1}\left[\|\mathbf{U}^0\|^2 + \frac{\tau^{2\alpha}\Gamma^2(2-\alpha)}{a_{n-1}^2}\|\mathbf{F}^n\|^2\right]. \tag{3.8}$$

Following [1], we set

$$E_k := \|\mathbf{U}^0\|^2 + \tau^{2\alpha}\Gamma^2(2-\alpha) \max_{1 \le l \le k} \frac{1}{a_{l-1}^2}\|\mathbf{F}^l\|^2, \tag{3.9}$$

and use the method of mathematical induction to show that

$$\|\mathbf{U}^k\|^2 \le E_k \tag{3.10}$$

for all $1 \le k \le n$. Indeed, the inequality (3.10) is obviously valid for $k = 1$. Assume that (3.10) is valid for all $k \le n-1$. It follows from (3.9) that E_k is a nondecreasing function of k. Taking into account (3.8), one obtains

$$\begin{aligned}\|\mathbf{U}^n\|^2 &\le \sum_{l=1}^{n-1}(a_{n-l-1} - a_{n-l})E_n + a_{n-1}\left[\|\mathbf{U}^0\|^2 + \frac{\tau^{2\alpha}\Gamma^2(2-\alpha)}{a_{n-1}^2}\|\mathbf{F}^n\|^2\right] \\ &\le \sum_{l=1}^{n-1}(a_{n-l-1} - a_{n-l})E_n + a_{n-1}E_n = E_n,\end{aligned}$$

and the inequality (3.10) is proved. Further, by Lemma 3.4,

$$\tau^\alpha a_{n-1}^{-1} < \frac{T^\alpha}{1-\alpha}.$$

Therefore, the inequality (3.10) can be rewritten as

$$\|\mathbf{U}^n\| \le \|\mathbf{U}^0\| + \frac{T^\alpha}{1-\alpha}\Gamma(2-\alpha)\max_{1\le l\le n}\|\mathbf{F}^l\|,$$

which finishes the proof. □

4. Error Estimates

It is well-known that the study of numerical methods for boundary values problems is more demanding than the treatment of periodic problems. In time independent problems, the error estimates are derived by considering the matrix of biharmonic discrete operator [12]. Following this work, we transformed nonhomogeneous boundary conditions into homogeneous ones. For simplicity, here we assume that the problem under consideration is periodic, so that in spatial direction(s) the biharmonic scheme has the fourth order accuracy. On the other hand, numerical simulations show that such accuracy can be also achieved for non-periodic boundary value problems.

Discretising the homogeneous boundary value problem (2.2), we write

$$\begin{aligned}&D^\alpha \mathbf{u}^n + S\mathbf{u}^n = \mathbf{F}^n + \mathbf{R}^n,\\ &u_0^n = u_M^n = 0.\end{aligned}$$

The vector $\mathbf{R}^n$ can be represented in the form

$$\mathbf{R}^n = (D^\alpha \mathbf{u}^n - {}_0^C D_t^\alpha \mathbf{u}^n) + \left(S\mathbf{u}^n - \frac{\partial^4 \mathbf{u}^n}{\partial x^4}\right) \equiv \mathbf{R}_1^n + \mathbf{R}_2^n,$$

and the summands in the left-hand side of the last formula admit the estimates

$$\|\mathbf{R}_1^n\|_\infty = \mathscr{O}(\tau^{2-\alpha}), \quad \|\mathbf{R}_2^n\|_\infty = \mathscr{O}(h^4),$$

which yield

$$\|\mathbf{R}^n\|_\infty = \mathscr{O}(\tau^{2-\alpha}) + \mathscr{O}(h^4). \tag{4.1}$$

Remark 4.1. The term $\|\mathbf{R}_2^n\|_\infty$ is estimated in [3], while the estimate for $\|\mathbf{R}_1^n\|_\infty$ follows from Lemma 2.1.

The convergence of our numerical method is described by the following theorem.

Theorem 4.1. *Let $H(x,t)$ be the Hermite polynomial* (2.1)*, $v(x,t)$ the exact solution of the periodic problem* (1.2)-(1.3) *and U_i^n the approximate solution obtained by Scheme I. If $v(x,t)\in C_{x,t}^{8,2}([x_L,x_R]\times[0,T])$, then*

$$\|v^n-(\mathbf{U}^n+H^n)\|\le \frac{T^\alpha}{1-\alpha}\Gamma(2-\alpha)\big(C_1\tau^{2-\alpha}+C_2h^4\big), \tag{4.2}$$

where

$$\begin{aligned} C_1&=\frac{\sqrt{2}}{\Gamma(2-\alpha)}\left[\frac{1-\alpha}{12}+\frac{2^{2-\alpha}}{2-\alpha}-(1+2^{-\alpha})\right]\max_{0\le t\le T}\left|\frac{\partial^2 v}{\partial t^2}\right|,\\ C_2&=\sqrt{2}(x_R-x_L)\max_{x_L\le x\le x_R}\left|\frac{\partial^8 v}{\partial x^8}\right|. \end{aligned}$$

