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Numerical Method for the Time Fractional Fokker-Planck Equation

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Abstract. In this paper, a new numerical algorithm for solving the time fractional Fokker-Planck equation is proposed. The analysis of local truncation error and the stability of this method are investigated. Theoretical analysis and numerical experiments show that the proposed method has higher order of accuracy for solving the time fractional Fokker-Planck equation.

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1 Introduction

The Fokker-Planck equation (FPE) is a widely used equation in statistical physics and describes the time evolution of a test particle under the influence of an external force field. The solution of a FPE is the probability of the particle at a certain position at a given time. It has been observed (cf. [3]) that in presence of an highly non-homogeneous medium the anomalous diffusion is not adequately described by the conventional FPE and models based on fractional derivational were proposed. Fractional Fokker-Planck equation (FFPE) can arise in which the temporal derivative and/or spatial derivative operators are fractional, see, e.g., [1, 2, 9–12, 14–16, 25–27]), but in this paper, we only consider fractional derivative operators with respect to time.

Consider the following time fractional Fokker-Planck equation (FFPE)

$$\frac{\partial}{\partial t}P(x,t) = {}_0 D_t^{1-\alpha}\left[\frac{\partial}{\partial x}\frac{U'(x)}{\eta_\alpha} + \kappa_\alpha\frac{\partial^2}{\partial x^2}\right]P(x,t), \tag{1.1}$$

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where $P(x,t)$ denotes the probability density, $U(x)$ indicates the potential of over-damped Brownian motion, κ_α denotes the anomalous diffusion coefficient, η_α represents the generalized friction coefficient, ${}_0D_t^{1-\alpha}P(x,t)$ stands for the Riemann-Liouville fractional derivative of order $1-\alpha$, which is defined by

$$ {}_0D_t^{1-\alpha}P(x,t)=\frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t\frac{P(x,\tau)}{(t-\tau)^{1-\alpha}}d\tau. $$

Some works have been done on developing numerical methods for the time FFPE (1.1) in the literature. Deng (cf. [6]) combined the Predictor-Corrector approach with the method of line, and presented a numerical algorithm for solving FFPE with the numerical error $\mathcal{O}(h^{\min(1+2\alpha,2)})+\mathcal{O}(\tau^2)$ (where and later h is time stepsize, τ is spatial stepsize), and got the corresponding stability condition. Chen et al. (cf. [5]) considered the Grünwald-Letnikov expansion and the L_1-approximation for fractional time derivative, while the first-order spatial derivative is approximated by the backward Euler implicit scheme, or the central or backward difference implicit scheme, then three numerical schemes were given. The local truncation errors $\mathcal{O}(h+\tau)$, $\mathcal{O}(h^{2-\alpha}+\tau^2)$ and $\mathcal{O}(h^{2-\alpha}+\tau)$ were obtained respectively. The objective of this paper is to achieve the efficient numerical methods with higher accuracy.

The outline of this paper is as follows. In Section 2, the numerical method for FFPE is given. Then, in Sections 3 and 4, the analysis of local truncation error and stability of this numerical method are obtained. Section 5 is used to present numerical results, comparing the fixed stepsize implementation of several methods on FFPE problem. The advantages of the new numerical method is readily apparent in computational accuracy and efficiency.

2 Numerical method

In this section, we introduce new numerical scheme for solving the FFPE

$$ \frac{\partial}{\partial t}P(x,t)={}_0D_t^{1-\alpha}\left[\frac{\partial}{\partial x}\frac{U'(x)}{\eta_\alpha}+\kappa_\alpha\frac{\partial^2}{\partial x^2}\right]P(x,t) \tag{2.1} $$

with the initial condition and the boundary conditions

$$ P(x,0)=\psi(x), \tag{2.2} $$
$$ P(c,t)=\varphi_1(t),\qquad P(d,t)=\varphi_2(t), \tag{2.3} $$

where $x\in[c,d], t\in[0,T]$, $0<\alpha<1$, $\psi(x)$, $\varphi_1(t)$ and $\varphi_2(t)$ are given function.

Eq. (2.1) can be written as

$$ \frac{\partial}{\partial t}P(x,t)={}_0D_t^{1-\alpha}\left[\frac{U''(x)}{\eta_\alpha}P(x,t)+\frac{U'(x)}{\eta_\alpha}\frac{\partial}{\partial x}P(x,t)+\kappa_\alpha\frac{\partial^2}{\partial x^2}P(x,t)\right]. \tag{2.4} $$

For a numerical calculation of $P(x,t)$, we first make a spatial and temporal discretization. For the space interval $[c,d]$ and time interval $[0,T]$, we choose the grid point as follow

$$\begin{cases} x_j = c + j\tau, & j = 0,1,\cdots,M, \\ t_k = kh, & k = 0,1,\cdots,N, \end{cases}$$

where $\tau = \frac{d-c}{M}$ denotes spatial stepsize, $h = \frac{T}{N}$ denotes time stepsize.

