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ANALYSIS AND NUMERICAL RESULTS FOR BOUNDARY OPTIMAL CONTROL PROBLEMS APPLIED TO TURBULENT BUOYANT FLOWS

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Abstract. In this work, we introduce the mathematical analysis of the optimal control for the Navier-Stokes system coupled with the energy equation and a k-ω turbulence model. While the optimal control of the Navier-Stokes system has been widely studied in past works, only a few works are based on the analysis of the turbulent flows. Moreover, the optimal control of turbulent buoyant flows are usually not taken into account due to the difficulties arising from the analysis and the numerical implementation of the optimality system. We first prove the existence of the solution of the boundary value problem associated with the studied system. Then we use an optimization method that relies on the Lagrange multiplier formalism to obtain the first-order necessary condition for optimality. We derive the optimality system and we solve it using a gradient descent algorithm that allows uncoupling state, adjoint, and optimality conditions. Some numerical results are then reported to validate the presented theoretical analysis.

Key words. Optimal control, buoyancy, turbulence modeling, k-ω model.

1. Introduction

In recent years, the optimal control of the energy and Navier-Stokes equations has gained attention in a variety of engineering fields. The optimal design of natural or mixed convection systems is crucial in many contexts, ranging from the thermal-hydraulics of nuclear reactors to semiconductor production processes where buoyancy forces control crystal growth.

In past years, considerable progress has been made in the mathematical analysis of the optimal control of Navier-Stokes and energy equations. Several works have been focused on the optimal control of the heat transfer in forced convection flows, where the coupling between the Navier-Stokes and energy equations is a one-way coupling, see for example [1, 2] and citations therein. In the case of natural or mixed convection flows, the mathematical analysis of the optimal control for the Oberbeck-Boussinesq system has been considered in several works focusing on stationary distributed and boundary controls [3, 4, 5, 6]. Distributed controls are very effective, but they are not feasible in many real cases. In the case of distributed controls, a feedback control can be applied over long period of time to obtain steady desired solutions, see for example [7, 8].

In this paper, we consider only boundary steady optimal control problems for turbulent flows in mixed or natural convection. The mathematical analysis and numerical simulations of the optimal control for turbulent flows without considering the temperature dependence have been investigated in past works [9, 10, 11]. An adjoint approach for the optimal control of turbulent buoyancy-driven flows has been proposed in [12], however a mathematical analysis has not been presented.

In this work, we consider the Reynolds averaged Navier-Stokes and energy system. The state is defined by the average velocity, total pressure field $(\mathbf{u}, p)$, the temperature field T and closed with a k-ω turbulence model [13], where k is the

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turbulent kinetic energy and ω its specific dissipation rate. We introduce the symmetric deformation tensor $\mathbf{S}(\mathbf{u})$ and its squared norm $\mathbf{S}^2(\mathbf{u})$ as

$$\mathbf{S}(\mathbf{u}) = \nabla \mathbf{u} + \nabla \mathbf{u}^T\,, \quad \mathbf{S}^2(\mathbf{u}) = \mathbf{S}(\mathbf{u}) : \mathbf{S}(\mathbf{u})\,.$$

The k-ω dynamical production terms S_k and S_ω for turbulence equations are defined by

$$S_k = \nu_t \mathbf{S}(\mathbf{u}) : \nabla \mathbf{u} = \frac{1}{2}\nu_t \mathbf{S}^2(\mathbf{u})\,, \tag{1}$$

$$S_\omega = \frac{\eta\omega}{k}\nu_t \mathbf{S}(\mathbf{u}) : \nabla \mathbf{u} = \frac{1}{2}\eta \mathbf{S}^2(\mathbf{u})\,. \tag{2}$$

We model the flow as incompressible according to the Oberbeck-Boussinesq approximation neglecting fluid density variations risen by the temperature in the advective term. Density temperature dependence cannot be neglected in the buoyancy force and a linear dependence is taken into account through the fluid coefficient of expansion γ around the reference temperature T_0 in the following specific form of the buoyancy force

$$\mathbf{f}_b = \gamma \mathbf{g}(T - T_0)\,,$$

where $\mathbf{g}$ is the gravitational acceleration vector.

The production terms due to buoyancy in k-ω equations are modeled according to [14, 15]. The source terms depending on the interaction between gravity and the turbulent heat flux components are modeled as

$$S_{k,b} = \frac{\gamma\nu_t}{Pr_t}\mathbf{g} \cdot \nabla T\,, \tag{3}$$

$$S_{\omega,b} = \frac{\eta\gamma}{Pr_t}\mathbf{g} \cdot \nabla T\,. \tag{4}$$

The coefficients η, β and β^* are model constants [13].

Under this framework, we consider an open bounded domain Ω with boundary Γ and the following governing state equations

$$(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nabla \cdot [(\nu + \nu_t)\mathbf{S}(\mathbf{u})] = \mathbf{f} - \gamma \mathbf{g}(T - T_0)\,, \tag{5}$$

$$\nabla \cdot \mathbf{u} = 0\,, \tag{6}$$

$$(\mathbf{u} \cdot \nabla)T = \nabla \cdot [(\alpha + \alpha_t)\nabla T]\,, \tag{7}$$

$$(\mathbf{u} \cdot \nabla)k - \nabla\left[(\nu + \sigma_k \nu_t) \cdot \nabla k\right] = S_k + S_{k,b} - \beta^*\, k\, \omega\,, \tag{8}$$

$$(\mathbf{u} \cdot \nabla)\omega - \nabla\left[(\nu + \sigma_\omega \nu_t) \cdot \nabla \omega\right] = S_\omega + S_{k,b} - \beta\omega^2\,, \tag{9}$$

where ν is the kinematic viscosity, α thermal diffusivity of the fluid, $\nu_t = k/\omega$ is the eddy kinematic viscosity and $\alpha_t = \nu_t/Pr_t$ is the eddy thermal diffusivity, where the turbulent Prandtl number Pr_t is assumed to be constant. The coefficients σ_k and σ_ω are model constants [13]. The system of equations (5)-(9) defines the state variable $(\mathbf{u}, p, T, k, \omega)$ when this is completed with suitable boundary conditions. However, the above system may not have a solution in many physical situations when k and ω become too large or too small. The k and ω equations have the typical pattern of the diffusion-reaction equations and therefore, introducing some assumptions, their solutions can be constrained inside a precise interval limited by the roots of the equation defined only by the right-hand-side non-linear terms. In an infinite medium or when advection and diffusion terms are negligible the equations (8)-(9) reduce to the non-linear right-hand-side terms

$$S_k + S_{k,b} - \beta^* k\omega = 0\,, \tag{10}$$

$$S_\omega + S_{\omega,b} - \beta\omega^2 = 0\,, \tag{11}$$

that implies

$$k = \sqrt{\nu_t \frac{S_k + S_{k,b}}{\beta^*}}\,, \qquad \omega = \sqrt{\frac{S_\omega + S_{\omega,b}}{\beta}}\,. \tag{12}$$

While $S_k \geq 0$ and $S_\omega \geq 0$, the sign of $S_{k,b}$ and $S_{\omega,b}$ is not defined *a priori*, so we must impose $S_k + S_{k,b} \geq 0$ and $S_\omega + S_{\omega,b} \geq 0$.

In order to limit the behavior of the k and ω variables, it is convenient to introduce two new variables P_k and P_ω

$$(\mathbf{u} \cdot \nabla)k - \nabla\left[(\nu + \sigma_k \nu_t) \cdot \nabla k\right] = P_k - \beta^*\, k\, \omega\,, \tag{13}$$

$$(\mathbf{u} \cdot \nabla)\omega - \nabla\left[(\nu + \sigma_\omega \nu_t) \cdot \nabla \omega\right] = P_\omega - \beta\omega^2\,. \tag{14}$$

Now we must define the source terms P_k and P_ω by taking into account their positiveness. To consider these lower bounds, we can define

$$S'_k = \max\left[S_k + S_{k,b}, 0\right]\,, \tag{15}$$

$$S'_\omega = \max\left[S_\omega + S_\omega, 0\right]\,. \tag{16}$$

Moreover, to keep the Navier-Stokes solutions in standard functional classes and have turbulent fields bounded in well-defined intervals, we must regularize the modeling of the turbulence sources. This is achieved by limiting the total turbulence production terms $P_k = S'_k$ and $P_\omega = S'_\omega$ under the maximum value of the respective dissipation terms. Therefore, given arbitrary limiting values k_1 and ω_1 we define

$$P_k = \min\left[S'_k,\, \beta^* k_1 \omega\right]\,, \tag{17}$$

$$P_\omega = \min\left[S'_\omega, \beta\omega_1^2\right]\,. \tag{18}$$

In the rest of the paper, we label k_1 and ω_1 with k_{max} and ω_{max} since they will be proved to be the limits for k and ω fields. The two relations (17) and (18) assure that, in the case of unbounded gradient velocity, the dissipation term can cope with the turbulence sources and keep k and ω limited.

The definition of ν_t can lead to singularities when $\omega \approx 0$. For this reason we bound the value of ν_t as

$$\nu_t = \min\left[\frac{k}{\omega}, \nu_{max}\right]\,. \tag{19}$$

Note that the introduced constants k_{max}, ω_{max} and ν_{max} can be chosen as large as needed in order to assure the regularity of the problem together with the accuracy of the physical solution. By doing so the solution of Navier-Stokes equations remains unchanged while only the turbulence source terms are modeled to avoid singularities.

In this work, we aim to control the temperature T_c on a portion of the boundary $\Gamma_C \subset \Gamma$ and minimize the cost functional

$$\begin{aligned} \mathcal{J}(\mathbf{u}, k, T_c) = {} & \frac{\alpha_u}{2} \int_{\Omega_d} (\mathbf{u} - \mathbf{u}_d)^2 d\mathbf{x} + \frac{\alpha_k}{2} \int_{\Omega_d} (k - k_d)^2 d\mathbf{x} \\ & + \frac{\lambda_1}{2} \int_{\Gamma_C} T_c^2 d\mathbf{x} + \frac{\lambda_2}{2} \int_{\Gamma_C} (\nabla T_c)^2 d\mathbf{x}\,, \end{aligned} \tag{20}$$

under the constraints (5)-(9) in order to have a desired velocity $\mathbf{u}_d$ or a desired turbulence kinetic energy k_d located over a certain domain $\Omega_d \subseteq \Omega$. The constants α_u, α_k and λ_2 are non-negative, while λ_1 is a positive constant. In particular, when $\alpha_u = 0$ or $\alpha_k = 0$ the objective functional can be used to control only the turbulent kinetic energy or the velocity field, respectively. Both λ_1 and λ_2 are regularization parameters, and the choice of them is a key point for the numerical solution of the problem because high values of λ_1 and λ_2 can result in a poor control, while low values can lead to convergence issues due to the enlargement of the functional space of the control variable T_c.

The rest of this paper is organized as follows. In the next section, we introduce the weak formulations and we prove the existence of the solution of the Navier S-tokes, energy, and turbulence equations. We also prove the existence of a solution of the associated boundary value problem. In section 3, we formulate the optimization problem and prove the existence of optimal solutions. In section 4, we show that the Lagrange multiplier approach is well-posed and we derive the final optimality system. In section 5, we introduce a numerical algorithm for the numerical implementation of the optimality system in a finite element framework and we present some numerical results.

2. Variational formulation of the state problem

We use standard notation $H^s(\mathcal{O})$ for Sobolev space of order s with respect to the set $\mathcal{O}$, which can be the flow domain $\Omega \subset \Re^n$, with $n = 2, 3$, or its boundary Γ or a part of it. The inner product over $H^m(\mathcal{O})$ is denoted by $(f, g)_m$, whenever m is a non-negative integer. We define, for $(fg) \in L^1(\mathcal{O})$ and $(\mathbf{u} \cdot \mathbf{v}) \in L^1(\mathcal{O})$

$$(f,g)_{\mathcal{O}} = \int_{\mathcal{O}} fg d\mathbf{x}\,, \qquad (\mathbf{u},\mathbf{v})_{\mathcal{O}} = \int_{\mathcal{O}} \mathbf{u}\cdot\mathbf{v} d\mathbf{x}\,.$$

We associate to $H^m(\mathcal{O})$ its natural norm $\|f\|_{m,\mathcal{O}} = \sqrt{(f,f)_m}$. We will neglect the domain label when $\mathcal{O} \equiv \Omega$.

For vector-valued functions and spaces, we use boldface notation. For example, $\mathbf{H}^s(\Omega) = [H^s(\Omega)]^n$ denotes the space of $\Re^n$-valued functions such that each component belongs to $H^s(\Omega)$. Of special interest is the space

$$\mathbf{H}^1(\Omega) = \left\{ u_i \in L^2(\Omega) \Big| \frac{\partial u_i}{\partial x_j} \in L^2(\Omega) \quad \text{for } i,j = 1,\dots,n \right\}$$

equipped with the norm $\|\mathbf{u}\|_1 = (\sum_{i,j}(\|u_i\|^2 + \|\partial u_i/\partial x_j\|^2))^{1/2}$. We also define the space

$$\mathbf{V}(\Omega) = \{\mathbf{u} \in \mathbf{H}^1(\Omega) \,|\, \nabla\cdot\mathbf{u} = 0\}\,.$$

Let Γ_s be a subset of Γ, we consider the subspace

$$\mathbf{H}^1_{\Gamma_s}(\Omega) = \{\mathbf{u} \in \mathbf{H}^1(\Omega) \,|\, \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_s\}\,.$$

Also, we write $\mathbf{H}^1_0(\Omega) = \mathbf{H}^1_\Gamma(\Omega)$. Let $\mathbf{H}^{1*}_{\Gamma_s}(\Omega)$ denote the dual space of $\mathbf{H}^1_{\Gamma_s}(\Omega)$. Note that $\mathbf{H}^{1*}_{\Gamma_s}(\Omega)$ is a subspace of $\mathbf{H}^{-1}(\Omega)$, where the latter is the dual space of $\mathbf{H}^1_0(\Omega)$.