Proof. We consider the vector $\mathbf{e}^n=(e_1^n,e_2^n,\cdots,e_{M-1}^n)^T$ with the components $e_i^n=U_i^n-(v(x_i,t^n)-H(x_i,t^n))$, describing the error at the interior mesh points. Then

$$\mathbf{A}\mathbf{e}^n=\frac{1}{\tau^\alpha\Gamma(2-\alpha)}\sum_{l=1}^{n-1}(a_{n-l-1}-a_{n-l})\mathbf{e}^l+\mathbf{R}^n,\quad 1\le n\le N.$$

It follows from (4.1) that the local truncation error $\mathbf{R}^n$ can be estimated as

$$\|\mathbf{R}^n\|\le C_1\tau^{2-\alpha}+C_2h^4.$$

Taking into account the relation $\mathbf{e}^0=0$ and the estimates (4.1) and (3.2), we obtain

$$\|\mathbf{e}^n\|\le\frac{T^\alpha}{1-\alpha}\Gamma(2-\alpha)(C_1\tau^{2-\alpha}+C_2h^4),$$

which implies the inequality (4.2). □

5. Two-Dimensional Problems

We now turn our attention to approximate solution of two-dimensional time-fractional fourth-order problems. Our approach is based on the fourth-order compact finite difference scheme for two-dimensional biharmonic problems [3]. For simplicity, here we only consider the problems with homogeneous boundary conditions — viz.

$$\begin{aligned} &{}_0^CD_tu(x,y,t)+\Delta^2u(x,y,t)=f(x,y,t), && (x,y)\in\Omega,\quad t\in(0,T],\\ &u(x,y,0)=\omega(x,y), && (x,y)\in\Omega,\\ &u(x,y,t)=0,\quad \frac{\partial u}{\partial\vec{n}}(x,y,t)=0, && (x,y)\in\partial\Omega,\quad t\in(0,T], \end{aligned} \tag{5.1}$$

where $\Omega=(0,L)^2$, $\vec{n}$ is the unit outwards normal to the boundary of Ω and Δ^2 the biharmonic operator,

$$\Delta^2 u(x,y,t)=\frac{\partial^4 u}{\partial x^4}(x,y,t)+\frac{\partial^4 u}{\partial y^4}(x,y,t)+2\frac{\partial^4 u}{\partial x^2\partial y^2}(x,y,t). \tag{5.2}$$

5.1. Compact finite difference scheme in two dimensions

We consider a difference scheme on a uniform mesh — viz. we set $x_i:=ih$, $y_j:=jh$, $0\le i,j\le M$ with $h=L/M$ and $t_n=n\tau$, $0\le n\le N$ with $\tau=T/N$. The nine-point discrete Stephenson approximation of the biharmonic operator (5.2) has the form

$$\Delta_h^2 u_{ij}^n=\tilde{\delta}_x^4 u_{ij}^n+\tilde{\delta}_y^4 u_{ij}^n+2\delta_x^2\delta_y^2 u_{ij}^n,$$

where

$$\delta_x^2 u_{ij}^n=\frac{u_{i-1,j}^n-2u_{ij}^n+u_{i+1,j}^n}{h^2},$$
$$\delta_y^2 u_{ij}^n=\frac{u_{i,j-1}^n-2u_{ij}^n+u_{i,j+1}^n}{h^2},$$

and $\delta_x^2\delta_y^2:=\delta_x^2\circ\delta_y^2$.

Let $L_{h,0}^2$ be the space of sequences $\{u_{ij}\}$, $0\le i,j\le M$ with the zero boundary conditions $u_{ij}=0$, $\{i,j\}\in\{0,M\}$. If I denotes the identity operator, then the Hermitian gradient $(v,w)\in(L_{h,0}^2)^2$ is defined by

$$\left(I+\frac{h^2}{6}\delta_x^2\right)v_{ij}^n=\Delta_x u_{ij}^n,\quad 1\le i,j\le M-1,$$
$$\left(I+\frac{h^2}{6}\delta_y^2\right)w_{ij}^n=\Delta_y u_{ij}^n,\quad 1\le i,j\le M-1,$$

and one-dimensional Stephenson operators $\tilde{\delta}_x^4$ by $\tilde{\delta}_y^4$ in the $x-$ and $y-$ directions are

$$\tilde{\delta}_x^4 u:=\frac{12}{h^2}\left(\Delta_x v-\delta_x^2 u\right),$$
$$\tilde{\delta}_y^4 u:=\frac{12}{h^2}\left(\Delta_y w-\delta_y^2 u\right).$$

It is known that the above operators approximate partial derivatives of u with the fourth order accuracy, but

$$\delta_x^2\delta_y^2 u_{ij}-\left(\frac{\partial^4 u}{\partial x^2\partial y^2}\right)_{ij}=\mathscr{O}(h^2).$$

Nevertheless, the biharmonic equation $\Delta^2 u=f$ can be approximated with the fourth order accuracy — viz.

$$\tilde{\Delta}_h^2 u_{ij}:=\tilde{\delta}_x^4\left(I-\frac{h^2}{6}\delta_y^2\right)u_{ij}+\tilde{\delta}_y^4\left(I-\frac{h^2}{6}\delta_x^2\right)u_{ij}+2\delta_x^2\delta_y^2 u_{ij}=f_{ij}. \tag{5.3}$$

We note that the right hand side of (5.3) contains only the values f_{ij}, which can happen when the function f is known at interior points only [2,3].