Using the second order central difference scheme to discretize the spatial partial derivatives of $P(x,t)$ at point x_j, we obtain

$$\begin{aligned} \frac{\partial}{\partial x}P(x_j,t) &= \frac{P(x_{j+1},t) - P(x_{j-1},t)}{2\tau} + R_{11}^{[j]}, \\ \frac{\partial^2}{\partial x^2}P(x_j,t) &= \frac{P(x_{j+1},t) - 2P(x_j,t) + P(x_{j-1},t)}{\tau^2} + R_{12}^{[j]}, \end{aligned}$$

where

$$\begin{aligned} &R_{11}^{[j]} = -\frac{\tau^2}{6}\frac{\partial^3}{\partial x^3}P(\xi_j,t), \qquad R_{12}^{[j]} = -\frac{\tau^2}{12}\frac{\partial^4}{\partial x^4}P(\eta_j,t), \\ &\xi_j \in (x_{j-1},x_{j+1}), \qquad \eta_j \in (x_{j-1},x_{j+1}), \qquad j = 1,2,\cdots,M-1. \end{aligned}$$

Then Eq. (2.4) can be written into the following semidiscrete form

$$\begin{aligned} \frac{d}{dt}P_j(t) = {}_0D_t^{1-\alpha}\Bigg[&\Big(\frac{\kappa_\alpha}{\tau^2} - \frac{U'(x_j)}{2\tau\eta_\alpha}\Big)P_{j-1}(t) + \Big(\frac{U''(x_j)}{\eta_\alpha} - \frac{2\kappa_\alpha}{\tau^2}\Big)P_j(t) \\ &+ \Big(\frac{U'(x_j)}{2\tau\eta_\alpha} + \frac{\kappa_\alpha}{\tau^2}\Big)P_{j+1}(t)\Bigg], \qquad j = 1,2,\cdots,M-1, \end{aligned} \tag{2.5}$$

where $P_j(t) \equiv P(x_j,t)$. Let

$$\hat{P}(t) = (P_1(t),P_2(t),\cdots,P_{M-1}(t))^T,$$

$$A = \begin{pmatrix} \frac{U_1''}{\eta_\alpha} - \frac{2\kappa_\alpha}{\tau^2} & \frac{U_1'}{2\tau\eta_\alpha} + \frac{\kappa_\alpha}{\tau^2} & & & \\ \frac{\kappa_\alpha}{\tau^2} - \frac{U_2'}{2\tau\eta_\alpha} & \frac{U_2''}{\eta_\alpha} - \frac{2\kappa_\alpha}{\tau^2} & \frac{U_2'}{2\tau\eta_\alpha} + \frac{\kappa_\alpha}{\tau^2} & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \frac{U_{M-2}'}{2\tau\eta_\alpha} + \frac{\kappa_\alpha}{\tau^2} \\ & & & \frac{\kappa_\alpha}{\tau^2} - \frac{U_{M-1}'}{2\tau\eta_\alpha} & \frac{U_{M-1}''}{\eta_\alpha} - \frac{2\kappa_\alpha}{\tau^2} \end{pmatrix},$$

$$g(t) = \Bigg(\Big(\frac{\kappa_\alpha}{\tau^2} - \frac{U_1'}{2\tau\eta_\alpha}\Big)\varphi_1(t),\ 0,\cdots,0,\ \Big(\frac{U_{M-1}'}{2\tau\eta_\alpha} + \frac{\kappa_\alpha}{\tau^2}\Big)\varphi_2(t)\Bigg)^T,$$

where $U_j' = U'(x_j)$, $U_j'' = U''(x_j)$, $j = 1, 2, \cdots, M-1$.

Then Eq. (2.5) can be rewritten as

$$\frac{d}{dt}\hat{P}(t) = {}_0D_t^{1-\alpha}(A\hat{P}(t) + g(t)), \tag{2.6}$$

with the initial condition is set as

$$\hat{P}(0) = (\psi(x_1), \psi(x_2), \cdots, \psi(x_{M-1}))^T. \tag{2.7}$$

So the numerical algorithm for solving the FFPE (2.1), (2.2) and (2.3) is transformed into to construct the numerical algorithm for solving the FODE (2.6) and (2.7).

Now consider the following fractional ordinary differential equations

$$\begin{cases} \dfrac{d}{dt}\hat{P}(t) = {}_0D_t^{1-\alpha} f(t, \hat{P}(t)), \\ \hat{P}(0) = \eta, \end{cases} \tag{2.8}$$

where $f(t, \hat{P}(t)) = A\hat{P}(t) + g(t)$, $\eta = (\psi(x_1), \psi(x_2), \cdots, \psi(x_{M-1}))^T$.

Based on the property of composition of Riemann-Liouville fractional derivative with integer-order derivative, that is

$${}_0D_t^{1-\alpha} f(t, \hat{P}(t)) = \frac{d}{dt}\left({}_0D_t^{-\alpha} f(t, \hat{P}(t))\right).$$

Integrating both sides of first equation in (2.8) from t_k to t_{k+1}, we get

$$\begin{aligned} &\hat{P}(t_{k+1}) - \hat{P}(t_k) \\ =&{}_0D_{t_{k+1}}^{-\alpha} f(t_{k+1}, \hat{P}(t_{k+1})) - {}_0D_{t_k}^{-\alpha} f(t_k, \hat{P}(t_k)) \\ =&\frac{1}{\Gamma(\alpha)}\int_0^{t_{k+1}} (t_{k+1}-\theta)^{\alpha-1} f(\theta, \hat{P}(\theta))d\theta - \frac{1}{\Gamma(\alpha)}\int_0^{t_k} (t_k-\theta)^{\alpha-1} f(\theta, \hat{P}(\theta))d\theta \\ =&\frac{1}{\Gamma(\alpha)}\sum_{m=0}^{k}\int_{t_m}^{t_{m+1}} (t_{k+1}-\theta)^{\alpha-1} f(\theta, \hat{P}(\theta))d\theta - \frac{1}{\Gamma(\alpha)}\sum_{m=0}^{k-1}\int_{t_m}^{t_{m+1}} (t_k-\theta)^{\alpha-1} f(\theta, \hat{P}(\theta))d\theta. \end{aligned}$$