Since the pressure is only determined up to an additive constant by the Navier-Stokes system with velocity boundary conditions, we define the space of square integrable functions having zero mean over Ω as

$$L^2_0(\Omega) = \{p \in L^2(\Omega) | \int_\Omega p d\mathbf{x} = 0\}\,.$$

We introduce the following continuous bilinear and trilinear forms useful to derive the weak form of the introduced system

$$a(\nu;\mathbf{u},\mathbf{v}) = \frac{1}{2}\int_\Omega \nu\mathbf{S}(\mathbf{u}) : \mathbf{S}(\mathbf{v}) d\mathbf{x} \qquad \forall \mathbf{u},\mathbf{v} \in \mathbf{H}^1(\Omega)\,, \tag{21}$$

$$b(\mathbf{u},\psi) = -\int_\Omega \psi\nabla\cdot\mathbf{u}\, d\mathbf{x} \qquad \forall\psi \in L^2_0(\Omega)\,,\, \forall\mathbf{u} \in \mathbf{H}^1(\Omega)\,, \tag{22}$$

$$a(k;T,\varphi) = k \int_\Omega \nabla T \cdot \nabla\varphi \, d\mathbf{x} \qquad \forall T,\varphi \in H^1(\Omega)\,. \tag{23}$$

Also we introduce the following continuous trilinear forms

$$c(\mathbf{w};\mathbf{u},\mathbf{v}) = \frac{1}{2}\left[\int_\Omega (\mathbf{w}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\,d\mathbf{x} - \int_\Omega (\mathbf{w}\cdot\nabla)\mathbf{v}\cdot\mathbf{u}\,d\mathbf{x}\right] \tag{24}$$

$$c(\mathbf{u},T,\varphi) = \int_\Omega (\mathbf{u}\cdot\nabla T)\varphi\,d\mathbf{x}\,, \tag{25}$$

for all $\mathbf{w} \in \mathbf{V}(\Omega)$, $\mathbf{u} \in \mathbf{H}^1(\Omega)\,, \mathbf{v} \in \mathbf{H}^1(\Omega)\,, T \in H^1(\Omega)$ and $\varphi \in H^1(\Omega)$. It is clear that $c(\mathbf{w};\mathbf{v},\mathbf{v}) = 0$ for all $\mathbf{w} \in \mathbf{V}(\Omega)$ and $c(\mathbf{u},\phi,\phi) = 0$ for all $\varphi \in H^1(\Omega)$. A detailed discussion on these trilinear forms can be found in [16].

We consider the following formulation of the direct problem for the Navier Stokes and energy system (5)-(7).

$$\begin{aligned}
&a(\nu+\nu_t;\mathbf{u},\mathbf{v}) + c(\mathbf{u};\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = (\mathbf{f},\mathbf{v}) - (\gamma(T-T_0)\mathbf{g},\mathbf{v}) \;\; \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega)\\
&b(\mathbf{u},q) = 0 \qquad \forall q \in L_0^2(\Omega)\\
&(\mathbf{u},\mathbf{s})_\Gamma = (\mathbf{g}_u,\mathbf{s})_\Gamma \qquad \forall \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma)\\
&a\big(\alpha + \frac{\nu_t}{Pr_t};T,\varphi\big) + c(\mathbf{u};T,\varphi) = (\varphi, g_{T,N})_{\Gamma_N} \qquad \forall \varphi \in H^1_{\Gamma_D}(\Omega)\\
&(T,s_T)_{\Gamma_D} \;=\; (g_T,s_T)_{\Gamma_D} \qquad \forall s_T \in H^{-1/2}(\Gamma_D)\,.
\end{aligned} \tag{26}$$

We also consider the following formulation of the direct problem for turbulence equations (8)-(9).

$$\begin{aligned}
&c(\mathbf{u};k,\psi) + a(\nu+\nu_t\sigma_k;k,\psi) = (P_k,\psi) - (\beta^*\,k\omega,\psi) \qquad \forall \psi \in H_0^1(\Omega)\\
&(k,s_k)_\Gamma \;=\; (g_k,s_k)_\Gamma \qquad \forall s_k \in H^{-1/2}(\Gamma)\\
&c(\mathbf{u};\omega,\phi) + a(\nu+\nu_t\sigma_\omega;\omega,\phi) = (P_\omega,\phi) - (\beta\,\omega^2,\phi) \qquad \forall \phi \in H_0^1(\Omega)\\
&(\omega,s_\omega)_\Gamma \;=\; (g_\omega,s_\omega)_\Gamma \qquad \forall s_\omega \in H^{-1/2}(\Gamma)\,.
\end{aligned} \tag{27}$$

Note that Γ_D and Γ_N are the portion of the boundary where Dirichlet and Neumann boundary conditions on the temperature field are imposed, respectively. Moreover, one may compute the normal flux on Γ_D in the normal direction as

$$q_n = \Big(\alpha + \frac{\nu_t}{Pr_t}\Big)\nabla T\cdot\widehat{\mathbf{n}}|_{\Gamma_D} \in H^{-1/2}(\Gamma_D)\,.$$

The existence of the solution of system (26) has been proved in [17], Theorem 3.1. Here we report the cited theorem.

Theorem 1. *Let Ω be an open, bounded set with Lipschitz-continuous boundary Γ. Let ν_t be a non-negative function in $L^\infty(\Omega)$, $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$, $\mathbf{g} \in L^\infty(\Omega)$, $\mathbf{g}_u \in \mathbf{H}^{1/2}(\Gamma)$, $g_T \in H^{1/2}(\Gamma_D)$ and $g_{T,N} \in L^2(\Gamma_N)$. Then,*

(i) the system (26) has at least one solution $(\mathbf{u},p,T) \in \mathbf{H}^1(\Omega)\times L^2(\Omega)\times H^1(\Omega)$.
(ii) there exist constants C_1, $C_2 > 0$ such that

$$\|T\|_1 + \|\mathbf{u}\|_1 \leq C_1\big(\|\mathbf{f}\|_{-1} + \|g_{T,N}\|_{0,\Gamma_N}\big) + C_2\,, \tag{28}$$

where C_2 depends on $\mathbf{g}_u$ and g_T. In particular, if $\mathbf{g}_u = \mathbf{0}$ and $g_T = 0$ then $C_2 = 0$.

We now introduce the existence of the solution for the k-ω turbulence system.

Theorem 2. *Let Ω be an open, bounded set with Lipschitz-continuous boundary Γ. Let $\mathbf{u} \in \mathbf{V}(\Omega)$, g_k and g_ω in $H^1(\Omega)\cup L^\infty(\Omega)$ and ν_t, P_k, P_ω as defined in (19), (17) and (18), respectively. Then*

(i) there exists at least one solution $(k,\omega) \in H^1(\Omega) \times H^1(\Omega)$ *of* (27)*;*
(ii) let ω_{max} *and* k_{max} *be positive real constants and*

$$k_{sup} = \sup\{\sup_{\Gamma}\{g_k\}, k_{max}\}\,, \tag{29}$$

$$\omega_{inf} = \inf\{\inf_{\Gamma}\{g_\omega\}, \inf_{\Omega}\{\sqrt{P_\omega/\beta}\}\}\,, \tag{30}$$

$$\omega_{sup} = \sup\{\sup_{\Gamma}\{g_\omega\}, \omega_{max}\}\,, \tag{31}$$

then

$$0 \le k \le k_{sup}\,, \tag{32}$$

$$0 \le \omega_{inf} \le \omega \le \omega_{sup}\,. \tag{33}$$

Proof. The proof of this Theorem can be found in [9], Theorem 2.

□

By using previous theorems we can prove the existence of the solution of the associated boundary value problem.

Theorem 3. *There exists a solution* $(\mathbf{u}, p, T, k, \omega)$ *of the associated boundary value problem in* (26)-(27)*.*

Proof. To prove the existence of the solution, we rely on the Schauder's fixed point theorem and we follow standard techniques (e.g. see [18]). To simplify the notation, we consider now the presented physical system with $\mathbf{g}_u = \mathbf{0}$ and $g_T = g_{T,N} = g_k = g_\omega = 0$. For a given set $(\mathbf{u}_1, T_1, k_1, \omega_1) \in \mathbf{H}_0^1(\Omega) \times H^1_{\Gamma_D}(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$, $(\mathbf{u}, p, T)$ and (k,ω) are the state of the following Navier-Stokes-k-ω and energy system

$$\begin{aligned}
& a(\nu + \nu_{t1}; \mathbf{u}, \mathbf{v}) + c(\mathbf{u}_1; \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}) \\
& \qquad\qquad\qquad - (\gamma(T_1 - T_0)\mathbf{g}, \mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega) \\
& b(\mathbf{u}, q) = 0 \qquad \forall\, q \in L_0^2(\Omega) \\
& a\Big(\alpha + \frac{\nu_{t1}}{Pr_t}; T, \varphi\Big) + c(\mathbf{u}_1; T, \varphi) = 0 \qquad \forall \varphi \in H^1_{\Gamma_D}(\Omega) \\
& c(\mathbf{u}_1; k, \psi) + a(\nu + \nu_{t1}\sigma_k; k, \psi) = (P_{k1}, \psi) - (\beta^*\, k\omega_1, \psi) \qquad \forall \psi \in H_0^1(\Omega) \\
& c(\mathbf{u}_1; \omega, \phi) + a(\nu + \nu_{t1}\sigma_\omega; \omega, \phi) = (P_{\omega 1}, \phi) - (\beta\, \omega\omega_1, \phi) \qquad \forall \phi \in H_0^1(\Omega)
\end{aligned} \tag{34}$$

where $\nu_{t1} = \nu_t(k_1, \omega_1)$, $P_{k1} = P_k(\mathbf{u}_1, T_1, k_1, \omega_1)$, $P_{\omega 1} = P_\omega(\mathbf{u}_1, T_1)$. Under the imposed hypotheses, we can now prove the existence of the solution of the split system (34). In fact, from Theorem 1 we have that $\|T\|_1 + \|\mathbf{u}\|_1$ is uniformly bounded. Moreover, from Theorem 2 we have that $\|k\|_1$ and $\|\omega\|_1$ are uniformly bounded by the constants C_k and C_ω as a function of the given values k_{max} and ω_{max}.

Let $D = \mathbf{H}_0^1(\Omega) \times H^1_{\Gamma_D}(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$ and $A = \mathbf{H}_0^1(\Omega) \times H^1_{\Gamma_D}(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$. We consider now the mapping $\mathcal{T} : D \to A$ such that

$$\begin{aligned}
\mathbf{u} &= \mathbf{u}(\mathbf{u}_1, T_1, k_1, \omega_1) \\
T &= T(\mathbf{u}_1, T_1, k_1, \omega_1) \\
k &= k(\mathbf{u}_1, T_1, k_1, \omega_1) \\
\omega &= \omega(\mathbf{u}_1, T_1, k_1, \omega_1)\,.
\end{aligned} \tag{35}$$

We endow the product space $\mathbf{H}_0^1(\Omega) \times H^1_{\Gamma_D}(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$ with the norm $\|(\mathbf{u}_1, T_1, k_1, \omega_1)\| = \|\mathbf{u}_1\|_1 + \|T_1\|_1 + \|k_1\|_1 + \|\omega_1\|_1$. It can be proved that $\mathcal{T}$ is a continuous mapping with respect to the introduced norm. Let R denote the constant $R = C_{u,T} + C_k + C_\omega$, where $C_{u,T} = C_1\big(\|\mathbf{f}\|_1 + \|g_{T,N}\|_{0,\Gamma_N}\big) + C_2$. For all $(\mathbf{u}_1, T_1, k_1, \omega_1) \in D$ we have $\|(\mathbf{u}, T, k, \omega)\| = \|\mathbf{u}\|_1 + \|T\|_1 + \|k\|_1 + \|\omega\|_1 < C_{u,T} + C_k + C_\omega = R$. Therefore

$$\mathcal{T}(B_R) \subset B_R\,, \tag{36}$$

where B_R is the ball of radius R.