Remark 5.1. The approximation properties of the operator $\tilde{\Delta}_h^2$ have been studied in [3], where Taylor series expansions are used. Here we propose another proof of the fact that the discrete difference operator $\tilde{\Delta}_h^2$ provides a fourth order approximation for the differential operator Δ^2. Thus the mixed derivative $\partial^4 u/\partial x^2 \partial y^2$ can be approximated by the operator

$$\left[\delta_x^2\left(I+\frac{h^2}{12}\delta_x^2\right)^{-1}\right]\circ\left[\delta_y^2\left(I+\frac{h^2}{12}\delta_y^2\right)^{-1}\right]$$

with the fourth order accuracy. In other words, we use the usual compact finite differences $\delta_x^2(I+h^2\delta_x^2/12)^{-1}$ and $\delta_y^2(I+h^2\delta_y^2/12)^{-1}$ to, respectively, approximate the partial derivatives $\partial^2/\partial x^2$ and $\partial^2/\partial y^2$. Since δ_x^2 and δ_y^2 commute, it follows that

$$\begin{aligned}
&\left[\delta_x^2\left(I+\frac{h^2}{12}\delta_x^2\right)^{-1}\right]\circ\left[\delta_y^2\left(I+\frac{h^2}{12}\delta_y^2\right)^{-1}\right]\\
=&\left[\delta_x^2\left(I-\frac{h^2}{12}\delta_x^2\right)\right]\circ\left[\delta_y^2\left(I-\frac{h^2}{12}\delta_y^2\right)\right]+\mathcal{O}(h^4)I\\
=&\,\delta_x^2\delta_y^2\left(I-\frac{h^2}{12}\delta_x^2-\frac{h^2}{12}\delta_y^2\right)+\mathcal{O}(h^4)I.
\end{aligned}$$

Consequently, the relations

$$(\delta_x^2)^2=\left(I+\frac{h^2}{6}\delta_x^2\right)\tilde{\delta}_x^4,\quad (\delta_y^2)^2=\left(I+\frac{h^2}{6}\delta_y^2\right)\tilde{\delta}_y^4$$

yield

$$\begin{aligned}
\Delta^2&=\frac{\partial^4}{\partial x^4}+\frac{\partial^4}{\partial y^4}+2\frac{\partial^4}{\partial x^2\partial y^2}\\
&=\tilde{\delta}_x^4+\tilde{\delta}_y^4+2\left[\delta_x^2\left(I+\frac{h^2}{12}\delta_x^2\right)^{-1}\right]\circ\left[\delta_y^2\left(I+\frac{h^2}{12}\delta_y^2\right)^{-1}\right]+\mathcal{O}(h^4)I\\
&=\tilde{\delta}_x^4+\tilde{\delta}_y^4+2\delta_x^2\delta_y^2\left(I-\frac{h^2}{12}\delta_x^2-\frac{h^2}{12}\delta_y^2\right)+\mathcal{O}(h^4)I\\
&=\tilde{\delta}_x^4+\tilde{\delta}_y^4+2\delta_x^2\delta_y^2-\frac{h^2}{6}\left(\delta_x^2\right)^2\delta_y^2-\frac{h^2}{6}\delta_x^2\left(\delta_y^2\right)^2+\mathcal{O}(h^4)I\\
&=\tilde{\delta}_x^4\left(I-\frac{h^2}{6}\delta_y^2\right)+\tilde{\delta}_y^4\left(I-\frac{h^2}{12}\delta_x^2\right)+2\delta_x^2\delta_y^2+\mathcal{O}(h^4)I\\
&=\tilde{\Delta}_h^2+\mathcal{O}(h^4)I.
\end{aligned}$$

Using the compact scheme for steady problems and (2.4), we write the following fully discrete scheme for the problem (5.1):

Scheme II. Find $\{U_{ij}^n\} \in L^2_{h,0}$, $0 \le i,j \le M,\ 0 \le n \le N$ such that

$$
\begin{aligned}
&D^\alpha U_{ij}^n + \tilde{\Delta}_h^2 U_{ij}^n = f_{ij}^n, && 1 \le i,j \le M-1, \quad 1 \le n \le N,\\
&U_{ij}^0 = \omega_{ij}, && 1 \le i,j \le M-1,\\
&U_{ij}^n = 0, && i,j \in \{0,M\}.
\end{aligned}
\tag{5.4}
$$