By the linear interpolation for $f(\theta, \hat{P}(\theta))$ at integral interval $[t_m, t_{m+1}]$, we obtain

$$\begin{aligned} {}_0D_{t_{k+1}}^{-\alpha} f(t_{k+1}, \hat{P}(t_{k+1})) &= \frac{1}{\Gamma(\alpha)}\sum_{m=0}^{k}\int_{t_m}^{t_{m+1}} (t_{k+1}-\theta)^{\alpha-1} f(\theta, \hat{P}(\theta))d\theta \\ &= \frac{1}{\Gamma(\alpha)}\sum_{m=0}^{k}\int_{t_m}^{t_{m+1}} (t_{k+1}-\theta)^{\alpha-1}\Big(\frac{\theta - t_{m+1}}{t_m - t_{m+1}} f(t_m, \hat{P}(t_m)) \\ &\quad + \frac{\theta - t_m}{t_{m+1} - t_m} f(t_{m+1}, \hat{P}(t_{m+1}))\Big)d\theta + R_{k+1}^{\alpha} \\ &= \frac{h^{\alpha}}{\Gamma(\alpha+2)}\sum_{m=0}^{k+1} \tilde{a}_{m,k+1} f(t_m, \hat{P}(t_m)) + R_{k+1}^{\alpha}, \end{aligned}$$

where

$$\tilde{a}_{m,k+1}=\begin{cases}(k+1)^{\alpha}(\alpha-k)+k^{\alpha+1}, & m=0,\\ (k-m)^{\alpha+1}-2(k+1-m)^{\alpha+1}+(k+2-m)^{\alpha+1}, & 1\le m\le k,\\ 1, & m=k+1,\end{cases} \tag{2.9}$$

$$R^{\alpha}_{k+1}=\frac{1}{\Gamma(\alpha)}\sum_{m=0}^{k}\int_{t_m}^{t_{m+1}}\frac{f''(\xi_m,\hat{P}(\xi_m))(\theta-t_m)(\theta-t_{m+1})}{2(t_{k+1}-\theta)^{1-\alpha}}d\theta,\quad \xi_m\in[t_m,t_{m+1}]. \tag{2.10}$$

Then we yield

$$\begin{aligned}&\hat{P}(t_{k+1})-\hat{P}(t_k)\\ =&\frac{h^{\alpha}}{\Gamma(\alpha+2)}\sum_{m=0}^{k+1}b_{m,k+1}f(t_m,\hat{P}(t_m))+R^{(1)}_{k+1},\qquad k=0,1,\cdots,N-1,\end{aligned} \tag{2.11}$$

where $b_{0,1}=\alpha,\ b_{1,1}=1$, while $k>0$ are

$$b_{m,k+1}=\begin{cases}(k+1)^{\alpha}(\alpha-k)+k^{\alpha}(2k-\alpha-1)-(k-1)^{\alpha+1}, & m=0,\\ (k+2-m)^{\alpha+1}-3(k+1-m)^{\alpha+1} & \\ \quad +3(k-m)^{\alpha+1}-(k-1-m)^{\alpha+1}, & 1\le m\le k-1,\\ 2^{\alpha+1}-3, & m=k,\\ 1, & m=k+1,\end{cases} \tag{2.12}$$

$$R^{(1)}_{k+1}=\begin{cases}R^{\alpha}_1, & k=0,\\ R^{\alpha}_{k+1}-R^{\alpha}_k, & k>0.\end{cases} \tag{2.13}$$

It follows from (2.11) and (2.12) that the numerical formula for solving the FODE (2.8) is devised as follow

$$\hat{P}_{k+1}-\hat{P}_k=\frac{h^{\alpha}}{\Gamma(\alpha+2)}\sum_{m=0}^{k+1}b_{m,k+1}f(t_m,\hat{P}_m),\qquad k=0,1,\cdots,N-1, \tag{2.14}$$

where $\hat{P}_k$ denotes the approximation of $\hat{P}(t_k)$.

Applying (2.14) to solve the FFPE (2.1), (2.2) and (2.3), the fully discrete scheme is as follow

$$P^{k+1}_j-P^k_j=\sum_{m=0}^{k+1}b_{m,k+1}(K_jP^m_{j-1}+W_jP^m_j+V_iP^m_{j+1}), \tag{2.15}$$

where P^k_j denotes the approximation of solution $P(x_j,t_k)$, $j=1,2,\cdots,M-1$, $k=0,1,\cdots,N-1$,

$$\begin{aligned}&K_j=\frac{h^{\alpha}}{\Gamma(\alpha+2)}\left(\frac{\kappa_{\alpha}}{\tau^2}-\frac{U'(x_j)}{2\tau\eta_{\alpha}}\right),\qquad W_j=\frac{h^{\alpha}}{\Gamma(\alpha+2)}\left(\frac{U''(x_j)}{\eta_{\alpha}}-\frac{2\kappa_{\alpha}}{\tau^2}\right),\\ &V_j=\frac{h^{\alpha}}{\Gamma(\alpha+2)}\left(\frac{U'(x_j)}{2\tau\eta_{\alpha}}+\frac{\kappa_{\alpha}}{\tau^2}\right).\end{aligned}$$

3 Error analysis

In order to obtain the error analysis, we first introduce some lemmas as follow.