The condition (36) derives from Theorem 1 and 2, and represents a mandatory hypothesis for the Schauder's fixed point theorem. In fact, the theorem provides that for a separated topological vector space D, a convex subset $B_R \subset D$, a continuous mapping of B_R into itself $\mathcal{T}$, such that $\mathcal{T}(B_R)$ is contained in a compact subset of B_R, equipped with the topology inherited from D, then $\mathcal{T}$ has a fixed point, namely, there exists $x \in B_R$ such that $\mathcal{T}(x) = x$. In conclusion, we can now apply the fixed point theorem to the system (34), and therefore there exists a solution of the system. □

3. The optimal control problem

In this section, we present the model for the optimal control of the presented state system, and prove the existence of an optimal solution. We first recall that, according to the Theorem 2, the set of all admissible functions k and ω is determined by

$$\mathcal{T}_{ad} = \big\{(k, \omega) \in H^1(\Omega) \times H^1(\Omega)\,|\, 0 \le \omega_{inf} \le \omega \le \omega_{sup} \text{ and } 0 \le k \le k_{sup}\big\}\,, \tag{37}$$

where ω_{inf}, ω_{sup} and k_{sup} have been introduced above.

In this work, we aim to control the temperature $T = g_T + T_c$ on a portion of the boundary $\Gamma_C \subseteq \Gamma_D \subseteq \Gamma$ to have a desired velocity $\mathbf{u}_d$ or a desired turbulence kinetic energy k_d on a certain domain $\Omega_d \subseteq \Omega$. The optimal control problem can be summarized as follows

> Given g_k, $g_\omega \in H^{\frac{1}{2}}(\Gamma)$, $g_T \in H^{\frac{1}{2}}(\Gamma_D)$ and $\mathbf{g}_u \in \mathbf{H}^{\frac{1}{2}}(\Gamma)$, find a state-control set $(\mathbf{u}, p, T, T_c, k, \omega, P_k, P_\omega, \nu_t) \in \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times H^1(\Omega) \times H^1_0(\Gamma_C) \times \mathcal{T}_{ad} \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ which minimizes the cost functional (20) under the constraints (26)-(27).

We also recall that P_k, P_ω and ν_t are defined in (17), (18) and (19), respectively. We can now define the admissible set of states and controls as

$$\begin{aligned}\mathcal{S}_{ad} = &\big\{(\mathbf{u}, p, T, T_c, k, \omega, P_k, P_\omega, \nu_t) \in \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times H^1(\Omega) \times H^1_0(\Gamma_C) \times \\ &\times \mathcal{T}_{ad} \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \text{ such that } \mathcal{J}(\mathbf{u}, k, T_c) < \infty\big\}\,.\end{aligned} \tag{38}$$

Since the main statement of the optimal control problem is the minimization of the functional (20), the problem can be reformulated as follows. We say that $(\widehat{\mathbf{u}}, \widehat{p}, \widehat{T}, \widehat{T}_c, \widehat{k}, \widehat{\omega}, \widehat{P}_k, \widehat{P}_\omega, \widehat{\nu}_t) \in \mathcal{S}_{ad}$ is an *optimal solution* if there exists $M > 0$ such that

$$\begin{aligned}&\mathcal{J}(\widehat{\mathbf{u}}, \widehat{T}, \widehat{T}_c, \widehat{k}) < \mathcal{J}(\mathbf{u}, k, T_c),\ \forall\,(\mathbf{u}, p, T, T_c, k, \omega, P_k, P_\omega, \nu_t) \in \mathcal{S}_{ad} \\ &\quad \text{satisfying } \|\mathbf{u} - \widehat{\mathbf{u}}\|_1 + \|p - \widehat{p}\|_0 + \|T - \widehat{T}\|_1 + \|k - \widehat{k}\|_1 + \|\omega - \widehat{\omega}\|_1 \\ &\qquad + \|\nu_t - \widehat{\nu}_t\|_0 + \|P_k - \widehat{P}_k\|_0 + \|P_\omega - \widehat{P}_\omega\|_0 + \|T_c - \widehat{T}_c\|_{1,\Gamma_C} < M\,.\end{aligned} \tag{39}$$

We now turn to the question of the existence of optimal solutions for the problem (39).

Theorem 4. *Let $\mathcal{S}_{ad}$ be not empty. There exists at least one optimal solution $(\widehat{\mathbf{u}}, \widehat{p}, \widehat{T}, \widehat{T}_c, \widehat{k}, \widehat{\omega}, \widehat{P}_k, \widehat{P}_\omega, \widehat{\nu}_t) \in \mathcal{S}_{ad}$.*

Proof. The proof of the existence of an optimal solution is obtained with standard techniques, and the interested reader can consult [19, 20]. We consider $T_c = 0$, i.e. $T = g_T$ on Γ_C, then we can find the solution $(\mathbf{u}, p, T, 0, k, \omega, P_k, P_\omega, \nu_t)$. This implies that $\mathcal{S}_{ad}$ is not empty. Therefore, since the set of the values of $\mathcal{J}$ is bounded from below, there exists a minimizing sequence $(\mathbf{u}_m, p_m, T_m, T_{cm}, k_m, \omega_m, P_{km}, P_{\omega m}, \nu_{tm}) \in \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times H^1(\Omega) \times H^1_0(\Gamma_C) \times \mathcal{T}_{ad} \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$. As defined in (17), (18) and (19), the sequences P_{km}, $P_{\omega m}$ and ν_{tm} are uniformly bounded. Since P_{km} and $P_{\omega m}$ are bounded, then also k_m and ω_m are uniformly bounded in $\mathcal{T}_{ad}$. From Theorem 1, we can also state that $\mathbf{u}_m$ and T_m are uniformly bounded in $\mathbf{H}^1(\Omega)$ and $H^1(\Omega)$, respectively. Following standard techniques, we can now extract subsequences $(\mathbf{u}_n, p_n, T_n, T_{cn}, k_n, \omega_n, P_{kn}, P_{\omega n}, \nu_{tn})$ converging to $(\widehat{\mathbf{u}}, \widehat{p}, \widehat{T}, \widehat{T}_c, \widehat{k}, \widehat{\omega}, \widehat{P}_k, \widehat{P}_\omega, \widehat{\nu}_t)$. To prove that the limit of the sub-sequence satisfies the problem we pass to the limit the equation problem. Following [18, 19], we can state that the solution of all the linear and the nonlinear operators converges to the solution of the equation problem. $\square$

4. The Lagrange multiplier method

4.1. Preliminaries. In this section, we show that the Lagrange multiplier technique is well-posed and can be used to obtain the first-order necessary condition. In particular, we introduce the Lagrangian map and we show that it is strictly differentiable.

We recall the inequality constraints introduced in (15)-(19) and define auxiliary variables which allows us to transform them into equality constraints [21]. Let us consider the source $S'_k = \max[S_k + S_{k,b}, 0]$ defined by (15). It is easy to show that finding S'_k from (15) is equivalent to solve the following system of equation

$$S'_k(S'_k - (S_k + S_{k,b})) = 0\,, \tag{40}$$

$$r^2_{k1} - S'_k - (S'_k - (S_k + S_{k,b})) = 0\,. \tag{41}$$

From (15) we have $S'_k = S_k + S_{kb}$ or $S'_k = 0$ that satisfies (40). When $S_k + S_{kb} \geq 0$ we have $S'_k = S_k + S_{kb} = r^2_{k1} \geq 0$ for some real number r^2_{k1}. Otherwise, when $S_k + S_{kb} \leq 0$ we have $S'_k = 0$ and $(S_k + S_{kb}) = -r^2_{k1} \leq 0$ for some real number r^2_{k1} that satisfies (41). The value $r^2_{k1} = 0$ is attained when $S'_k = (S_k + S_{k,b}) = 0$. Vice-versa from (40) we have $S'_k = 0$ and/or $S'_k = S_k + S_{k,b}$. From (41), S'_k is zero when $S_k + S_{k,b} \leq 0$ and $S'_k = S_k + S_{k,b}$ when $S'_k \geq 0$. With the same arguments the source P_k, defined in (17), satisfies

$$(S'_k - P_k)(\beta^* k_{max}\omega - P_k) = 0\,, \tag{42}$$

$$r^2_{k2} - (S'_k - P_k) - (\beta^* k_{max}\omega - P_k) = 0\,, \tag{43}$$

for some $r_{k2} \in L^2(\Omega)$. By using similar arguments, finding P_k in (17) it is equivalent to solve (42) and find a real r^2_{k2} in (43).

Similarly, let us consider the definition of $S'_\omega = \max[S_\omega + S_{\omega,b}, 0]$ in (18). When $S_\omega + S_{\omega b} \geq 0$ we have $S'_\omega = S_\omega + S_{\omega b}$ and $S'_\omega - (S_\omega + S_{\omega b}) = r^2_{\omega 1} \geq 0$ for some real number $r^2_{\omega 1}$. Otherwise, when $S_\omega + S_{\omega b} \leq 0$ we have $S'_\omega = 0$ and $-(S_\omega + S_{\omega b}) = r^2_{\omega 1} \geq 0$ for some real number $r^2_{\omega 1}$. In this case, we have that S'_ω satisfies

$$S'_\omega(S'_\omega - (S_\omega + S_{\omega,b})) = 0\,, \tag{44}$$

$$r^2_{\omega 1} - S'_\omega - (S'_\omega - (S_\omega + S_{\omega,b})) = 0\,, \tag{45}$$

for some $r_{\omega 1} \in L^2(\Omega)$. Vice-versa, when (44) is satisfied and there exists a $r^2_{\omega 1}$ we have $S'_\omega = S_\omega + S_{\omega b}$ with $S'_\omega \geq 0$ or $S'_\omega = 0$ with $S_\omega + S_{\omega,b} \leq 0$ which implies (18).

With the same arguments the source P_ω, defined in (18), satisfies

$$(S'_\omega - P_\omega)(\beta\omega^2_{max} - P_\omega) = 0\,, \tag{46}$$

$$r^2_{\omega 2} - (S'_\omega - P_\omega) - (\beta\omega^2_{max} - P_\omega) = 0\,, \tag{47}$$

for some $r_{\omega 2} \in L^2(\Omega)$. Finding P_ω from (15) is equivalent to solve (46)-(47).

Finally, the inequality (19) can be replaced by

$$(k - \nu_t\omega)(\nu_{max} - \nu_t) = 0\,, \tag{48}$$

$$r^2_\nu - (k - \nu_t\omega) - \omega(\nu_{max} - \nu_t) = 0\,, \tag{49}$$

for some $r_\nu \in L^2(\Omega)$.

Now we consider all the constraint equations and the functional in two mappings in order to study their differential properties. It is convenient to define the following functional spaces

$$\begin{aligned} &\mathbf{B_{1e}} = \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times H^1(\Omega) \times H^1_0(\Gamma_C) \times H^{-\frac{1}{2}}(\Gamma_D) \times \mathcal{T}_{ad}\,,\\ &\mathbf{B_{1c}} = (L^2(\Omega))^4 \times (L^2(\Omega))^4 \times (L^2(\Omega))^2\,, \quad \mathbf{B_1} = \mathbf{B_{1e}} \times \mathbf{B_{1c}}\,, \end{aligned} \tag{50}$$

$$\begin{aligned} &\mathbf{B_{2e}} = \mathbf{H}^{-1}(\Omega) \times L^2_0(\Omega) \times H^{1*}(\Omega) \times H^{\frac{1}{2}}(\Gamma_D) \times H^{-1}(\Omega) \times H^{-1}(\Omega)\,,\\ &\mathbf{B_{2c}} = (L^2(\Omega))^4 \times (L^2(\Omega))^4 \times (L^2(\Omega))^2\,, \quad \mathbf{B_2} = \mathbf{B_{2e}} \times \mathbf{B_{2c}}\,, \end{aligned} \tag{51}$$

$$\begin{aligned} &\mathbf{B_{3e}} = \mathbf{H}^1_0(\Omega) \times L^2_0(\Omega) \times H^1_{\Gamma_D \setminus \Gamma_C}(\Omega) \times H^1_0(\Gamma_C) \times H^{-\frac{1}{2}}(\Gamma_D) \times (H^1_0(\Omega))^2\,,\\ &\mathbf{B_{3c}} = (L^2(\Omega))^4 \times (L^2(\Omega))^4 \times (L^2(\Omega))^2\,, \quad \mathbf{B_3} = \mathbf{B_{3e}} \times \mathbf{B_{3c}}\,, \end{aligned} \tag{52}$$

and equip $\mathbf{B_1}, \mathbf{B_2}, \mathbf{B_3}$ with the usual graph norms for the product spaces involved. Given $\mathbf{z}_0 = (\mathbf{u}, p, T, T_c, q_n, k, \omega, S'_k, r_{k1}, P_k, r_{k2}, S'_\omega, r_{\omega 1}, P_\omega, r_{\omega 2}, \nu_t, r_\nu) \in \mathbf{B_1}$, we can now define the nonlinear mapping $M : \mathbf{B_1} \to \mathbf{B_2}$ at $\mathbf{z}_0$ by $\mathbf{M}(\mathbf{z}_0) \cdot \mathbf{z}_0 = \mathbf{b}$ with $\mathbf{b} = (\mathbf{l}_1, l_2, l_3, l_4, l_5, l_6, \mathbf{l}_k, \mathbf{l}_\omega, \mathbf{l}_\nu)$ if and only if