Having determined the unknowns U_{ij}, we then derive $\{V_{ij}^n\}$ and $\{W_{ij}^n\}$ from the systems of equations

$$
\begin{aligned}
&\left(I + \frac{h^2}{6}\delta_x^2\right) V_{ij}^n = \Delta_x U_{ij}^n, \quad 1 \le i,j \le M-1,\\
&\left(I + \frac{h^2}{6}\delta_y^2\right) W_{ij}^n = \Delta_y U_{ij}^n, \quad 1 \le i,j \le M-1
\end{aligned}
$$

with the boundary condition $V_{ij}^n = W_{ij}^n = 0$ for $\{i,j\} \in \{0,M\}$.

5.2. Matrix form of Scheme II

The two-dimensional finite difference operators acting on the space $L^2_{h,0}$ can be associated with an $(M-1)\times(M-1)$, $M \ge 2$ matrix operators acting on vectors $u_{i,j} \in L^2_{h,0}$. We recall [10, 15] that the tensor (Kronecker) product $A \otimes B$ of the matrices $A \in M_{m,n}$ and $B \in M_{p,q}$ is a matrix in $M_{m\times p, n\times q}$ defined by

$$
A \otimes B = \begin{bmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,n}B \\ a_{2,1}B & a_{2,2}B & \cdots & a_{2,n}B \\ \vdots & \vdots & \cdots & \vdots \\ a_{m,1}B & a_{m,2}B & \cdots & a_{m,n}B \end{bmatrix}.
$$

Let us write $\{U_{i,j}\}$ for the column vector

$$
\mathbf{U} = \left[U_{1,1}, \cdots, U_{1,M-1}, U_{2,1}, \cdots, U_{2,M-1}, \cdots, U_{M-1,1}, \cdots, U_{M-1,M-1}\right]^\mathsf{T} \in R^{(M-1)^2}.
$$

Using the so defined matrices $\mathbf{T}, \mathbf{P}$ and vectors v_1, v_2, we represent the bidimensional Hermitian gradient as

$$
\mathbf{V} = \frac{3}{h}[\mathbf{I} \otimes \mathbf{P}^{-1}\mathbf{K}]\mathbf{U}, \quad \mathbf{W} = \frac{3}{h}[\mathbf{P}^{-1}\mathbf{K} \otimes \mathbf{I}]\mathbf{U}, \tag{5.5}
$$

the mixed derivative $\delta_x^2\delta_y^2$ as

$$
\delta_x^2\delta_y^2 = \frac{1}{h^4}\mathbf{T}\otimes\mathbf{T}
$$

and two fourth order difference operators in two dimensions as

$$
\begin{aligned}
\tilde{\delta}_x^4 &= \frac{12}{h^2}\mathbf{I} \otimes \left(\frac{3}{2h^2}\mathbf{K}\mathbf{P}^{-1}\mathbf{K} + \frac{1}{h^2}\mathbf{T}\right),\\
\tilde{\delta}_y^4 &= \frac{12}{h^2}\left(\frac{3}{2h^2}\mathbf{K}\mathbf{P}^{-1}\mathbf{K} + \frac{1}{h^2}\mathbf{T}\right) \otimes \mathbf{I}.
\end{aligned}
\tag{5.6}
$$

This differs from the notation used in [3] — cf. Remark 5.2 below. In addition, an equivalent representation of the operator $\mathbf{S} = 6(3\mathbf{K}\mathbf{P}^{-1}\mathbf{K} + 2\mathbf{T})/h^4$ is provided in Lemma 3.1. Thus the matrix form of the fourth order difference operator $\tilde{\Delta}_h^2$ in (5.3) is

$$\begin{aligned}\tilde{\Delta}_h^2 = &\frac{1}{h^4}\left[6\left(\mathbf{I}_{M-1} + \frac{1}{6}\mathbf{T}\right)\otimes \mathbf{P}^{-1}\mathbf{T}^2 + 6\mathbf{P}^{-1}\mathbf{T}^2 \otimes \left(\mathbf{I}_{M-1} + \frac{1}{6}\mathbf{T}\right) + 2\mathbf{T}\otimes\mathbf{T}\right] \\ &+ \frac{36}{h^4}\left(\mathbf{I}_{M-1} + \frac{1}{6}\mathbf{T}\right)\otimes[v_1\ v_2]\begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix} + \frac{36}{h^4}[v_1\ v_2]\begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}\otimes\left(\mathbf{I}_{M-1} + \frac{1}{6}\mathbf{T}\right). \end{aligned} \tag{5.7}$$