Lemma 3.1. *If $y(t) \in C^2[0,T]$, and*

$$\begin{aligned}
{}_0D_{t_k}^{-\alpha} y(t_k) &= \frac{1}{\Gamma(\alpha)} \sum_{m=0}^{k-1} \int_{t_m}^{t_{m+1}} (t_k-\theta)^{\alpha-1} y(\theta) d\theta \\
&= \frac{1}{\Gamma(\alpha)} \sum_{m=0}^{k-1} \int_{t_m}^{t_{m+1}} \frac{(t_{m+1}-\theta)y(t_m) + (\theta - t_m) y(t_{m+1})}{h(t_k-\theta)^{1-\alpha}} d\theta + E_k \\
&= \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{m=0}^{k} \tilde{a}_{m,k} y(t_m) + E_k, \qquad (3.1)
\end{aligned}$$

then

$$E_k = \frac{1}{\Gamma(\alpha)} \sum_{m=0}^{k-1} \int_{t_m}^{t_{m+1}} \frac{y''(\xi_m)(\theta - t_m)(\theta - t_{m+1})}{2(t_k-\theta)^{1-\alpha}} d\theta,$$

which satisfies

$$| E_k | \le Ch^2,$$

where $\tilde{a}_{m,k}$ are defined by (2.9), C is a constant which depend only on parameter and bound for certain derivatives of $y(t)$.

Lemma 3.2. *IF $y(t) \in C^2[0,T]$, and*

$$\begin{aligned}
&{}_0D_{t_{k+1}}^{-\alpha} y(t_{k+1}) - {}_0D_{t_k}^{-\alpha} y(t_k) \\
=& \frac{1}{\Gamma(\alpha)} \sum_{m=0}^{k} \int_{t_m}^{t_{m+1}} (t_{k+1}-\theta)^{\alpha-1} y(\theta) d\theta - \frac{1}{\Gamma(\alpha)} \sum_{m=0}^{k-1} \int_{t_m}^{t_{m+1}} (t_k-\theta)^{\alpha-1} y(\theta) d\theta \\
=& \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{m=0}^{k+1} b_{m,k+1} y(t_m) + E_{k+1}^{(1)}, \qquad (3.2)
\end{aligned}$$

then $E_{k+1}^{(1)} = E_{k+1} - E_k$, which satisfies

$$| E_{k+1}^{(1)} | \le C_1 h^2,$$

where $b_{m,k+1}$ be defined by (2.12), C_1 is a constant which depend only on parameter and bound for certain derivatives of $y(t)$.

Theorem 3.1. *If the numerical method (2.15) is used to solve the time fractional Fokker-Planck equation (2.1), (2.2), (2.3), then the local truncation error is $\mathcal{O}(\tau^2 + h^2)$.*

Proof. Substituting the exact solution $p(x_j,t_k)$ for P_j^k in (2.15), we obtain

$$
\begin{aligned}
&P(x_j,t_{k+1})-P(x_j,t_k)\\
=&\frac{h^\alpha}{\Gamma(\alpha+2)}\sum_{m=0}^{k+1}b_{m,k+1}\Big[\Big(\frac{\kappa_\alpha}{\tau^2}-\frac{U'(x_j)}{2\tau\eta_\alpha}\Big)P(x_{j-1},t_m)+\Big(\frac{U''(x_j)}{\eta_\alpha}-\frac{2\kappa_\alpha}{\tau^2}\Big)P(x_j,t_m)\\
&+\Big(\frac{U'(x_j)}{2\tau\eta_\alpha}+\frac{\kappa_\alpha}{\tau^2}\Big)P(x_{j+1},t_m)\Big]+R_j^{k+1},
\end{aligned}
\tag{3.3}
$$

for $j=1,2,\cdots,M-1$, $k=0,1,\cdots,N-1$, where

$$
R_j^{k+1}=R_1+R^{(1)}_{k+1,j},\qquad R_1=({}_0D_{t_{k+1}}^{-\alpha}-{}_0D_{t_k}^{-\alpha})\Big(\frac{U'(x_j)}{\eta_\alpha}R_{11}^{[j]}+\kappa_\alpha R_{12}^{[j]}\Big),
$$

$R^{(1)}_{k+1,j}$ denotes the j component of $R^{(1)}_{k+1}$.

Suppose that $U'(x_j)$ $(j=1,2,\cdots,M-1)$ are bounded, then

$$
\Big|\frac{U'(x_j)}{\eta_\alpha}R_{11}^{[j]}+\kappa_\alpha R_{12}^{[j]}\Big|=\Big|-\frac{U'(x_j)}{\eta_\alpha}\frac{\tau^2}{6}\frac{\partial^3}{\partial x^3}P(\xi_j,t)-\kappa_\alpha\frac{\tau^2}{12}\frac{\partial^4}{\partial x^4}P(\eta_j,t)\Big|\le C_2\tau^2.
$$

Consequently,

$$
\begin{aligned}
|\,R_1\,|=&\Big|({}_0D_{t_{k+1}}^{-\alpha}-{}_0D_{t_k}^{-\alpha})\Big(\frac{U'(x_j)}{\eta_\alpha}R_{11}^{[j]}+\kappa_\alpha R_{12}^{[j]}\Big)\Big|\\
\le& C_2\tau^2(|{}_0D_{t_{k+1}}^{-\alpha}1\,|+|{}_0D_{t_k}^{-\alpha}1\,|)\\
\le& C_2\tau^2\Big(\frac{h^\alpha}{\Gamma(\alpha+1)}((k+1)^\alpha+(k)^\alpha)\Big)\\
\le& C_2\frac{2T^\alpha}{\Gamma(\alpha+1)}\tau^2 \;=\; C_3\tau^2.
\end{aligned}
\tag{3.4}
$$

According to Lemma 3.2 and $f(t,\hat{P}(t))=A\hat{P}(t)+g(t)$, we conclude that

$$
\begin{aligned}
|\,R^{(1)}_{k+1,j}\,|=&\Big|\sum_{m=1}^{M-1}a_{jm}E^{(1)'}_{k+1,m}+E^{(1)''}_{k+1,j}\Big|\\
\le&\|\,A\,\|_\infty\cdot\max_{1\le m\le M-1}|\,E^{(1)'}_{k+1,m}\,|+|\,E^{(1)''}_{k+1,j}\,|\\
\le&(\|\,A\,\|_\infty+1)C_1h^2\;=\;C_4h^2,
\end{aligned}
\tag{3.5}
$$

where $E^{(1)'}_{k+1,m}$, $E^{(1)''}_{k+1,j}$ denote the error of function $P_m(t)$ and $g_j(t)$ which defined by (3.2) respectively.