$$\begin{aligned} &a(\nu + \nu_t; \mathbf{u}, \mathbf{v}) + c(\mathbf{u}; \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) - (\mathbf{f}, \mathbf{v})\\ &\qquad\qquad + (\gamma(T - T_0)\mathbf{g}, \mathbf{v}) = (\mathbf{l}_1, \mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega)\\ &b(\mathbf{u}, q) = (l_2, q) \qquad \forall\, q \in L^2_0(\Omega)\\ &a\left(\alpha + \frac{\nu_t}{Pr_t}; T, \varphi\right) + c(\mathbf{u}; T, \varphi) - (q_n, \varphi)_{\Gamma_D}\\ &\qquad\qquad - (g_{T,N}, \varphi)_{\Gamma_N} = (l_3, \varphi) \qquad \forall \varphi \in H^1(\Omega)\\ &(T, s_T)_{\Gamma_D} - (T_c, s_T)_{\Gamma_C} - (g_T, s_T)_{\Gamma_D} = (l_4, s_T)_{\Gamma_D} \qquad \forall s_T \in H^{-1/2}(\Gamma_D)\\ &c(\mathbf{u}; k, \psi) + a(\nu + \nu_t\sigma_k; k, \psi) - (P_k, \psi) + (\beta^* k\omega, \psi) = (l_5, \psi) \quad \forall \psi \in H^1_0(\Omega)\\ &c(\mathbf{u}; \omega, \phi) + a(\nu + \nu_t\sigma_\omega; \omega, \phi) - (P_\omega, \phi) + (\beta\omega^2, \phi) = (l_6, \phi) \quad \forall \phi \in H^1_0(\Omega) \end{aligned} \tag{53}$$

and

$$\begin{aligned} &S'_k\left(S'_k - \frac{1}{2}\nu_t \mathbf{S}^2(\mathbf{u}) - \frac{\gamma\nu_t}{Pr_t}\mathbf{g}\cdot\nabla T\right) = l_{k0}\\ &r^2_{k1} - S'_k - \left(S'_k - \frac{1}{2}\nu_t \mathbf{S}^2(\mathbf{u}) - \frac{\gamma\nu_t}{Pr_t}\mathbf{g}\cdot\nabla T\right) = l_{k1}\\ &(S'_k - P_k)(\beta^* k_{max}\omega - P_k) = l_{k2}\,, \quad r^2_{k2} - (S'_k - P_k) - (\beta^* k_{max}\omega - P_k) = l_{k3}\\ &S'_\omega\left(S'_\omega - \frac{1}{2}\eta \mathbf{S}^2(\mathbf{u}) - \frac{\eta\gamma}{Pr_t}\mathbf{g}\cdot\nabla T\right) = l_{\omega 0} \end{aligned} \tag{54}$$

$$r_{\omega 1}^2 - S'_\omega - \left(S'_\omega - \frac{1}{2}\eta \mathbf{S}^2(\mathbf{u}) - \frac{\eta\gamma}{Pr_t}\mathbf{g}\cdot\nabla T \right) = l_{\omega 1}$$
$$(S'_\omega - P_\omega)(\beta\omega_{max}^2 - P_\omega) = l_{\omega 2}\,, \qquad r_{\omega 2}^2 - (S'_\omega - P_\omega) - (\beta\omega_{max}^2 - P_\omega) = l_{\omega 3}$$
$$(k - \nu_t\omega)(\nu_{max} - \nu_t) = l_{\nu 0}\,, \qquad r_\nu^2 - (k - \nu_t\omega) - \omega(\nu_{max} - \nu_t) = l_{\nu 1}$$

where all the equations of (54) hold in Ω. From the definition of $\mathbf{b}$, we can state that the set of constraint equations in our optimal control problem can be expressed as $\mathbf{M}(\mathbf{z}_0)\cdot\mathbf{z}_0 = \mathbf{0}$.

Given $\widehat{\mathbf{z}} = (\widehat{\mathbf{u}}, \widehat{p}, \widehat{T}, \widehat{T}_c, \widehat{q}_n, \widehat{k}, \widehat{\omega}, \widehat{S}'_k, \widehat{r}_{k1}, \widehat{P}_k, \widehat{r}_{k2}, \widehat{S}'_\omega, \widehat{r}_{\omega 1}, \widehat{P}_\omega, \widehat{r}_{\omega 2}, \widehat{\nu}_t, \widehat{r}_\nu) \in \mathbf{B}_1$, we define the nonlinear mapping $\mathbf{Q} : \mathbf{B}_1 \to \Re\times\mathbf{B}_2$. For $a \in \Re$ we set $\mathbf{Q}(\mathbf{z}_0)\cdot\widehat{\mathbf{z}} = (a, \mathbf{b})$ if and only if

$$\mathbf{Q}(\mathbf{z}_0)\cdot\widehat{\mathbf{z}} = \begin{pmatrix} \mathcal{J}(\mathbf{u}, k, T_c) - \mathcal{J}(\widehat{\mathbf{u}}, \widehat{k}, \widehat{T}_c) \\ \mathbf{M}(\mathbf{z}_0)\cdot\widehat{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} a \\ \mathbf{b} \end{pmatrix}. \tag{55}$$

4.2. Mapping differentiability. We now introduce the notion of map differentiability, and we show that the mappings M and Q introduced above are strictly differentiable. For the definition of the differentiability, see [22].

Lemma 1. *Let* $\mathbf{z}_0 \in \mathbf{B}_1$, $\widetilde{\mathbf{z}}_0 = (\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{T}, \widetilde{T}_c, \widetilde{q}_n, \widetilde{k}, \widetilde{\omega}, \widetilde{S}'_k, \widetilde{r}_{k1}, \widetilde{P}_k, \widetilde{r}_{k2}, \widetilde{S}'_\omega, \widetilde{r}_{\omega 1}, \widetilde{P}_\omega, \widetilde{r}_{\omega 2}, \widetilde{\nu}_t, \widetilde{r}_\nu) \in \mathbf{B}_3$ *and* $\bar{\mathbf{b}} = (\bar{\mathbf{l}}_1, \bar{l}_2, \bar{l}_3, \bar{l}_4, \bar{l}_5, \bar{l}_6, \bar{\mathbf{l}}_k, \bar{\mathbf{l}}_\omega, \bar{\mathbf{l}}_\nu) \in \mathbf{B}_2$. *Let consider, as derivative map, the bounded linear operator* $\mathbf{M}' : \mathbf{B_3} \to \mathbf{B_2}$, *where* $\mathbf{M}'(\mathbf{z}_0)\cdot\widetilde{\mathbf{z}}_0 = \bar{\mathbf{b}}$, *defined as*

$$\begin{aligned}
&a(\widetilde{\nu}_t; \mathbf{u}, \mathbf{v}) + a(\nu + \nu_t; \widetilde{\mathbf{u}}, \mathbf{v}) + c(\widetilde{\mathbf{u}}; \mathbf{u}, \mathbf{v}) + c(\mathbf{u}; \widetilde{\mathbf{u}}, \mathbf{v}) + b(\mathbf{v}, \widetilde{p}) \\
&\qquad + (\gamma\mathbf{g}\widetilde{T}, \mathbf{v}) = (\bar{\mathbf{l}}_1, \mathbf{v}) \qquad \forall\mathbf{v}\in\mathbf{H}_0^1(\Omega) \\
&b(\widetilde{\mathbf{u}}, q) = (\bar{l}_2, q) \qquad \forall\, q \in L_0^2(\Omega) \\
&a\left(\frac{\widetilde{\nu}_t}{Pr_t}; T, \varphi\right) + a\left(\alpha + \frac{\nu_t}{Pr_t}; \widetilde{T}, \varphi\right) + c(\widetilde{\mathbf{u}}; T, \varphi) + c(\mathbf{u}; \widetilde{T}, \varphi) \\
&\qquad - (\widetilde{q}_n, \varphi)_{\Gamma_D} = (\bar{l}_3, \varphi) \qquad \forall\varphi\in H^1(\Omega) \\
&(\widetilde{T}, s_T)_{\Gamma_D} - (\widetilde{T}_c, s_T)_{\Gamma_C} = (\bar{l}_4, s_T)_{\Gamma_D} \qquad \forall s_T \in H^{-1/2}(\Gamma_D) \qquad (56)\\
&c(\widetilde{\mathbf{u}}; k, \psi) + c(\mathbf{u}; \widetilde{k}, \psi) + a(\widetilde{\nu}_t\sigma_k; k, \psi) + a(\nu + \nu_t\sigma_k; \widetilde{k}, \psi) \\
&\qquad - (\widetilde{P}_k, \psi) + (\beta^*\widetilde{k}\omega, \psi) + (\beta^* k\widetilde{\omega}, \psi) = (\bar{l}_5, \psi) \qquad \forall\psi\in H_0^1(\Omega) \\
&c(\widetilde{\mathbf{u}}; \omega, \phi) + c(\mathbf{u}; \widetilde{\omega}, \phi) + a(\widetilde{\nu}_t\sigma_\omega; \omega, \phi) + a(\nu + \nu_t\sigma_\omega; \widetilde{\omega}, \phi) \\
&\qquad - (\widetilde{P}_\omega, \phi) + 2(\beta\omega\widetilde{\omega}, \phi) = (\bar{l}_6, \phi) \qquad \forall\phi\in H_0^1(\Omega)
\end{aligned}$$

and

$$\begin{aligned}
&\widetilde{S}'_k\Big(S'_k - (S_k + S_{k,b})\Big) + \\
&\qquad S'_k\left(\widetilde{S}'_k - \frac{1}{2}\widetilde{\nu}_t\mathbf{S}^2(\mathbf{u}) - \nu_t\mathbf{S}(\mathbf{u}):\mathbf{S}(\widetilde{\mathbf{u}}) - \frac{\gamma\widetilde{\nu}_t}{Pr_t}\mathbf{g}\cdot\nabla T - \frac{\gamma\nu_t}{Pr_t}\mathbf{g}\cdot\nabla\widetilde{T}\right) = \bar{l}_{k0} \\
&2r_{k1}\widetilde{r}_{k1} - 2\widetilde{S}'_k + \frac{1}{2}\widetilde{\nu}_t\mathbf{S}^2(\mathbf{u}) + \nu_t\mathbf{S}(\mathbf{u}):\mathbf{S}(\widetilde{\mathbf{u}}) + \frac{\gamma\widetilde{\nu}_t}{Pr_t}\mathbf{g}\cdot\nabla T + \frac{\gamma\nu_t}{Pr_t}\mathbf{g}\cdot\nabla\widetilde{T} = \bar{l}_{k1} \\
&(\widetilde{S}'_k - \widetilde{P}_k)(\beta^* k_{max}\omega - P_k) + (S'_k - P_k)(\beta^* k_{max}\widetilde{\omega} - \widetilde{P}_k) = \bar{l}_{k2} \\
&2r_{k2}\,\widetilde{r}_{k2} - \widetilde{S}'_k - \beta^* k_{max}\widetilde{\omega} + 2\widetilde{P}_k = \bar{l}_{k3} \\
&\widetilde{S}'_\omega\Big(S'_\omega - (S_\omega + S_{\omega,b})\Big) + S'_\omega\left(\widetilde{S}'_\omega - \eta\mathbf{S}(\mathbf{u}):\mathbf{S}(\widetilde{\mathbf{u}}) - \frac{\eta\gamma}{Pr_t}\mathbf{g}\cdot\nabla\widetilde{T}\right) = \bar{l}_{\omega 0} \qquad (57)
\end{aligned}$$

$$
\begin{aligned}
&2r_{\omega 1}\,\widetilde{r}_{\omega 1} - 2\widetilde{S}'_\omega + \eta \mathbf{S}(\mathbf{u}) : \mathbf{S}(\widetilde{\mathbf{u}}) + \frac{\eta\gamma}{Pr_t}\mathbf{g}\cdot\nabla\widetilde{T} = \bar{l}_{\omega 1}\\
&(\widetilde{S}'_\omega - \widetilde{P}_\omega)(\beta\omega^2_{max} - P_\omega) - \widetilde{P}_\omega(S'_\omega - P_\omega) = \bar{l}_{\omega 2}\\
&2r_{\omega 2}\,\widetilde{r}_{\omega 2} - \widetilde{S}'_\omega + 2\widetilde{P}_\omega = \bar{l}_{\omega 3}\\
&(\widetilde{k} - \widetilde{\nu}_t\omega - \nu_t\widetilde{\omega})(\nu_{max} - \nu_t) - (k - \nu_t\omega)\widetilde{\nu}_t = \bar{l}_{\nu 0}\\
&2r_\nu\widetilde{r}_\nu - \widetilde{k} + 2\widetilde{\nu}_t\omega + 2\nu_t\widetilde{\omega} - \widetilde{\omega}\nu_{max} = \bar{l}_{\nu 1}
\end{aligned}
$$

Consider the nonlinear operator $\mathbf{Q}' : \mathbf{B_3} \to \Re \times \mathbf{B_2}$, *where* $\mathbf{Q}'(\mathbf{z}_0)\cdot\widetilde{\mathbf{z}}_0 = (\bar{a}, \bar{\mathbf{b}})$ *for* $\bar{a} \in \Re$. *If we set*

$$
\begin{aligned}
\mathcal{J}'(\mathbf{u}, k, T_c)\cdot\widetilde{\mathbf{z}}_0 = {}&\alpha_u\int_{\Omega_d}(\mathbf{u}-\mathbf{u}_d)\cdot\widetilde{\mathbf{u}}d\mathbf{x} + \alpha_k\int_{\Omega_d}(k-k_d)\widetilde{k}d\mathbf{x}\\
&+\lambda_1\int_{\Gamma_C} T_c\widetilde{T}_c d\mathbf{x} + \lambda_2\int_{\Gamma_C}\nabla T_c\cdot\nabla\widetilde{T}_c d\mathbf{x}\,,
\end{aligned}
\tag{58}
$$

then the strict derivative of $\mathbf{Q}$ *at a point* $\mathbf{z}_0$ *is given by* $\mathbf{Q}'$ *if and only if*

$$
\begin{pmatrix} \mathcal{J}'(\mathbf{u}, k, T_c)\cdot\widetilde{\mathbf{z}}_0 \\ \mathbf{M}'(\mathbf{z}_0)\cdot\widetilde{\mathbf{z}}_0 \end{pmatrix} = \begin{pmatrix} \bar{a} \\ \bar{\mathbf{b}} \end{pmatrix}. \tag{59}
$$