Consequently, with vectors $\{U_{ij}^n\} \in L_{h,0}^2$, Scheme II can be written as

$$\begin{aligned} &\mathbf{A}\mathbf{U}^1 = \frac{1}{\tau^\alpha\Gamma(2-\alpha)}\mathbf{U}^0 + \mathbf{F}^1, \\ &\mathbf{A}\mathbf{U}^n = \frac{1}{\tau^\alpha\Gamma(2-\alpha)}\sum_{l=1}^{n-1}(a_{n-l-1} - a_{n-l})\mathbf{U}^l + \frac{1}{\tau^\alpha\Gamma(2-\alpha)}a_{n-1}\mathbf{U}^0 + \mathbf{F}^n, \\ &\mathbf{U}^0 = \left(\omega_{1,1}^0, \cdots, \omega_{1,M-1}^0, \cdots, \omega_{M-1,1}^0, \cdots, \omega_{M-1,M-1}^0\right)^T, \end{aligned} \tag{5.8}$$

where

$$\mathbf{A} = \tilde{\Delta}_h^2 + \frac{1}{\tau^\alpha\Gamma(2-\alpha)}\mathbf{I},$$

and

$$\mathbf{F}^n = \left(f_{1,1}^n, \cdots, f_{1,M-1}^n, \cdots, f_{M-1,1}^n, \cdots, f_{M-1,M-1}^n\right)^T.$$

Similar to the one-dimensional case, we first derive $\{\mathbf{U}^n\}$ from (5.8) and then use (5.5) to determine $\{\mathbf{V}^n\}$ and $\{\mathbf{W}^n\}$.

Remark 5.2. The ordering of the column vector $\mathbf{U}$ is by rows first and then by columns, and the order of appearance of the matrices in the tensor products in (5.5)-(5.7) is different from those in [3]. Due to the symmetric form of the biharmonic operator (5.2), the fourth order difference operator $\tilde{\Delta}_h^2$ in (5.7) is unchanged — cf. [3, relation (67)]. Therefore, the final coefficient matrix A in (5.8) remains the same.

Analogously to Theorem 4.1, one can show that the compact finite difference scheme (5.4) converges in discrete L_2-norm as $\mathcal{O}(\tau^{2-\alpha}) + \mathcal{O}(h^4)$.

Theorem 5.1. *Let $u(x,y,t)$ and $\{U_{ij}^n\}$ be, respectively, the exact solution of the periodic problem* (5.1) *and the numerical solution obtained by Scheme II. Moreover, let $\mathbf{u}^n$ be defined on the mesh points similar to $\mathbf{U}^n$. If $u(x,y,t) \in C_{x,y,t}^{8,8,2}([0,L]^2 \times [0,T])$, then*

$$\|\mathbf{u}^n - \mathbf{U}^n\| \le C_1\tau^{2-\alpha} + C_2h^4, \tag{5.9}$$

where the constants C_1, C_2 depend on u, L, T, and α, but not on τ and h.

6. Numerical Experiments

In this section, we consider a few numerical examples to test the convergence of our schemes. In all examples we fix $T = 1$. Moreover, in the first two examples $x_L = 0$ and $x_R = 1$. In Example 6.3 we set $L = 1$. As was already mentioned, if $\{\mathbf{U}^n\}$ is known, then $\{\mathbf{V}^n\}$ can be also found. Two-dimensional problems are handled analogously — i.e. if $\{\mathbf{U}^n\}$ is known, then $\{\mathbf{V}^n\}$ and $\{\mathbf{W}^n\}$ can be easily computed, which is the advantage of the Stephenson scheme. In one-dimensional setting we estimate the errors using the following discrete L_2-norm and $W^{1,2}$, $W^{1,\infty}$ semi-norms:

$$\|\mathbf{e}^N\| = \|\mathbf{e}^N\|_{l^2} = \left(h\sum_{i=1}^{M-1}\left(e_i^N\right)^2\right)^{1/2},$$

$$|\tilde{\mathbf{e}}^N|_{1,2} = \left(h\sum_{i=1}^{M-1}\left(V_i^N - \frac{\partial u}{\partial x}(x_i,t_N)\right)^2\right)^{1/2},$$

$$|\tilde{\mathbf{e}}^N|_{1,\infty} = \max_{1\le i\le M-1}\left|V_i^N - \frac{\partial u}{\partial x}(x_i,t_N)\right|.$$

In two-dimensional problems, the corresponding terms are evaluated in similar way — viz.

$$\|\mathbf{e}^N\| = \|\mathbf{e}^N\|_{l^2} = \left(h^2\sum_{i=1}^{M-1}\sum_{j=1}^{M-1}\left(e_{ij}^N\right)^2\right)^{1/2},$$

$$|\tilde{\mathbf{e}}^N|_{1,2} = h^2\sum_{i=1}^{M-1}\sum_{j=1}^{M-1}\left[\left(V_{ij}^N - \frac{\partial u}{\partial x}(x_i,y_j,t_N)\right)^2 + \left(W_{ij}^N - \frac{\partial u}{\partial y}(x_i,y_j,t_N)\right)^2\right]^{1/2},$$

$$|\tilde{\mathbf{e}}^N|_{1,\infty} = \max_{1\le i,j\le M-1}\left\{\left|V_{ij}^N - \frac{\partial u}{\partial x}(x_i,y_j,t_N)\right|, \left|W_{ij}^N - \frac{\partial u}{\partial y}(x_i,y_j,t_N)\right|\right\}.$$