From (3.4) and (3.5), we get

$$
\begin{aligned}
|\,R_j^{k+1}\,|=&|\,R_1+R^{(1)}_{k+1,j}\,|\le|\,R_1\,|+|\,R^{(1)}_{k+1,j}\,|\\
\le& C_3\tau^2+C_4h^2\;\le\;C_5(\tau^2+h^2),
\end{aligned}
\tag{3.6}
$$

where $C_5=\max(C_3,C_4)$.

4 Stability analysis

In this section, we given the stability analysis using the approach of Fourier analysis.

Let P_j^k and $\hat{P}_j^k$ be two parallel approximation solution sequences for the implicit numerical method (2.15), and define that

$$\varepsilon_j^k = P_j^k - \hat{P}_j^k, \quad j = 0, 1, \cdots, M, \quad k = 0, 1, \cdots, N,$$

and

$$E^k = (\varepsilon_1^k, \varepsilon_2^k, \cdots, \varepsilon_{M-1}^k)^T,$$

respectively. Then we obtain the following roundoff error equation

$$\varepsilon_j^{k+1} - \varepsilon_j^k = \sum_{m=0}^{k+1} b_{m,k+1}(K_j \varepsilon_{j-1}^m + W_j \varepsilon_j^m + V_j \varepsilon_{j+1}^m), \tag{4.1}$$

where $j = 1, 2, \cdots, M-1$, $k = 0, 1, 2, \cdots, N-1$, $\varepsilon_0^{k+1} = \varepsilon_M^{k+1} = 0$.

For $k = 0, 1, \cdots, N$, we define the following grid function $\varepsilon^k(x)$

$$\varepsilon^k(x) = \begin{cases} \varepsilon_j^k, & x_j - \frac{\tau}{2} \le x \le x_j + \frac{\tau}{2}, \ j = 1, 2, \cdots, M-1, \\ 0, & c \le x \le c + \frac{\tau}{2}, \ d - \frac{\tau}{2} \le x \le d. \end{cases} \tag{4.2}$$

Then $\varepsilon^k(x)$ has the Fourier series expansion

$$\varepsilon^k(x) = \sum_{l=-\infty}^{+\infty} \nu_k(l) e^{i2\pi l x/Q},$$

where $Q = d - c$ is the period of function $\varepsilon^k(x)$, $i = \sqrt{-1}$,

$$\nu_k(l) = \frac{1}{Q} \int_c^d \varepsilon^k(x) e^{-i2\pi l x/Q} dx.$$

Based on the definition of norm, then we have

$$\| E^k \|_2^2 = \sum_{j=1}^{M-1} \tau \mid \varepsilon_j^k \mid^2 = \int_c^d \mid \varepsilon^k(x) \mid^2 dx. \tag{4.3}$$

Using the Parseval equalities, we get

$$\int_c^d \mid \varepsilon^k(x) \mid^2 dx = \sum_{l=-\infty}^{+\infty} \mid \nu_k(l) \mid^2. \tag{4.4}$$

Assume that the solution of the difference equation (4.1) has the form

$$\varepsilon_j^k = \nu_k e^{i\omega j \tau}, \quad j = 1, 2, \cdots, M-1, \tag{4.5}$$

where $\omega = 2\pi l/Q$. Substituting (4.5) into (4.1), gives

$$(1+\lambda_j)v_{k+1} = v_k - \lambda_j \sum_{m=0}^{k} b_{m,k+1} v_m, \quad k = 0,1,\cdots,N-1, \tag{4.6}$$

where

$$Re(\lambda_j) = \frac{4h^\alpha \kappa_\alpha}{\Gamma(\alpha+2)\tau^2} \sin^2\left(\frac{\omega\tau}{2}\right) - \frac{h^\alpha U''(x_j)}{\Gamma(\alpha+2)\eta_\alpha}, \tag{4.7a}$$

$$Im(\lambda_j) = -\frac{h^\alpha U'(x_j)}{\Gamma(\alpha+2)\tau\eta_\alpha} \sin(\omega\tau). \tag{4.7b}$$

Lemma 4.1. *If k is positive integer, then the coefficients $b_{m,k+1}$ determined by (2.12) satisfy*

(1) $b_{m,k+1} < 0$ $(m = 0,1,\cdots,k-1)$;

(2) $\sum\limits_{m=0}^{k+1} b_{m,k+1} = (\alpha+1)((k+1)^\alpha - k^\alpha)$.

For convenience, Symbol C_α denotes a α constant value which satisfy $2^{\alpha+1} - 3 = 0$.