Proof. The linearity and the boundedness of the operators $\mathbf{M}'$ and $\mathbf{Q}'$ follows from the continuity of the forms $a(\cdot;\cdot,\cdot)$, $b(\cdot,\cdot)$ and $c(\cdot;\cdot,\cdot)$ for both scalar and vector functions. The proof that $\mathbf{M}'$ is the strict derivative of the mapping $\mathbf{M}$ also follows from the continuity of the trilinear form $c(\cdot;\cdot,\cdot)$ and bilinear form $a(\cdot;\cdot,\cdot)$. The procedure is standard, and similar proofs have been reported in [19, 23]. Indeed, it can be proved that for a given $\mathbf{z}_0 = (\mathbf{u}, p, T, T_c, q_n, k, \omega, S'_k, r_{k1}, P_k, r_{k2}, S'_\omega, r_{\omega 1}, P_\omega, r_{\omega 2}, \nu_t, r_\nu) \in \mathbf{B}_1$, then for any $\varepsilon > 0$, and considering $\mathbf{z}_1, \mathbf{z}_2 \in \mathbf{B}_1$ such that, for an appropriate $\delta = \delta(\varepsilon)$, we have $\|\mathbf{z}_0 - \mathbf{z}_1\|_{\mathbf{B}_1} < \delta$ and $\|\mathbf{z}_0 - \mathbf{z}_2\|_{\mathbf{B}_1} < \delta$, we obtain

$$
\|\mathbf{M}(\mathbf{z}_1) - \mathbf{M}(\mathbf{z}_2) - \mathbf{M}'(\mathbf{z}_0)\cdot(\mathbf{z}_1 - \mathbf{z}_2)\|_{\mathbf{B}_2} \le \varepsilon\|\mathbf{z}_1 - \mathbf{z}_2\|_{\mathbf{B}_1}\,.
$$

This proves that the mapping $\mathbf{M}$ is strictly differentiable on all $\mathbf{B}_1$ and its strict derivative is given by $\mathbf{M}'$.

Using again standard techniques, it is easy to show that the mapping $\mathbf{Q}$ is strictly differentiable and that its strict derivative is given by $\mathbf{Q}'$ [19, 23]. ☐

We now recall the fact that the introduced variables r_{k1}, $r_{\omega 1}, r_{k2}$, $r_{\omega 2}$, and r_ν are equal to zero when the turbulence sources in k and ω satisfy both limits at the same time in all the relations (15)-(19). This may be a problem for the optimization if this is verified over domain with positive measure. However, this is not a problem if this happens over points or boundary regions with zero measure. For this reason we introduce the following subsets

$$
\Omega_{P_k} = \Big\{\mathbf{x}\in\Omega \text{ such that } S'_k = S_k + S_{k,b} = 0 \text{ or } P_k = \beta^* k_{max}\,\omega = S'_k\Big\}, \tag{60}
$$

$$
\Omega_{P_\omega} = \Big\{\mathbf{x}\in\Omega \text{ such that } S'_\omega = S_\omega + S_{\omega,b} = 0 \text{ or } P_\omega = \beta\omega^2_{max} = S'_\omega\Big\}, \tag{61}
$$

$$
\Omega_\nu = \Big\{\mathbf{x}\in\Omega \text{ such that } \nu_t = \nu_{max} = k/\omega\Big\}. \tag{62}
$$

These sets are used to assure the validity of the Lagrange multiplier technique around the region where the minimum point should be searched.

The differential operator $\mathbf{M}'$ is rather complex. Many equations in this operator are non-coercive elliptic equations with advection term driven by the velocity field $\mathbf{u} \in \mathbf{H}^1(\Omega)$. The existence result for this class of equations can be obtained not in the Lax-Milgram setting, but by using a Leray-Schauder Topological Degree argument. In order to deal with these equations, we introduce the following theorem.

Theorem 5. *Let $\Omega \subset \Re^n$ be a bounded open subset with boundary Γ. Let $\Gamma_D \subset \Gamma$ be a set with positive measure and $\Gamma_N \subseteq \Gamma \setminus \Gamma_D$. Consider*

$$
\begin{aligned}
-\nabla \cdot (A^T \nabla y) + (\mathbf{u} \cdot \nabla) y + by &= f \qquad && \text{in } \Omega \\
y &= y_1 \qquad && \text{on } \Gamma_D \\
A^T \nabla y \cdot \mathbf{n} &= y_n \qquad && \text{on } \Gamma_N \,,
\end{aligned} \tag{63}
$$

with $b \in L^{n_*/2}(\Omega)$, $b \geq 0$ a.e. on Ω, $\mathbf{u} \in \mathbf{L}^{n_*}(\Omega)$, and $f \in H^{1*}_{\Gamma_D}(\Omega)$ where $n_* = n$ when $n \geq 3$, $n_* \in]2, \infty[$ when $n = 2$. If A is a function which satisfies these two properties:

(1) $\exists\, \alpha_A > 0$ such that $A(x)\xi \cdot \xi \geq \alpha_A |\xi|^2$ for a.e. $x \in \Omega$ and for all $\xi \in \Re$;
(2) $\exists\, \Lambda_A > 0$ such that $|A(x)| \leq \Lambda_A$ for a.e. $x \in \Omega$;

then, there exists a unique solution $y \in H^1(\Omega)$ of (63).

Proof. The proof of this result is based on a Leray-Schauder Topological Degree argument and can be found in [24]. □

We remark that the Theorem 5 is also valid with only Dirichlet boundary conditions, namely when the boundary Γ_N is empty.

Lemma 2. *Let $\mathbf{z}_0 \in \mathbf{B_1}$. Then, if the region $\Omega_{P_k} \cup \Omega_{P_\omega} \cup \Omega_{S_\nu}$ has zero measure, we have that*

(i) the operator $\mathbf{M}'(\mathbf{z}_0)$ has closed range in $\mathbf{B_2}$,
(ii) the operator $\mathbf{Q}'(\mathbf{z}_0)$ has closed range in $\Re \times \mathbf{B_2}$,
(iii) the operator $\mathbf{Q}'(\mathbf{z}_0)$ is not onto in $\Re \times \mathbf{B_2}$.

Proof. In order to prove (i) we can split the range operator $\mathbf{M}'(z_0)$ in a product of range spaces for all its components and apply well known results. The operator range of $\mathbf{M}'$ can be split into four parts: the Navier-Stokes, the temperature, the k-ω model and the turbulence source constraint derivative equations. First, let us consider the Navier-Stokes derivative operator

$$
\begin{aligned}
& a(\nu + \nu_t; \widetilde{\mathbf{u}}, \mathbf{v}) + c(\widetilde{\mathbf{u}}; \mathbf{u}, \mathbf{v}) + c(\mathbf{u}; \widetilde{\mathbf{u}}, \mathbf{v}) + b(\mathbf{v}, \widetilde{p}) = (\bar{\mathbf{l}}_1^*, \mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega) \\
& (\bar{\mathbf{l}}_1^*, \mathbf{w}) = (\bar{\mathbf{l}}_1, \mathbf{w}) - (\gamma \mathbf{g} \widetilde{T}, \mathbf{w}) - a(\widetilde{\nu}_t; \mathbf{u}, \mathbf{w}) \qquad \forall \mathbf{w} \in \mathbf{H}^1_0(\Omega) \\
& b(\widetilde{\mathbf{u}}, q) = (\bar{l}_2, q) \qquad \forall\, q \in L^2_0(\Omega)
\end{aligned} \tag{64}
$$

with $\nu_t \in L^\infty(\Omega)$ and $\nu + \nu_t > 0$. The question of the closeness of the range $(\bar{\mathbf{l}}_1^*, \bar{l}_2)$ in $H^{-1}(\Omega) \times L^2_0(\Omega)$ of (64) is discussed in many papers, see for examples [25, 9, 1].

Since $\mathbf{z}_0$ is an optimal solution, $\widetilde{T}$ and $\widetilde{q}_n$ solve the equations

$$
\begin{aligned}
& a\left(\alpha + \frac{\nu_t}{Pr_t}; \widetilde{T}, \varphi\right) + c(\mathbf{u}; \widetilde{T}, \varphi) - (\widetilde{q}_n, \varphi)_{\Gamma_D} = (\bar{l}_3^*, \varphi) \qquad \forall \varphi \in H^1(\Omega) \\
& (\widetilde{T}, s_T)_{\Gamma_D} = (\bar{l}_4^*, s_T)_{\Gamma_D} \qquad \forall s_T \in H^{-1/2}(\Gamma_D)
\end{aligned} \tag{65}
$$

with

$$
\begin{aligned}
& (\bar{l}_3^*, \varphi) = (\bar{l}_3, \varphi) - a\left(\frac{\widetilde{\nu}_t}{Pr_t}; T, \varphi\right) - c(\widetilde{\mathbf{u}}; T, \varphi) \qquad \forall \varphi \in H^1(\Omega) \\
& (\bar{l}_4^*, s_T)_{\Gamma_D} = (\widetilde{T}_c, s_T)_{\Gamma_C} + (\bar{l}_4, s_T)_{\Gamma_D} \qquad \forall s_T \in H^{-1/2}(\Gamma_D)
\end{aligned} \tag{66}
$$

For $(\bar{l}_3, \bar{l}_4) \in H^{1*}(\Omega) \times H^{1/2}(\Gamma)$ we have $(\bar{l}_3^*, \bar{l}_4^*) \in H^{1*}(\Omega) \times H^{1/2}(\Gamma)$. By using the result in Theorem 5 for each $(\bar{l}_3^*, \bar{l}_4^*) \in H^{1*}(\Omega) \times H^{1/2}(\Gamma)$ we have a solution and therefore the range of the mapping $\mathbf{M}'(z_0)$ for the energy equation is onto.

Now we consider the k-ω system in $\mathbf{M}'$. Since $\mathbf{z}_0$ is an optimal solution, reduces to

$$\begin{aligned}
&a(\nu + \nu_t \sigma_k; \widetilde{k}, \phi) + c(\mathbf{u}; \widetilde{k}, \phi) + (\beta^* \omega \widetilde{k}, \phi) = (\bar{l}_5^* \phi) \qquad \forall \phi \in H_0^1(\Omega) \\
&(\bar{l}_5^* \phi) = (\bar{l}_5 \phi) - a(\widetilde{\nu}_t \sigma_k; k, \phi) - c(\widetilde{\mathbf{u}}; k, \phi) - (\beta^* \widetilde{\omega} k, \phi) + (\widetilde{P}_k, \phi) \quad \forall \phi \in H_0^1(\Omega) \\
&a(\nu + \nu_t \sigma_\omega; \widetilde{\omega}, \psi) + c(\mathbf{u}; \widetilde{\omega}, \psi) + (2\beta \omega \widetilde{\omega}, \psi) = (\bar{l}_6^* \psi) \qquad \forall \psi \in H_0^1(\Omega) \\
&(\bar{l}_6^* \psi) = (\bar{l}_6 \psi) - a(\widetilde{\nu}_t \sigma_\omega; \omega, \psi) - c(\widetilde{\mathbf{u}}; \omega, \psi) + (\widetilde{P}_\omega, \psi) \qquad \forall \psi \in H_0^1(\Omega)
\end{aligned} \tag{67}$$

with homogeneous Dirichlet boundary conditions. It is possible to show that $\widetilde{\omega}$-equation in (67) has a solution for all $\bar{l}_6^*$ and also that $\widetilde{k}$-equation can be solved for all $\bar{l}_4^*$. In fact, since $\nu + \nu_t$ is a positive function in $L^\infty(\Omega)$ and thanks to the Sobolev compact embeddings $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^q(\Omega)$ which holds for $1 \le q < \infty$ when $n = 2$ and for $1 \le q \le 6$ when $n = 3$, we have that $\mathbf{u} \in \mathbf{H}^1(\Omega)$ verifies the hypothesis in Theorem 5 both with $n = 2$ and with $n = 3$.