The approximate discrete $W^{1,2}$, $W^{1,\infty}$ semi-norms estimate the difference between the gradient of the exact solutions and the corresponding numerical approximations $\{\mathbf{V}^n\}$, $\{\mathbf{W}^n\}$. Recalling that $\|\mathbf{e}^N\|$ depends on τ and h, we write it as $\|\mathbf{e}(\tau,h,t^N)\|$ for clarity. Since N is the integer part $[1/\tau]$ of $1/\tau$, we have $N\tau \approx 1$. To test the numerical convergence, we will follow the approach of [9]. Noting the theoretical estimate (5.9), we can expect that the replacing τ by $\tau/2^{4/(2-\alpha)}$ and h by $h/2$, makes the numerical error estimate 16 times smaller since

$$\begin{aligned}\left\|\mathbf{e}\left(\tau/2^{4/(2-\alpha)}, h/2, t^{\tilde{N}}\right)\right\| &\approx C_1\left(\tau/2^{4/(2-\alpha)}\right)^{2-\alpha} + C_2(h/2)^4\\ &= \frac{1}{16}\left(C_1\tau^{2-\alpha} + C_2h^4\right) \approx \frac{1}{16}\left\|\mathbf{e}(\tau,h,t^N)\right\|\end{aligned}$$

for N and $\widetilde{N}$ such that $N\tau \approx \tilde{N}\tau/2^{4/(2-\alpha)} \approx 1$. In addition, one of the parameters τ or h can be chosen sufficiently small to ignore its influence on the error estimate. It allows to test spatial and temporal convergence rates separately by using the relation

$$\left\|\mathbf{e}(\tau,h,t^N)\right\| = \mathscr{O}\left(\tau^{2-\alpha}\right) + \mathscr{O}\left(h^4\right) \le C_1\tau^{2-\alpha} + C_2h^4.$$

Of course, such a test depends on the assumption that the lower and upper bounds for the ratio C_1/C_2 are known in advance, so that we can decide which parameter shall be small. Here we combine these two tests into one by an appropriate choice of mesh sizes. Consequently, in the numerical tests the experimental convergence order $r = r(\tau,h)$ is calculated as

$$r(\tau,h)=\log_2\Big(\left\|\mathbf{e}\left(\tau,h,t^{N_1}\right)\right\|_*/\left\|\mathbf{e}\left(\tau/2^{4/(2-\alpha)},h/2,t^{N_2}\right)\right\|_*\Big),$$

where $N_1 = [1/\tau]$ and $N_2 = [2^{4/(2-\alpha)}/\tau]$. As already mentioned, we expect that $r = r(\tau,h) \approx \log_2 16 = 4$. Let us also note that in the numerical simulations, we consider the cases $\alpha = 0.25$, $\alpha = 0.5$ and $\alpha = 0.75$.

Example 6.1 (Homogeneous boundary conditions)**.** We consider the problem (2.2), which has the solution $u(x,t) = t^3\sin^2(\pi x)$. In this case,

$$u(0,t)=\frac{\partial u}{\partial x}(1,t)=0,$$

and

$$f(x,t)=\frac{6t^{3-\alpha}}{\Gamma(4-\alpha)}\sin^2(\pi x)-8\pi^4t^3\cos(2\pi x).$$

The errors of the compact scheme and the experimental convergence order are shown in Tables 1-3. We note that they are consistent with theoretical results.

Example 6.2 (Nonhomogeneous boundary conditions)**.** We consider problem (1.2)-(1.3), which has the solution $v(x,t) = t^3e^x$. In this case,

$$g(x,t)=(\frac{6t^{3-\alpha}}{\Gamma(4-\alpha)}+t^3)e^x,$$

and the Hermite interpolation function has the form

$$H(x,t)=[(1+3x)(x-1)^2+(2-x)x^2e]t^3.$$

The corresponding errors of the compact scheme are presented in Tables 4-6 and the accuracy meets our expectations.

Example 6.3 (Two-dimensional problem. Homogeneous boundary conditions)**.** We consider the problem (5.1), which has the solution $u(x,y,t) = t^3\sin^2(\pi x)\sin^2(\pi y)$. In this case,

$$\begin{aligned}f(x,y,t)=&\frac{3t^{3-\alpha}}{2\Gamma(4-\alpha)}(1-\cos(2\pi x))(1-\cos(2\pi y))\\&+4\pi^4t^3(4\cos(2\pi x)\cos(2\pi y)-\cos(2\pi x)-\cos(2\pi y)).\end{aligned}$$

The corresponding errors of the compact scheme are presented in Tables 7-9 and they are consistent with the theoretical estimates.