Lemma 4.2. *If v_{k+1} be the solution of (4.6), and $\min\{Re(\lambda_j), j = 1,2,\cdots,M-1\} \geq C_\lambda > 0$, $0 < \alpha \leq C_\alpha$, then exist the constant $\widetilde{M} > 0$ such that*

$$|\, v_{k+1} \,| \leq e^{\widetilde{M}kh} \,|\, v_0 \,|, \quad k = 0,1,\cdots,N-1. \tag{4.8}$$

Proof. Using the mathematical induction, we can verify that the result. Considering on the ratio of time grid and spatial grid is bounded, then exist a constant $\widetilde{M}$ such that

$$\frac{1}{C_\lambda h} \leq \widetilde{M}.$$

For $k = 0$, we get from (4.6)

$$|\, v_1 \,| = \left| \frac{1-\alpha\lambda_j}{1+\lambda_j} v_0 \right|.$$

Because $0 < Re(\lambda_j)$, $0 < \alpha \leq C_\alpha < 1$, then we have

$$|\, v_1 \,| \leq \left| \frac{1-\alpha\lambda_j}{1+\lambda_j} \right| \cdot |\, v_0 \,| \leq |\, v_0 \,| = e^{\widetilde{M}\cdot 0h} \,|\, v_0 \,|.$$

For $k = 1$, from (4.6) and $3 - 2^{\alpha+1} \geq 0$, it follows that

$$\begin{aligned} |\, v_2 \,| &= \left| \frac{1+(3-2^{\alpha+1})\lambda_j}{1+\lambda_j} \cdot v_1 + \frac{\lambda_j}{1+\lambda_j}(1-\alpha)(2^\alpha - 1) \cdot v_0 \right| \\ &\leq \left| \frac{1+(3-2^{\alpha+1})\lambda_j}{1+\lambda_j} \right| \cdot e^{\widetilde{M}\cdot 0h} \,|\, v_0 \,| + (1-\alpha)(2^\alpha - 1) \left| \frac{\lambda_j}{1+\lambda_j} \right| \,|\, v_0 \,| \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1+(3-2^{\alpha+1}) \mid \lambda_j \mid}{\mid 1+\lambda_j \mid} \cdot \mid v_0 \mid + (2^{\alpha}-1)\left|\frac{\lambda_j}{1+\lambda_j}\right| \mid v_0 \mid \\
&\leq \frac{1+(2-2^{\alpha}) \mid \lambda_j \mid}{\mid 1+\lambda_j \mid} \cdot \mid v_0 \mid \\
&\leq \frac{1+\mid \lambda_j \mid}{\mid 1+\lambda_j \mid} \cdot \mid v_0 \mid \leq \frac{1+\mid \lambda_j \mid}{\mid \lambda_j \mid} \cdot \mid v_0 \mid \\
&\leq \left(1+\frac{1}{Re(\lambda_j)}\right) \cdot \mid v_0 \mid \leq \left(1+\frac{1}{C_\lambda}\right) \cdot \mid v_0 \mid \\
&\leq (1+\widetilde{M}h) \cdot \mid v_0 \mid \leq e^{\widetilde{M}h} \cdot \mid v_0 \mid .
\end{aligned}
$$

Assume that

$$
\mid v_m \mid \leq e^{\widetilde{M}(m-1)h} \mid v_0 \mid, \qquad m = 0, 1, \cdots, k.
$$

In view of Lemma 4.1, Eq. (4.6) leads to

$$
\begin{aligned}
\mid v_{k+1} \mid \leq & \left|\frac{1+(3-2^{\alpha+1})\lambda_j}{1+\lambda_j}\right| \cdot \mid v_k \mid + \left|\frac{\lambda_j}{1+\lambda_j}\right| \sum_{m=1}^{k-1} \mid b_{m,k+1} \mid\mid v_m \mid \\
& + \left|\frac{\lambda_j}{1+\lambda_j}\right| \mid b_{0,k+1} \mid\mid v_0 \mid \\
\leq & \left|\frac{1+(3-2^{\alpha+1})\lambda_j}{1+\lambda_j}\right| \cdot e^{\widetilde{M}(k-1)h} \mid v_0 \mid \\
& + \left|\frac{\lambda_j}{1+\lambda_j}\right| \sum_{m=1}^{k-1} \mid b_{m,k+1} \mid \cdot e^{\widetilde{M}(m-1)h} \mid v_0 \mid + \left|\frac{\lambda_j}{1+\lambda_j}\right| \mid b_{0,k+1} \mid\mid v_0 \mid \\
\leq & \left|\frac{1+(3-2^{\alpha+1})\lambda_j}{1+\lambda_j}\right| \cdot e^{\widetilde{M}(k-1)h} \mid v_0 \mid \\
& + \left|\frac{\lambda_j}{1+\lambda_j}\right| \sum_{m=1}^{k-1} \mid b_{m,k+1} \mid \cdot e^{\widetilde{M}(k-1)h} \mid v_0 \mid + \left|\frac{\lambda_i}{1+\lambda_i}\right| \mid b_{0,k+1} \mid \cdot e^{\widetilde{M}(k-1)h} \mid v_0 \mid \\
\leq & \left(\left|\frac{1+(3-2^{\alpha+1})\lambda_j}{1+\lambda_j}\right| + \left|\frac{\lambda_j}{1+\lambda_j}\right| (1-(3-2^{\alpha+1}))\right) \cdot e^{\widetilde{M}(k-1)h} \mid v_0 \mid \\
\leq & \frac{1+\mid \lambda_j \mid}{\mid 1+\lambda_j \mid} \cdot e^{\widetilde{M}(k-1)h} \mid v_0 \mid \leq \frac{1+\mid \lambda_j \mid}{\mid \lambda_j \mid} \cdot e^{\widetilde{M}(k-1)h} \mid v_0 \mid \\
\leq & \left(1+\frac{1}{Re(\lambda_j)}\right) \cdot e^{\widetilde{M}(k-1)h} \mid v_0 \mid \leq \left(1+\frac{1}{C_\lambda}\right) \cdot e^{\widetilde{M}(k-1)h} \mid v_0 \mid \\
\leq & (1+\widetilde{M}h) \cdot e^{\widetilde{M}(k-1)h} \mid v_0 \mid \leq e^{\widetilde{M}h} \cdot e^{\widetilde{M}(k-1)h} \mid v_0 \mid \\
= & e^{\widetilde{M}kh} \mid v_0 \mid .
\end{aligned}
$$

This verifies the conclusion (4.8).