Finally, we focus on the system (57) under the assumption that $\mathbf{z}_0$ is an optimal solution. From this we have that $\mathbf{S}^2(\mathbf{u})$ is bounded and $\nu_t \in L^\infty(\Omega)$. If we assume that the region $\Omega_\nu \cup \Omega_{P_k} \cup \Omega_{P_\omega}$ has a measure zero then $r_\nu, r_{k1}, r_{k2}, r_{\omega 1}, r_{\omega 2}$ cannot be zero a.e. on the domain Ω. Therefore the equations can be solved a.e in Ω for all $\mathbf{l}_\nu = (l_{\nu 0}, l_{\nu 1}) \in L^2(\Omega) \times L^2(\Omega)$, $\mathbf{l}_k = (l_{k0}, l_{k1}, l_{k2}) \in L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ and $\mathbf{l}_\omega = (l_{\omega 0}, l_{\omega 1}, l_{\omega 2}) \in L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ as a function of $\widetilde{\nu}_t$, $\widetilde{r}_{\nu 1}$, $\widetilde{k}$, $\widetilde{r}_{k1}$, $\widetilde{r}_{k2}$ and $\widetilde{\omega}$, $\widetilde{r}_{\omega 1}$ and $\widetilde{r}_{\omega 2}$, respectively.

Starting from (i), the proof of (ii) and (iii) can be found easily by using the standard techniques in [25, 19, 9]. □

Theorem 6. *Let $\widehat{\mathbf{z}} \in \mathbf{B}_1$ denote an optimal solution. Then there exists a nonzero Lagrange multiplier $(\Lambda, \widehat{\mathbf{z}}_a) = (\Lambda, \widehat{\mathbf{u}}_a, \widehat{p}_a, \widehat{T}_a, \widehat{q}_a, \widehat{k}_a, \widehat{\omega}_a, \widehat{S'}_{ka}, \widehat{r}_{k1a}, \widehat{P}_{ka}, \widehat{r}_{k2a}, \widehat{S'}_{\omega a}, \widehat{P}_{\omega a}, \widehat{r}_{\omega 1a}, \widehat{r}_{\omega 2a}, \widehat{\nu}_a, \widehat{r}_{\nu a}) \in \Re \times \mathbf{B}_2^*$ satisfying the Euler equations*

$$\Lambda \mathcal{J}'(\widehat{\mathbf{u}}, \widehat{k}, \widehat{T}_c) \cdot \widetilde{\mathbf{z}}_0 + \langle \widehat{\mathbf{z}}_a, \mathbf{M}'(\widehat{\mathbf{z}}) \cdot \widetilde{\mathbf{z}}_0 \rangle = 0 \qquad \forall \widetilde{\mathbf{z}}_0 \in \mathbf{B}_3 \tag{68}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\mathbf{B}_2$ and $\mathbf{B}_2^*$.

Proof. From Lemma 2, we have that the range of $\mathbf{Q}'(\widehat{\mathbf{z}})$ is a closed, proper subspace of $\Re \times \mathbf{B}_2$. Then, from the Hahn-Banach theorem, there exists a nonzero element of $\Re \times \mathbf{B}_2^*$ that nullifies the range of $Q'(\widehat{\mathbf{z}})$. Then, there exists $(\Lambda, \widehat{\mathbf{u}}_a, \widehat{p}_a, \widehat{T}_a, \widehat{q}_a, \widehat{k}_a, \widehat{\omega}_a, \widehat{S'}_{ka}, \widehat{r}_{k1a}, \widehat{P}_{ka}, \widehat{r}_{k2a}, \widehat{S'}_{\omega a}, \widehat{r}_{\omega 1a}, \widehat{P}_{\omega a}, \widehat{r}_{\omega 2a}, \widehat{\nu}_a, \widehat{r}_{\nu a}) \in \Re \times \mathbf{B}_2^*$ such that

$$\langle (\bar{a}, \bar{\mathbf{b}}), (\Lambda, \widehat{\mathbf{z}}_a) \rangle = 0 \,, \tag{69}$$

for all $(\bar{a}, \bar{\mathbf{b}}) = (\bar{a}, \bar{\mathbf{l}}_1, \bar{l}_2, \bar{l}_3, \bar{l}_4, \bar{l}_5, \bar{l}_6, \bar{\mathbf{l}}_\nu, \bar{\mathbf{l}}_k, \bar{\mathbf{l}}_\omega)$ belonging to the range of $\mathbf{Q}'(\widehat{\mathbf{z}})$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\Re \times \mathbf{B}_2$ and $\Re \times \mathbf{B}_2^*$. Note that $\Lambda \neq 0$ since otherwise we would have that $\langle \bar{\mathbf{b}}, \widehat{\mathbf{z}}_a \rangle = 0$ for all $\bar{\mathbf{b}} \in \mathbf{B}_2$. This would imply $\widehat{\mathbf{z}}_a = 0$ contradicting the fact that $(\Lambda, \widehat{\mathbf{z}}_a) \neq 0$. Clearly, using the definition of $\mathbf{Q}'(\widehat{\mathbf{z}})$, (68) and (69) are equivalent. □

4.3. The optimality system. Dropping the $\widehat{(\cdot)}$ notation for optimal solution, we derive now the optimality system using (68). Thus, we introduce the following equations

$$\lambda_1 (T_c, \widetilde{T}_c)_{\Gamma_C} + \lambda_2 (\nabla T_c, \nabla \widetilde{T}_c)_{\Gamma_C} = (q_a, \widetilde{T}_c)_{\Gamma_C} \,, \tag{70}$$

for all $\widetilde{T}_c \in H^1_0(\Gamma_C)$,

$$
(71)\quad
\begin{aligned}
&b(\mathbf{u}_a, \widetilde{p}) = 0\,,\\
&a(\nu+\nu_t; \widetilde{\mathbf{u}}, \mathbf{u}_a) + c(\mathbf{u}; \widetilde{\mathbf{u}}, \mathbf{u}_a) + c(\widetilde{\mathbf{u}}; \mathbf{u}, \mathbf{u}_a) + b(\widetilde{\mathbf{u}}, p_a) =\\
&\qquad = -\alpha_u \Lambda((\mathbf{u}-\mathbf{u}_d), \widetilde{\mathbf{u}})_{\Omega_d} - c(\widetilde{\mathbf{u}}; T, T_a) - c(\widetilde{\mathbf{u}}; k, k_a) - c(\widetilde{\mathbf{u}}; \omega, \omega_a)\\
&\qquad + a(\nu_t(r_{k1a} - S'_{ka}S'_k) + \eta(r_{\omega 1a} - S'_{\omega a}S'_\omega); \mathbf{u}, \widetilde{\mathbf{u}})\,,
\end{aligned}
$$

for all $(\widetilde{\mathbf{u}}, \widetilde{p}) \in \mathbf{H}^1_0(\Omega) \times L^2_0(\Omega)$,

$$
(72)\quad
\begin{aligned}
&a(\alpha + \frac{\nu_t}{Pr_t}; \widetilde{T}, T_a) + c(\mathbf{u}; \widetilde{T}, T_a) + (\widetilde{T}, q_a)_{\Gamma_D} = -(\gamma \mathbf{g}\widetilde{T}, \mathbf{u}_a)\\
&\qquad + \left(\frac{\gamma}{Pr_t}\mathbf{g}\cdot\nabla\widetilde{T}, \nu_t(r_{k1a} - S'_{ka}S'_k) + \eta(r_{\omega 1a} - S'_{\omega a}S'_\omega)\right)\\
&(T_a, \widetilde{q}_n)_{\Gamma_D} = 0\,,
\end{aligned}
$$

with

$$
(73)\qquad q_a = -\left(\alpha + \frac{\nu_t}{Pr_t}\right)\nabla T_a\cdot\widehat{\mathbf{n}} \quad \text{on } \Gamma_D\,,
$$

for all $(\widetilde{T}_{\Gamma_D \setminus \Gamma_C}, \widetilde{q}_n) \in H^1(\Omega)\times H^{-\frac{1}{2}}(\Gamma_D)$,

$$
(74)\quad
\begin{aligned}
&a(\nu+\nu_t\sigma_k; \widetilde{k}, k_a) + c(\mathbf{u}; \widetilde{k}, k_a) + (\beta^*\widetilde{k}\omega, k_a) = -\alpha_k\Lambda((k-k_d), \widetilde{k})_{\Omega_d}\\
&\qquad - (\widetilde{k}, \nu_a(\nu_{max} - \nu_t) - r_{\nu a})\,,\\
&a(\nu+\nu_t\sigma_\omega; \widetilde{\omega}, \omega_a) + c(\mathbf{u}; \widetilde{\omega}, \omega_a) + (2\beta\omega\widetilde{\omega}, \omega_a) =\\
&\qquad - (\beta^* k\widetilde{\omega}, k_a) - (P_{ka}(S'_k - P_k) - r_{k2a}, \beta^* k_{max}\widetilde{\omega})\\
&\qquad + (\nu_t\nu_a(\nu_{max} - \nu_t) - r_{\nu a}(2\nu_t - \nu_{max}), \widetilde{\omega})\,,
\end{aligned}
$$

for all $(\widetilde{k}, \widetilde{\omega}) \in H^1_0(\Omega)\times H^1_0(\Omega)$. We also introduce the algebraic system

$$
(75)\quad
\begin{aligned}
&\nu_a\omega\left(\nu_{max} + \frac{k}{\omega} - 2\nu_t\right) = \frac{\mathbf{S}(\mathbf{u}):\mathbf{S}(\mathbf{u}_a)}{2} + \frac{\nabla T\cdot\nabla T_a}{Pr_t} + \sigma_k\nabla k\cdot\nabla k_a\\
&\qquad + \sigma_\omega\nabla\omega\cdot\nabla\omega_a - (S'_{ka}S'_k - r_{k1a})\left(\frac{1}{2}\mathbf{S}^2(\mathbf{u}) + \frac{\gamma}{Pr_t}\mathbf{g}\cdot\nabla T\right) + 2r_{\nu a}\omega\,,\\
&S'_{ka}\left(2S'_k - \frac{1}{2}\nu_t\mathbf{S}^2(\mathbf{u}) - \frac{\gamma\nu_t}{Pr_t}\mathbf{g}\cdot\nabla T\right) = 2r_{k1a} + r_{k2a} - P_{ka}(\beta^* k_{max}\omega - P_k)\,,\\
&S'_{\omega a}\left(2S'_\omega - \frac{1}{2}\eta\mathbf{S}^2(\mathbf{u}) - \frac{\eta\gamma}{Pr_t}\mathbf{g}\cdot\nabla T\right) = 2r_{\omega 1a} + r_{\omega 2a} - P_{\omega a}(\beta\omega^2_{max} - P_\omega)\,,\\
&P_{ka}(\beta^* k_{max}\omega + S'_k - 2P_k) = 2r_{k2a} - k_a\,,\\
&P_{\omega a}(\beta\omega^2_{max} + S'_\omega - 2P_\omega) = 2r_{\omega 2a} - \omega_a\,.
\end{aligned}
$$

Lastly, we have

$$
(76)\qquad r_{k1a}r_{k1} = r_{k2a}r_{k2} = r_{\omega 1a}r_{\omega 1} = r_{\omega 2a}r_{\omega 2} = r_{\nu a}r_\nu = 0\,.
$$

Theorem 7. *Let $\mathbf{z}\in\mathbf{B}_1$ denote a solution of the optimal control problem. Then, if the region $\Omega_{P_k}\cup\Omega_{P_\omega}\cup\Omega_\nu$ has zero measure, the control variable $T_c\in H^1_0(\Gamma_C)$ is the solution of* (70).

Also, $(\mathbf{u}_a, p_a) \in \mathbf{H}^1_0(\Omega)\times L^2_0(\Omega)$ is solution of (71). *In addition, $(T_a, q_a) \in H^1(\Omega)\times H^{-\frac{1}{2}}(\Gamma_D)$ is the solution of* (72) *under the condition* (73). *Also, $(k_a, \omega_a) \in H^1_0(\Omega)\times H^1_0(\Omega)$ is solution of* (74).

Moreover, $(\nu_a, S'_{ka}, S'_{\omega a}, P_{ka}, P_{\omega a}) \in (L^2(\Omega))^5$ are solutions of the algebraic equations (75), *and $(r_{k1a}, r_{k2a}, r_{\omega 1a}, r_{\omega 2a}, r_{\nu a}) \in (L^2(\Omega))^5$ satisfy* (76).