Table 1: Error and experiment order of convergence of Scheme I for $\alpha = 0.25$ (Example 6.1).

M	N	$\|\mathbf{e}^N\|_\infty$	order	$\|\mathbf{e}^N\|$	order	$\|\tilde{\mathbf{e}}^N\|_{1,\infty}$	order	$\|\tilde{\mathbf{e}}^N\|_{1,2}$	order
5	5	0.0045	-	0.0031	-	0.0351	-	0.0261	-
10	24	2.3045e-4	4.2874	1.4118e-4	4.4567	0.0019	4.2074	0.0014	4.2206
20	119	1.4176e-5	4.0229	8.6857e-6	4.0227	1.2804e-4	3.8913	9.0523e-5	3.9510
40	580	8.7137e-7	4.0240	5.3391e-7	4.0240	7.9399e-6	4.0113	5.6135e-6	4.0113
80	2826	5.4190e-8	4.0072	3.3205e-8	4.0113	4.9415e-7	4.0061	3.4935e-7	4.0062

Table 2: Error and experiment order of convergence of Scheme I for $\alpha = 0.5$ (Example 6.1).

M	N	$\|\mathbf{e}^N\|_\infty$	order	$\|\mathbf{e}^N\|$	order	$\|\tilde{\mathbf{e}}^N\|_{1,\infty}$	order	$\|\tilde{\mathbf{e}}^N\|_{1,2}$	order
5	5	0.0047	-	0.0032	-	0.0347	-	0.0258	-
10	32	2.5662e-4	4.1950	1.5736e-4	4.3459	0.0020	4.1169	0.0015	4.1043
20	202	1.4814e-5	4.1146	9.0862e-6	4.1142	1.2662e-4	3.9814	8.9497e-5	4.0670
40	1280	9.0650e-7	4.0305	5.5603e-7	4.0304	7.8144e-6	4.0182	5.5231e-6	4.0183
80	8127	5.6431e-8	4.0057	3.4614e-8	4.0057	4.8753e-7	4.0026	3.4457e-7	4.0026

Table 3: Error and experiment order of convergence of Scheme I for $\alpha = 0.75$ (Example 6.1).

M	N	$\|\mathbf{e}^N\|_\infty$	order	$\|\mathbf{e}^N\|$	order	$\|\tilde{\mathbf{e}}^N\|_{1,\infty}$	order	$\|\tilde{\mathbf{e}}^N\|_{1,2}$	order
5	5	0.0049	-	0.0033	-	0.0339	-	0.0252	-
10	46	2.7107e-4	4.1760	1.6653e-4	4.3086	0.0019	4.1572	0.0014	4.1699
20	422	1.5904e-5	4.0912	9.7732e-6	4.0908	1.2196e-4	3.9615	8.6144e-5	4.0225
40	3880	9.8077e-7	4.0193	6.0272e-7	4.0193	7.5862e-6	4.0069	5.3584e-6	4.0069
80	35658	6.1088e-8	4.0050	3.7542e-8	4.0049	4.7347e-7	4.0020	3.3442e-7	4.0021

Table 4: Error and experiment order of convergence of Scheme I for $\alpha = 0.25$ (Example 6.2).

M	N	$\|\mathbf{e}^N\|_\infty$	order	$\|\mathbf{e}^N\|$	order	$\|\tilde{\mathbf{e}}^N\|_{1,\infty}$	order	$\|\tilde{\mathbf{e}}^N\|_{1,2}$	order
5	5	2.6849e-6	-	1.6494e-6	-	2.2274e-5	-	1.6646e-5	-
10	24	1.6341e-7	4.0383	9.7432e-8	4.0814	1.3638e-6	4.0297	9.9709e-7	4.0613
20	119	1.0842e-8	3.9138	6.3566e-9	3.9381	8.9968e-8	3.9221	6.5600e-8	3.9260
40	580	6.7303e-10	4.0098	3.9382e-10	4.0126	5.6259e-9	3.9993	4.0963e-9	4.0013
80	2826	4.1730e-11	4.0115	2.4366e-11	4.0146	3.5128e-10	4.0014	2.5551e-10	4.0029

Table 5: Error and experiment order of convergence of Scheme I for $\alpha = 0.5$ (Example 6.2).