Theorem 4.1. *If* $\min\{Re(\lambda_j),\ j=1,2,\cdots,M-1\} \geq C_\lambda > 0$, $0<\alpha\leq C_\alpha$, *then the numerical method* (2.15) *is used to solve time fractional Fokker-Planck equation* (2.1), (2.2), (2.3) *is stable.*

Proof. According to Lemma 4.2 and formula (4.8) we get

$$|\ v_{k+1}\ | \leq e^{\widetilde{M}kh}\ |\ v_0\ | \leq e^{\widetilde{M}T}\ |\ v_0\ |\ . \tag{4.9}$$

Based on (4.9), (4.3), (4.4), we obtain

$$\|\ E^k\ \|_2 \leq e^{\widetilde{M}T}\ \|\ E^0\ \|_2, \quad k=1,2,\cdots,N.$$

The proof of Theorem 4.1 is completed.

5 Numerical examples

Example 5.1. Consider the time fractional Fokker-Planck equation

$$\frac{\partial w(x,t)}{\partial t} = {}_0D_t^{1-\alpha}\Big[\frac{\partial}{\partial x}(-1)+\frac{\partial^2}{\partial x^2}\Big]w(x,t), \quad 0\leq x\leq 1, \quad t>0,$$

with the initial condition and the boundary conditions

$$\begin{aligned} w(x,0) &= x(1-x), && 0\leq x\leq 1,\\ w(0,t) &= -\frac{3t^\alpha}{\Gamma(1+\alpha)}-\frac{2t^{2\alpha}}{\Gamma(1+2\alpha)}, && t>0,\\ w(1,t) &= -\frac{t^\alpha}{\Gamma(1+\alpha)}-\frac{2t^{2\alpha}}{\Gamma(1+2\alpha)}, && t>0. \end{aligned}$$

The exact solution of the problem is

$$w(x,t) = x(1-x)+(2x-3)\frac{t^\alpha}{\Gamma(1+\alpha)}-\frac{2t^{2\alpha}}{\Gamma(1+2\alpha)}.$$

Define

$$error = \Big(\tau\sum_{j=1}^{M-1}(P(x_j,t_N)-P_j^N)^2\Big)^{\frac{1}{2}},$$

where $P_j^N, P(x_j,t_N)$ denote the numerical solution and the exact solution at grid point (x_j,t_N) respectively.

Table 1 gives the comparison of the computational error obtained by using two numerical methods for various time and spatial stepsize, in which *error*1 is obtained by the numerical method in [5] (in short L1-CDIA), *error*2 denotes the error results which obtained by the proposed method in this paper, where $t\in[0,100]$. From Table

Table 1: A comparison of error of two numerical methods for $\alpha = 0.5$.

τ	h	$error1$ (see [5])	$error2$
1/5	1/5	2.69×10^{-6}	4.4842×10^{-7}
1/10	1/10	1.02×10^{-6}	1.2197×10^{-7}
1/20	1/20	3.98×10^{-7}	3.6789×10^{-8}
1/40	1/40	1.59×10^{-7}	2.3228×10^{-8}

Table 2: The error result for $\alpha = 0.5$, $h = 0.01$ and 0.001.

τ	$error$ ($h = 0.01$)	$error$ ($h = 0.001$)
1/10	2.0065×10^{-6}	4.5377×10^{-8}
1/20	1.9977×10^{-6}	4.5105×10^{-8}
1/40	1.9955×10^{-6}	4.5036×10^{-8}
1/80	1.9950×10^{-6}	4.5023×10^{-8}
1/100	1.9949×10^{-6}	4.5019×10^{-8}

Table 3: The error results for $\alpha = 0.3$, $\tau = 0.01$ and 0.001.

h	$error$ ($\tau = 0.01$)	$error$ ($\tau = 0.001$)
1/10	2.7811×10^{-4}	2.7811×10^{-4}
1/20	9.3229×10^{-5}	9.3226×10^{-5}
1/40	3.1857×10^{-5}	3.1856×10^{-5}
1/80	1.1079×10^{-5}	1.1078×10^{-5}
1/100	7.9135×10^{-6}	7.9132×10^{-6}

Table 4: The error results for $\tau = 0.01$, $h = 0.01$ and various value α.

α	$error$
0.3	7.9135×10^{-6}
0.4	3.8423×10^{-6}
0.5	1.9949×10^{-6}
0.7	6.7050×10^{-7}
0.8	3.8687×10^{-7}
0.9	1.7783×10^{-7}

1, it show that the proposed method has advantage over the method (L1-CDIA) with the same stepsize implementation. The new method can obtain higher accuracy and efficiency.

In Tables 2-4, $t \in [0,1]$, Table 2 gives the behaviour of error with fixed time stepsize while spatial step is reduced. Table 3 gives the behaviour of error with fixed spatial stepsize while time step is reduced. Table 4 shows the error for various value α with fixed time and spatial stepsize implementation. Figs. 1 and 2 give the numerical solution, the exact solution and the error behaviour of Example 5.1 with $h = 0.01$, $\tau = 0.01$, $\alpha = 0.5$.