Proof. The Euler equations (68) are equivalent to

$$
\begin{aligned}
&\Lambda(\alpha_u((\mathbf{u}-\mathbf{u}_d),\widetilde{\mathbf{u}})_{\Omega_d}+\alpha_k((k-k_d),\widetilde{k})_{\Omega_d}+\lambda_1(T_c,\widetilde{T}_c)_{\Gamma_C}+\lambda_2(\nabla T_c,\nabla\widetilde{T}_c)_{\Gamma_C})\\
&+a(\widetilde{\nu}_t;\mathbf{u},\mathbf{u}_a)+a(\nu+\nu_t;\widetilde{\mathbf{u}},\mathbf{u}_a)+c(\widetilde{\mathbf{u}};\mathbf{u},\mathbf{u}_a)+c(\mathbf{u};\widetilde{\mathbf{u}},\mathbf{u}_a)\\
&+b(\mathbf{u}_a,\widetilde{p})+(\gamma\mathbf{g}\widetilde{T},\mathbf{u}_a)+b(\widetilde{\mathbf{u}},p_a)+c(\widetilde{\mathbf{u}};T,T_a)+c(\mathbf{u};\widetilde{T},T_a)\\
&+a\left(\frac{\widetilde{\nu}_t}{Pr_t};T,T_a\right)+a\left(\alpha+\frac{\nu_t}{Pr_t};\widetilde{T},T_a\right)-(\widetilde{q}_n,T_a)_{\Gamma_D}+(\widetilde{T},q_a)_{\Gamma_D}-(\widetilde{T}_c,q_a)_{\Gamma_C}\\
&+c(\widetilde{\mathbf{u}};k,k_a)+c(\mathbf{u};\widetilde{k},k_a)+a(\widetilde{\nu}_t\sigma_k;k,k_a)+a(\nu+\nu_t\sigma_k;\widetilde{k},k_a)-(\widetilde{P}_k,k_a)\\
&+(\beta^*\widetilde{k}\omega,k_a)+(\beta^*k\widetilde{\omega},k_a)+c(\widetilde{\mathbf{u}};\omega,\omega_a)+c(\mathbf{u};\widetilde{\omega},\omega_a)\\
&+a(\widetilde{\nu}_t\sigma_\omega;\omega,\omega_a)+a(\nu+\nu_t\sigma_\omega;\widetilde{\omega},\omega_a)-(\widetilde{P}_\omega,\omega_a)+2(\beta\omega\widetilde{\omega},\omega_a)\\
&+\left(S'_{ka},\widetilde{S}'_k\left(S'_k-\frac{1}{2}\nu_t\mathbf{S}^2(\mathbf{u})-\frac{\gamma\nu_t}{Pr_t}\mathbf{g}\cdot\nabla T\right)+S'_k\left(\widetilde{S}'_k-\frac{1}{2}\widetilde{\nu}_t\mathbf{S}^2(\mathbf{u})-\nu_t\mathbf{S}(\mathbf{u}):\mathbf{S}(\widetilde{\mathbf{u}})\right.\right.\\
&\left.\left.-\frac{\gamma\widetilde{\nu}_t}{Pr_t}\mathbf{g}\cdot\nabla T-\frac{\gamma\nu_t}{Pr_t}\mathbf{g}\cdot\nabla\widetilde{T}\right)\right)+\left(r_{k1a},2r_{k1}\widetilde{r}_{k1}-2\widetilde{S}'_k+\frac{1}{2}\widetilde{\nu}_t\mathbf{S}^2(\mathbf{u})+\nu_t\mathbf{S}(\mathbf{u}):\mathbf{S}(\widetilde{\mathbf{u}})\right.\\
&\left.+\frac{\gamma\widetilde{\nu}_t}{Pr_t}\mathbf{g}\cdot\nabla T+\frac{\gamma\nu_t}{Pr_t}\mathbf{g}\cdot\nabla\widetilde{T}\right)+(P_{ka},(\widetilde{S}'_k-\widetilde{P}_k)(\beta^*k_{max}\omega-P_k)\\
&+(S'_k-P_k)(\beta^*k_{max}\widetilde{\omega}-\widetilde{P}_k))+(r_{k2a},2r_{k2}\widetilde{r}_{k2}-\widetilde{S}'_k-\beta^*k_{max}\widetilde{\omega}+2\widetilde{P}_k)\\
&+\left(S'_{\omega a},\widetilde{S}'_\omega\left(S'_\omega-\frac{1}{2}\eta\mathbf{S}^2(\mathbf{u})-\frac{\eta\gamma}{Pr_t}\mathbf{g}\cdot\nabla T\right)+S'_\omega\left(\widetilde{S}'_\omega-\eta\mathbf{S}(\mathbf{u}):\mathbf{S}(\widetilde{\mathbf{u}})\right.\right.\\
&\left.\left.-\frac{\eta\gamma}{Pr_t}\mathbf{g}\cdot\nabla\widetilde{T}\right)\right)+\left(r_{\omega1a},2r_{\omega1}\widetilde{r}_{\omega1}-2\widetilde{S}'_\omega+\eta\mathbf{S}(\mathbf{u}):\mathbf{S}(\widetilde{\mathbf{u}})+\frac{\eta\gamma}{Pr_t}\mathbf{g}\cdot\nabla\widetilde{T}\right)\\
&+(P_{\omega a},(\widetilde{S}'_\omega-\widetilde{P}_\omega)(\beta\omega^2_{max}-P_\omega)-\widetilde{P}_\omega(S'_\omega-P_\omega))+(r_{\omega2a},2r_{\omega2}\widetilde{r}_{\omega2}-\widetilde{S}'_\omega+2\widetilde{P}_\omega)\\
&+(\nu_a,(\widetilde{k}-\widetilde{\nu}_t\omega-\nu_t\widetilde{\omega})(\nu_{max}-\nu_t)-(k-\nu_t\omega)\widetilde{\nu}_t)+(r_{\nu a},2r_\nu\widetilde{r}_\nu-\widetilde{k}\\
&+2\widetilde{\nu}_t\omega+2\nu_t\widetilde{\omega}-\widetilde{\omega}\nu_{max})=0\,,
\end{aligned}
$$

for all $\mathbf{z}\in\mathbf{B}_1$. In order to satisfy the integral on the boundary, we set homogeneous Dirichlet boundary conditions for the adjoint variables $(\mathbf{u}_a,k_a,\omega_a)$. By extracting the terms involved in the same variation, we obtain (70)-(76). ☐

If the region $\Omega_{P_k}\cup\Omega_{P_\omega}\cup\Omega_{S_\nu}$ has zero measure, then $r_{k1},r_{k2},r_{\omega1},r_{\omega2}$ and r_ν are almost everywhere different from zero. From (76) we note that if $r_{k1}\neq 0$, then $r_{k1a}=0$. This is true also for $r_{k2a},r_{\omega1a},r_{\omega2a}$ and $r_{\nu a}$. Therefore, the final adjoint system reduces to

$$
\begin{aligned}
&b(\mathbf{u}_a,\widetilde{p})=0\,,\\
&a(\nu+\nu_t;\widetilde{\mathbf{u}},\mathbf{u}_a)+c(\mathbf{u};\widetilde{\mathbf{u}},\mathbf{u}_a)+c(\widetilde{\mathbf{u}};\mathbf{u},\mathbf{u}_a)+b(\widetilde{\mathbf{u}},p_a)=\\
&\quad=-\alpha_u\Lambda((\mathbf{u}-\mathbf{u}_d),\widetilde{\mathbf{u}})_{\Omega_d}-c(\widetilde{\mathbf{u}};T,T_a)-c(\widetilde{\mathbf{u}};k,k_a)-c(\widetilde{\mathbf{u}};\omega,\omega_a)\\
&\quad-a(\nu_tS'_{ka}S'_k+\eta S'_{\omega a}S'_\omega;\mathbf{u},\widetilde{\mathbf{u}})\,,
\end{aligned}
\tag{77}
$$

for all $(\widetilde{\mathbf{u}},\widetilde{p})\in\mathbf{H}^1_0(\Omega)\times L^2_0(\Omega)$,

$$
\begin{aligned}
&a(\alpha+\frac{\nu_t}{Pr_t};\widetilde{T},T_a)+c(\mathbf{u};\widetilde{T},T_a)+(\widetilde{T},q_a)_{\Gamma_D}=-(\gamma\mathbf{g}\widetilde{T},\mathbf{u}_a)\\
&\quad-\left(\frac{\gamma}{Pr_t}\mathbf{g}\cdot\nabla\widetilde{T},\nu_tS'_{ka}S'_k+\eta S'_{\omega a}S'_\omega\right)
\end{aligned}
\tag{78}
$$

for all $\widetilde{T}_{\Gamma_D \setminus \Gamma_C} \in H^1(\Omega)$,

$$
\begin{aligned}
&a(\nu + \nu_t\sigma_k; \widetilde{k}, k_a) + c(\mathbf{u}; \widetilde{k}, k_a) + (\beta^* \widetilde{k}\omega, k_a) = -\alpha_k((k - k_d), \widetilde{k})_{\Omega_d} \\
&\quad - (\widetilde{k}, \nu_a(\nu_{max} - \nu_t)) \,, \\
&a(\nu + \nu_t\sigma_\omega; \widetilde{\omega}, \omega_a) + c(\mathbf{u}; \widetilde{\omega}, \omega_a) + (2\beta\omega\widetilde{\omega}, \omega_a) = -(\beta^* k\widetilde{\omega}, k_a) \\
&\quad - (P_{ka}(S'_k - P_k), \beta^* k_{max}\widetilde{\omega}) + (\nu_t \nu_a(\nu_{max} - \nu_t), \widetilde{\omega}) \,,
\end{aligned} \tag{79}
$$

for all $(\widetilde{k}, \widetilde{\omega}) \in H_0^1(\Omega) \times H_0^1(\Omega)$. Lastly, in the case in which $\Omega_{P_k} \cup \Omega_{P_\omega} \cup \Omega_{S_\nu}$ has zero measure, we have the following algebraic equations

$$
\begin{aligned}
\nu_a r_\nu^2 &= \frac{\mathbf{S}(\mathbf{u}) : \mathbf{S}(\mathbf{u}_a)}{2} + \frac{\nabla T \cdot \nabla T_a}{Pr_t} + \sigma_k \nabla k \cdot \nabla k_a \\
&\quad + \sigma_\omega \nabla\omega \cdot \nabla\omega_a - S'_{ka} S'_k \left(\frac{1}{2}\mathbf{S}^2(\mathbf{u}) + \frac{\gamma}{Pr_t}\mathbf{g} \cdot \nabla T \right) , \\
S'_{ka} r_{k1}^2 &= -P_{ka}(\beta^* k_{max}\omega - P_k) \,, \\
S'_{\omega a} r_{\omega 1}^2 &= -P_{\omega a}(\beta\omega_{max}^2 - P_\omega) \,, \\
P_{ka} r_{k2}^2 &= -k_a \,, \\
P_{\omega a} r_{\omega 2}^2 &= -\omega_a \,.
\end{aligned} \tag{80}
$$

Furthermore, in the case in which no bounds are reached, we have

$$
\nu_t = \frac{k}{\omega} \,, \quad P_k = S'_k = S_k + S_{kb} \,, \quad P_\omega = S'_\omega + S_{\omega b} \,, \tag{81}
$$

then the adjoint system (77)-(80) simplifies and in particular

$$
S'_{ka} S'_k = k_a \,, \quad S'_{\omega a} S'_\omega = \omega_a \,. \tag{82}
$$

5. Numerical Results

In this section, we report the results obtained by solving the optimality system (70) and (77)-(80). Since the coupled solution of the system is extremely expensive, we uncouple the state, adjoint and control equations by using the steepest descent algorithm described in Algorithm 1. In particular, a standard line search with backtracking strategy is performed [26]. When the functional decreases under a certain tolerance ε_{opt}, the optimal solution is found and the algorithm stops. We set $\varepsilon_{opt} = 10^{-6}$ for all tests in this section.

Algorithm 1 Description of the Steepest Descent algorithm.

1. Set $T_c^0 = 0$ and find a state $(\mathbf{u}^0, p^0, T^0, k^0, \omega^0, \nu_t^0)$ satisfying (26) and (27)
2. Compute the functional $\mathcal{J}^0$ in (20)
3. Set $r^0 = 1$

for $i = 1 \to i_{max}$ **do**
    4. Solve the system (77)-(80) to obtain the adjoint state $(\mathbf{u}_a^i, p_a^i, T_a^i, k_a^i, \omega_a^i, \nu_a^i)$
    5. Solve control equation (70) to obtain T_c^i
    6. Set $r^i = r^0$
    while $\mathcal{J}^i(T^{i-1}|_{\Gamma_C} - r^i T_c^i) > \mathcal{J}^{i-1}(T^{i-1}|_{\Gamma_C})$ **do** ▷ *Line search*
        7. Set $r^i = \rho\, r^i$
        8. Solve (26)-(27) for the state $(\mathbf{u}^i, p^i, T^i, k^i, \omega^i, \nu_t^i)$ with $T^i|_{\Gamma_C} = T^{i-1}|_{\Gamma_C} - r^i T_c^i$
        if $r^i < \varepsilon_r$ **then**
            Line search not successful ▷ *Unsuccessful end of algorithm*
        end if
        if $\big((\mathcal{J}^i(T^i|_{\Gamma_C}) - \mathcal{J}^i(T^{i-1}|_{\Gamma_C})\big)/\mathcal{J}^i(T^{i-1}|_{\Gamma_C}) < \varepsilon_{opt}$ **then**
            Optimal solution found ▷ *Successful end of algorithm*
        end if
    end while
end for

We study a two-dimensional cavity where the flow is driven by buoyancy forces. Let us consider the domain $\Omega = [0, L] \times [0, L] \in \Re^2$ reported in Figure 1. In our computations we consider $L = 0.01m$. Let be $\Gamma_D = \Gamma_1 \cup \Gamma_3$, $\Gamma_C = \Gamma_1$ and $\Gamma_N = \Gamma_2 \cup \Gamma_4$. We set $\mathbf{f} = \mathbf{0}$, $\mathbf{g}_u = \mathbf{0}$ and $g_{T,N} = 0$, while $g_k = a_1 \delta^2$ and $g_\omega = 2\nu/\beta^* \delta^2$ where δ is the distance from the wall. Moreover, the function g_T on Γ_D is given as

$$g_T = \begin{cases} 493\ K \text{ on } \Gamma_3 \\ 503\ K \text{ on } \Gamma_1\,. \end{cases} \tag{83}$$

For the reference case, we set $T_c = 0$ then on $\Gamma_C = \Gamma_3$ we have $T = g_T$.