M	N	$\|\mathbf{e}^N\|_\infty$	order	$\|\mathbf{e}^N\|$	order	$\|\tilde{\mathbf{e}}^N\|_{1,\infty}$	order	$\|\tilde{\mathbf{e}}^N\|_{1,2}$	order
5	5	2.4385e-6	-	1.3914e-6	-	2.3271e-5	-	1.6740e-5	-
10	32	1.5356e-7	3.9891	8.7746e-8	3.9871	1.5077e-6	3.9481	1.0782e-6	3.9566
20	202	9.3615e-9	4.0359	5.3191e-9	4.0441	9.3060e-8	4.0180	6.6349e-8	4.0224
40	1280	5.8366e-10	4.0035	3.2869e-10	4.0164	5.7912e-9	4.0062	4.1241e-9	4.0079
80	8127	3.6398e-11	4.0032	2.0487e-11	4.0039	3.6238e-10	3.9983	2.5778e-10	3.9999

Table 6: Error and experiment order of convergence of Scheme I for $\alpha = 0.75$ (Example 6.2).

M	N	$\|\mathbf{e}^N\|_\infty$	order	$\|\mathbf{e}^N\|$	order	$\lvert\tilde{\mathbf{e}}^N\rvert_{1,\infty}$	order	$\lvert\tilde{\mathbf{e}}^N\rvert_{1,2}$	order
5	5	1.9062e-6	-	1.1404e-6	-	2.5427e-5	-	1.7309e-5	-
10	46	1.1486e-7	4.0528	7.4062e-8	3.9447	1.6283e-6	3.9649	1.0998e-6	3.9762
20	422	7.1276e-9	4.0103	4.6332e-9	3.9987	1.0326e-7	3.9790	6.8573e-8	4.0035
40	3880	4.5173e-10	3.9799	2.9021e-10	3.9968	6.4670e-9	3.9970	4.2940e-9	3.9972
80	35658	2.8208e-11	4.0013	1.8147e-11	3.9993	4.0454e-10	3.9987	2.6849e-10	3.9994

Table 7: Error and experiment order of convergence of Scheme II for $\alpha = 0.25$ (Example 6.3).

M	N	$\|\mathbf{e}^N\|_\infty$	order	$\|\mathbf{e}^N\|$	order	$\lvert\tilde{\mathbf{e}}^N\rvert_{1,\infty}$	order	$\lvert\tilde{\mathbf{e}}^N\rvert_{1,2}$	order
5	5	0.0041	-	0.0019	-	0.0319	-	0.0227	-
10	24	2.2801e-4	4.1685	8.5543e-5	4.4732	0.0019	4.0695	0.0012	4.2416
20	119	1.3997e-5	4.0259	5.2517e-6	4.0258	1.2859e-4	3.8851	7.8731e-5	3.9300
40	580	8.5916e-7	4.0260	3.2238e-7	4.0260	7.9775e-6	4.0107	4.8842e-6	4.0107

Table 8: Error and experiment order of convergence of Scheme II for $\alpha = 0.5$ (Example 6.3).

M	N	$\|\mathbf{e}^N\|_\infty$	order	$\|\mathbf{e}^N\|$	order	$\lvert\tilde{\mathbf{e}}^N\rvert_{1,\infty}$	order	$\lvert\tilde{\mathbf{e}}^N\rvert_{1,2}$	order
5	5	0.0041	-	0.0019	-	0.0317	-	0.0226	-
10	32	2.4817e-4	4.0462	9.3199e-5	4.3495	0.0020	3.9864	0.0013	4.1197
20	202	1.4259e-5	4.1214	5.3562e-6	4.1210	1.2833e-4	3.9621	7.8544e-5	4.0489
40	1280	8.7118e-7	4.0328	3.2727e-7	4.0327	7.9231e-6	4.0176	4.8491e-6	4.0177

Table 9: Error and experiment order of convergence of Scheme II for $\alpha = 0.75$ (Example 6.3).

M	N	$\|\mathbf{e}^N\|_\infty$	order	$\|\mathbf{e}^N\|$	order	$\lvert\tilde{\mathbf{e}}^N\rvert_{1,\infty}$	order	$\lvert\tilde{\mathbf{e}}^N\rvert_{1,2}$	order
5	5	0.0042	-	0.0019	-	0.0314	-	0.0224	-
10	46	2.5040e-4	4.0681	9.4236e-5	4.3336	0.0020	3.9727	0.0013	4.1069
20	422	1.4592e-5	4.1010	5.4934e-6	4.1005	1.2600e-4	3.9885	7.7055e-5	4.0765
40	3880	8.9846e-7	4.0216	3.3826e-7	4.0215	7.8396e-6	4.0065	4.7943e-6	4.0065

7. Conclusion

We studied the convergence of a compact finite difference scheme for one- and two-dimensional time fractional fourth order equations with the first Dirichlet boundary conditions. In one-dimensional case, we used a Hermitian interpolation function to transform it into a problem with homogeneous boundary conditions. The Stephenson scheme is used for spatial derivatives discretisation. As a by-product of the method, the approximate values of normal derivatives are obtained. For periodic problems, the scheme is proved to be stable and convergent with the accuracy $\mathcal{O}(\tau^{2-\alpha}) + \mathcal{O}(h^4)$. In two-dimensional problems, the error estimates are similar. The results of numerical experiments are consistent with the theoretical analysis.

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