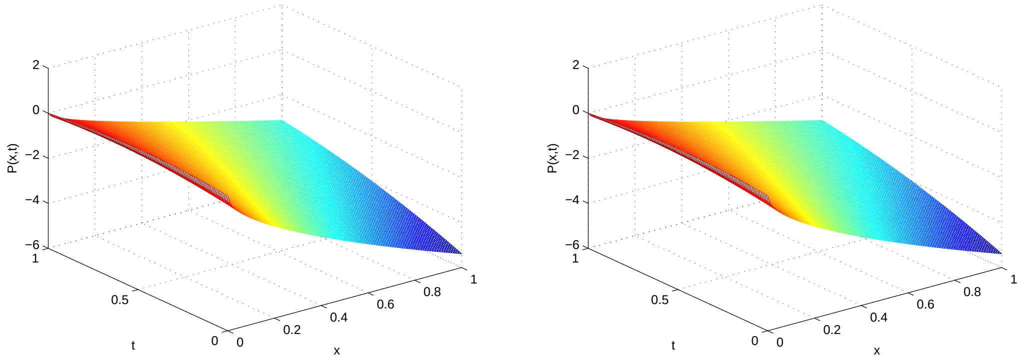


Figure 1: Numerical solution (left) and exact solution (right) of Example 5.1 with $h = 0.01$, $\tau = 0.01$, $\alpha = 0.5$.

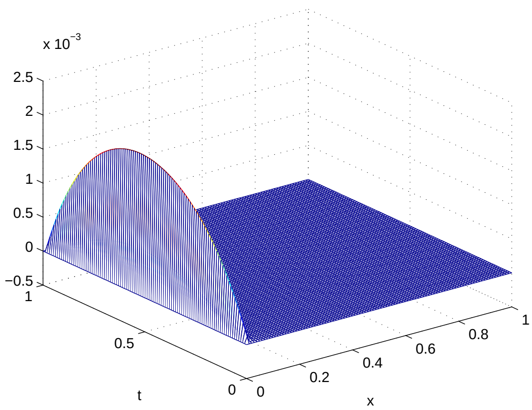


Figure 2: Example 5.1: Behaviour of error with $h = 0.01$, $\tau = 0.01$, $\alpha = 0.5$.

Example 5.2. Consider the time fractional Fokker-Planck equation (2.1) with the following parameter values

$$\begin{aligned} &U(x) = \cos x - 6x, && \eta_\alpha = 6, && \kappa_\alpha = 2, \\ &c = 1, && d = 11, && T = 0.3, \\ &\psi(x) = 0.10, && \varphi_1(t) = 0.10, && \varphi_2(t) = 0.10. \end{aligned}$$

Choose the numerical solution obtained with $h = 0.0006$, $\tau = 0.02$ as the exact solution. Let

$$error = \max_{j=1,2,\cdots,M-1} \mid P(x_j, 0.3) - P_{\tau,h}(x_j, 0.3) \mid .$$

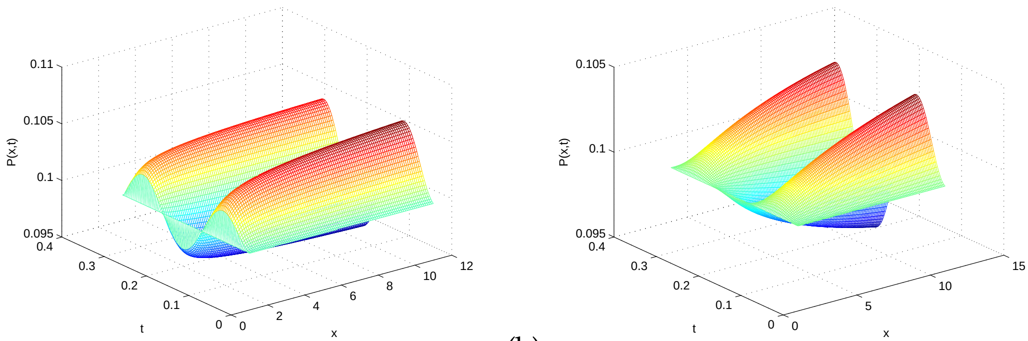
The numerical results are shown in Tables 5-6, which *Error*1 denotes the computational error which obtained by the predictor-corrector method in [6], *Error*2 denotes the error which obtained by the proposed method in this paper. The numerical solutions of Example 5.2 are displayed in Fig. 3.

Table 5: A comparison of error of two numerical methods for $\alpha = 0.8$ with time stepsize is fixed, spatial stepsize is decreased.

τ	$Error1$ ($h = 0.000015$) (see [6])	$Error2$ ($h = 0.03$)
1/2	8.8000×10^{-3}	8.0000×10^{-5}
1/4	2.2000×10^{-3}	2.0000×10^{-5}
1/8	5.1969×10^{-4}	1.0000×10^{-5}
1/16	1.0394×10^{-4}	1.0000×10^{-5}

Table 6: A comparison of error of two numerical methods for $\alpha = 0.8$ with spatial stepsize is fixed, time stepsize is decreased.

Method1 (see [6]) ($\tau = 0.0625$)		Method2 ($\tau = 0.125$)	
h	$Error1$	h	$Error2$
0.00012	2.0425×10^{-4}	0.03	8.0×10^{-5}
0.00006	1.4622×10^{-4}	0.015	2.0×10^{-5}
0.00003	1.1795×10^{-4}	0.0075	1.0×10^{-5}
0.000015	1.0394×10^{-4}	0.00375	1.0×10^{-5}



(a) (b)

Figure 3: Example 5.2: The numerical solution of Example 5.2 with $h = 0.003$, $\tau = 0.1$ for (a) $\alpha = 0.3$ and (b) $\alpha = 0.8$.

From Tables 5 and 6, it can be seen that the proposed method can obtain higher orders of accuracy than the predictor-corrector method in [6]. Consequently, the proposed method improves the computational efficiency.

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