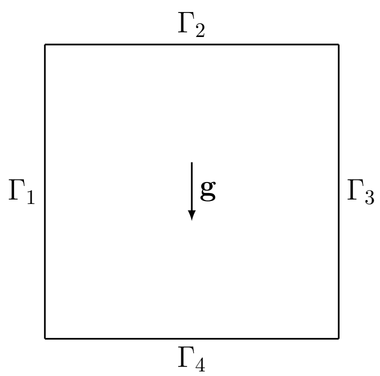

FIGURE 1. Computational domain $\Omega \in \Re^2$.

Since the presented optimal control problem allows controlling the velocity $\mathbf{u}_d$ and the turbulence kinetic energy k_d on Ω_d, we now consider two different control cases. In particular, we report a velocity matching case, and a turbulent kinetic energy enhancement case. In both cases we consider $\gamma = 2.5684 \cdot 10^{-4} K^{-1}$,

$Pr = 0.025$ and a kinematic viscosity $\nu = 1.750 \cdot 10^{-7} m^2/s$. We discretize the numerical problem in a finite element framework, and we consider a 30×30 uniform quadrangular mesh.

Velocity matching case. We first consider a velocity matching case, imposing $\alpha_u = 1$ and $\alpha_k = 0$ in (20). We set $\Omega_d = [0.2y^+, 0.3y^+] \times [0.4y^+, 0.6y^+]$, where $y^+ = y/L$. Moreover, considering $\mathbf{u}_d = (u_d, v_d)$ we aim to control the vertical component of the velocity $v_d = 8 \cdot 10^{-3} m/s$. In the reference case, the average vertical component of the velocity on Ω_d is $v_{d,ref} = 5.689 \cdot 10^{-3} m/s$, and the distance from the objective $\int_{\Omega_d} (\mathbf{u} - \mathbf{u}_d)^2 d\mathbf{x}$ assumes the value $5.8518 \cdot 10^{-12}$.

TABLE 1. Velocity matching case: objective functional and number of iterations of the optimization algorithm for different λ_1 and λ_2 values.

$10^{12} \cdot \int_{\Omega_d} (\mathbf{u} - \mathbf{u}_d)^2 d\mathbf{x}$ (iterations)

$\lambda_1 \backslash \lambda_2$	10^{-7}	10^{-8}	10^{-9}	10^{-10}	0
10^{-4}	3.0859 (117)	3.1189 (88)	3.2028 (53)	3.1681 (37)	3.0762 (34)
10^{-5}	3.0842 (10)	3.1023 (7)	3.1210 (5)	2.9488 (3)	2.6664 (2)
10^{-6}	1.5356 (3)	1.5304 (3)	1.4946 (3)	1.4160 (4)	1.3499 (4)
10^{-7}	1.4873 (4)	1.4942 (4)	1.5482 (3)	1.4322 (4)	1.3618 (4)
10^{-8}	1.4884 (5)	1.5019 (4)	1.5038 (3)	1.4576 (3)	1.3363 (4)

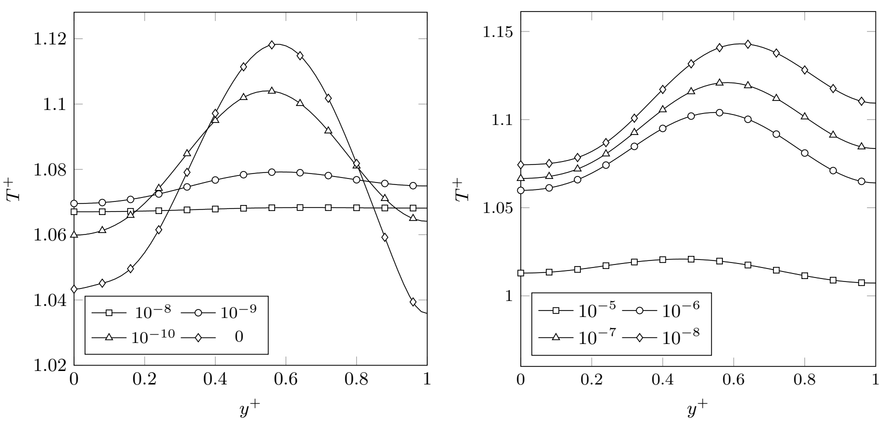

FIGURE 2. Velocity matching case: on the left temperature profiles on Γ_C for $\lambda_1 = 10^{-6}$ and $\lambda_2 = 10^{-8}$, 10^{-9}, 10^{-10}, 0; on the right temperature profiles on Γ_C for $\lambda_2 = 10^{-10}$ and $\lambda_1 = 10^{-5}$, 10^{-6}, 10^{-7}, 10^{-8}.

A brief analysis of the dependence on the regularization parameters of the implemented numerical algorithm is now reported. In particular, in Table 1 the values of the distance from the objective $\int_{\Omega_d} (\mathbf{u} - \mathbf{u}_d)^2 d\mathbf{x}$ are reported for different values of λ_1 and λ_2. The number of iterations needed by the implemented numerical algorithm is also reported in Table. We observe that for larger values of the penalty parameters λ_1 and λ_2, the number of iterations increases. However, in the standard control cases, the number of iterations is supposed to decrease for larger penalty parameters λ_1 and λ_2. This behavior concerns the numerical simulations where the optimality system is solved in a fully coupled fashion or when it is solved with an iterative scheme and a minimum value of the functional $\mathcal{J}$ is found for each iteration of the internal cycle (note that the “internal cycle” is the loop solving the state system, and the “external cycle” is the loop solving the adjoint equations, see Algorithm 1). However, the Steepest Descent algorithm used in our work is based on the exit condition $\mathcal{J}^i(T^{i-1}|_{\Gamma_C} - r^i T_c^i) < \mathcal{J}^{i-1}(T^{i-1}|_{\Gamma_C})$ for the internal cycle. Then, when this condition is satisfied, we solve the adjoint system in the external cycle to find a new minimization direction. Thus, we do not find the minimum of the functional for each loop of the internal cycle. For this reason, a general behavior based on the penalty parameters λ_1 and λ_2 cannot be established in the presented algorithm.

The effects of the regularization parameter λ_1 on the distance from the objective are negligible, with the exception of the cases with $\lambda_1 = 10^{-5}$ and $\lambda_1 = 10^{-4}$. In fact, for high values of the regularization parameter λ_1, the control is less effective, and the number of iterations needed by the algorithm increases. In particular, for $\lambda_1 = 10^{-4}$ the algorithm needs a significantly higher number of iterations to find the optimal solution. The variable λ_2 has a small impact on the results, however, the distance from the objective tends to decrease as the value of λ_2 decreases. More generally, we can assert that low values of the regularization parameters lead to a smaller distance from the objective. Note that in the presented tests the ratio between the distance from the objective of the reference solution and the optimal one is smaller than 10. This is due to the fact that the average reference velocity on Ω_d reported above is close to the desired one.

We now define the non-dimensional temperature field $T^+ = T/T_{ref}$, where T_{ref} is the temperature on Γ_C in the reference case. We consider $T_{ref} = 503K$. In Figure 2 we report T^+ on Γ_C for different values of the regularization parameter λ_2 (on the left) and λ_1 (on the right). Note that the choice of the regularization parameters affects the optimal solution on Γ_C. In particular, high values of the regularization parameters lead to flat profiles of the controlled temperature, as expected.

Turbulence kinetic energy enhancement case. To test the optimal control solver with $\alpha_u = 0$ and $\alpha_k = 1$ in (20) we consider a turbulence enhancement problem which consists in increasing the turbulent kinetic energy k. We set $k_d = 5.0 \cdot 10^{-4}\ m^2/s^2$ since in the reference case the turbulent kinetic energy is everywhere smaller than this value, in particular $k^0_{max} = 1.5 \cdot 10^{-5}\ m^2/s^2$. Let $\Omega_d = [0.45y^+, 0.55y^+] \times [0.45y^+, 0.55y^+]$ be the region where we aim to minimize the functional $\int_{\Omega_d} (k - k_d)^2 d\mathbf{x}$, which assumes the value $1.3338 \cdot 10^{-10}$ in the reference case. We solve the optimal control problem with $\lambda_1 = 10^{-4}$, 10^{-5}, 10^{-6}, 10^{-7} and $\lambda_2 = 10^{-9}$, 10^{-10}, 10^{-11}, 0. In Table 2 the objective functional and the number of algorithm iterations are reported for the different λ_1 and λ_2 values. The functional appears strongly decreased in all the considered cases. The decrease in the functional ranges from two orders of magnitude to four orders of magnitude. The test cases with a higher λ_1 value, in particular with $\lambda_1 = 10^{-4}$ and 10^{-5}, are characterized by a higher number of iterations to find the optimum.

In Figure 3 non-dimensional temperature T^+ profiles on the controlled boundary Γ_C are reported for different values of λ_1 and λ_2. In the left of Figure 3, the influence of the penalization coefficient λ_2 is shown. The plot illustrates the temperature profiles obtained with the optimization algorithm for $\lambda_1 = 10^{-6}$ and $\lambda_2 = 10^{-9}$, 10^{-10}, 10^{-11}, 0. The Figure evidences that the highest considered value of λ_2

Table 2. Turbulent kinetic energy enhancement case: objective functional and number of iterations of the optimization algorithm for different λ_1 and λ_2 values.

$10^{17} \cdot \int_{\Omega_d} (k - k_d)^2 d\mathbf{x}$ (iterations)

$\lambda_1 \backslash \lambda_2$	10^{-9}	10^{-10}	10^{-11}	0
10^{-4}	4.7800 (309)	19.314 (348)	21.211 (316)	21.642(307)
10^{-5}	13.228 (33)	19.950 (37)	21.404 (34)	23.679 (33)
10^{-6}	68.757 (5)	21.113 (6)	38.872 (5)	129.83 (5)
10^{-7}	179.87 (3)	241.72 (3)	114.29 (3)	17.216 (3)

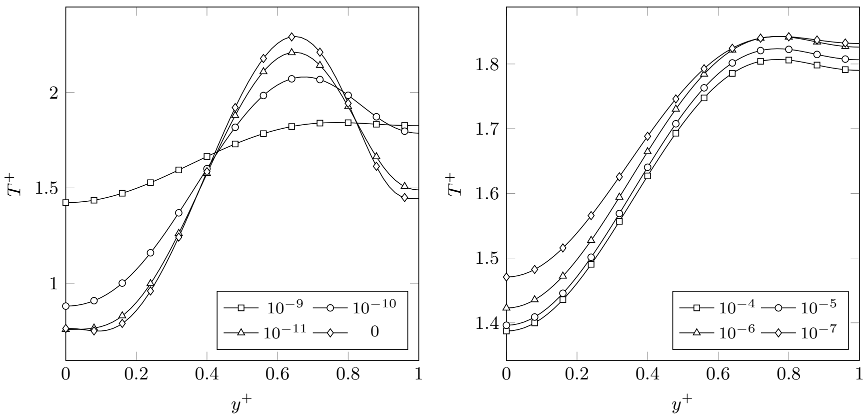

Figure 3. Turbulent kinetic energy enhancement case: on the left temperature profiles on Γ_C for $\lambda_1 = 10^{-6}$ and $\lambda_2 = 10^{-9}$, 10^{-10}, 10^{-11}, 0; on the right temperature profiles on Γ_C for $\lambda_2 = 10^{-9}$ and $\lambda_1 = 10^{-4}$, 10^{-5}, 10^{-6}, 10^{-7}.

brings to the smoothest temperature profile on Γ_C, while with low values of λ_2 the optimal solution is less regular and close to the solution obtained with $\lambda_2 = 0$. On the right of Figure 3 we aim to show the influence of the penalization coefficient λ_1. The results with $\lambda_2 = 10^{-9}$ and different values of λ_1, i.e. $\lambda_1 = 10^{-4}$, 10^{-5}, 10^{-6}, 10^{-7}, are reported. The temperature profiles are characterized by similar behavior in the cases with fixed λ_2. When λ_1 is small the boundary temperature assumes higher values, but the profiles have a similar trend. The penalization coefficient λ_2 in this test case influences strongly the optimal solution, while the coefficient λ_1 plays a minor role.

6. Conclusion

In this work, the analysis of a boundary optimal control problem for the Reynolds Averaged Navier-Stokes and energy system coupled with a two-equation turbulence model in a k-ω formulation has been presented. In particular, by starting from the existence of the solution of the Navier-Stokes system coupled with the energy equation, and the existence of the solution of the k-ω turbulence model, we have proved the existence of the coupled associated boundary value problem. In order to do so, we have introduced some bounds on P_k, P_ω and ν_t. Moreover, we have introduced a boundary optimal control problem to obtain a desired velocity and/or a desired turbulence kinetic energy on a domain Ω_d, by controlling the temperature on a boundary Γ_C. The optimal control system has been obtained through the Lagrange multiplier method. In particular, we have proved that the Lagrange multiplier technique is well-posed and can be used to obtain the first-order necessary condition.

Lastly, we have introduced a numerical Steepest Descent algorithm for the numerical implementation of the proposed optimality system in a FEM framework. Then, some numerical results have been shown, presenting both velocity matching and turbulent kinetic energy enhancement cases. In particular, the dependence on the regularization parameters has been analyzed in order to show consistency with the expectations.

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