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An Intrinsic Formulation of the von Kármán Equations

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Abstract. To begin with, we identify the intrinsic equations of a von Kármán nonlinearly elastic plate, which allow to directly compute the stresses inside the plate without having to first compute the displacement field, by contrast with the classical displacement approach. Then we establish that these intrinsic equations possess weak solutions, which are the bending moments and stress resultants of the middle surface of the plate.

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1 Introduction

When an elastic plate is subjected to specific boundary conditions and applied forces, its reference configuration, i.e., the portion of space it occupies in the absence of forces, becomes a deformed configuration. One of the central themes of plate theory then consists in providing equations that allow to determine the displacement vector at each point of the reference configuration. The unknown is thus a vector field defined over the reference configuration, whose components are those of the unknown displacement field. These equations originally took the

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form of boundary value problems, i.e., partial differential equations on the middle surface of the plate, complemented by suitable boundary conditions. However, it was subsequently recognised that minimising an ad hoc energy functional over an appropriate set of admissible displacements of the middle surface of the plate was the most efficient way to obtain existence theorems.

From the computational viewpoints, however, this classical approach is not fully satisfactory, in that the unknowns of primary interest in practical applications are not so much the components of the displacement field, but instead those of the stress tensor field inside the body, since large stresses, rather than large displacements, are more likely to provoke the collapse of an elastic structure. But computing the stresses from the displacement, by means of the constitutive equation of the elastic material constituting the structure, involves computing derivatives, a procedure well-known to be unstable numerically.

By contrast, in an intrinsic approach, it is the components of the stress tensor, or more generally of any bona fide measure of stress, that are the only unknowns, instead of the components of the displacement vector field. Although he idea of using such intrinsic approaches for the modelling of elastic bodies goes back to Chien [6, 7] and, more recently, to Antman [1], it is only in the last two decades that mathematical foundations of such methods have been investigated, first for the three-dimensional model of elasticity, both linear and nonlinear (see [10]), then for two-dimensional models for plates and shells, but only for linear models (see [11, 12]), or for the nonlinear Kirchhoff-Love model (see [13–15]).

The objective of this paper is to address one of the missing cases, namely the von Kármán equations for a nonlinearly elastic plate (see [20]). The new nonlinear plate model, which is defined in Theorem 4.1 below, is equivalent to von Kármán's model, but has the advantage of being defined solely in terms of the bending moments and stress resultants of the middle surface of the plate, instead of the vertical component of the displacement field and of the Airy function in the original von Kármán equations. This is of importance in applications where lower-dimensional models for thin elastic bodies are most of the time used to predict the stresses that may appear in them (see, e.g., [17–19]).

The paper is organized as follows. In the Section 2, we first introduce the notation and all relevant notions from differential geometry of surfaces that we will need. Then we describe the type of applied forces and boundary conditions that correspond to the von Kármán equations.

In Section 3, we state the classical two-dimensional von Kármán equations for a nonlinearly elastic plate and give a brief, but self-contained, account of the derivation of these equations as the limit as the thickness of the plate approaches zero of the three-dimensional equations of the plate.

In Section 4, we define the new intrinsic von Kármán equations. Then we show in Theorems 4.1 and 4.2 that the intrinsic von Kármán equations and the classical von Kármán equations are equivalent.

In Section 5, we define the notion of weak solution of the intrinsic von Kármán equations. Then we justify this definition in Theorem 5.1 by showing that a classical solution is a weak solution that is sufficiently smooth. Finally, we prove in Theorem 5.2 that the intrinsic von Kármán equations defined in Section 4 possess a weak solution.

2 Notation and assumptions

Greek indices vary in the set $\{1,2\}$, save in the notation ∂_τ and ∂_ν used for the tangential and normal derivatives along the boundary of a two-dimensional domain, while Latin indices vary in the set $\{1,2,3\}$. The summation convention for repeated indices is used in conjunction to these rules for indices. Vectors and vector-valued functions are denoted by boldface letters to distinguish them from scalar functions.

Throughout this paper, $\mathbb{E}^3$ denotes the three-dimensional Euclidean space. The inner-product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{E}^3$ is denoted $\boldsymbol{a}\cdot\boldsymbol{b}$. The vector product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{E}^3$ is denoted $\boldsymbol{a}\wedge\boldsymbol{b}$. The Euclidean norm of a vector $\boldsymbol{a}\in\mathbb{E}^3$ is denoted $|\boldsymbol{a}|$.

The space of all real 2×2 symmetric matrices is denoted $\mathbb{S}^2$. The Frobenius norm of a real matrix $\boldsymbol{A}$ is denoted $|\boldsymbol{A}|$.

A Cartesian frame in $\mathbb{E}^3$ is given once and for all and its vectors are denoted $\boldsymbol{e}_i$. Thus a point $x\in\mathbb{E}^3$ is identified with its Cartesian coordinates by the relation $x=y_\alpha\boldsymbol{e}_\alpha+x_3\boldsymbol{e}_3$, where $(y,x_3)\in\mathbb{R}^3$ and $y=(y_\alpha)\in\mathbb{R}^2$. Partial derivative operators with respect to these Cartesian coordinates y_α and x_3 are denoted $\partial_\alpha:=\partial/\partial y_\alpha$, $\partial_3:=\partial/\partial x_3$, $\partial_{\alpha\beta}:=\partial^2/\partial y_\alpha\partial y_\beta$ and $\Delta:=\partial_{11}+\partial_{22}$.

A plate is an elastic body whose natural state, which is by definition a stress-free configuration of the plate, is of the form

$$\Omega:=\big\{(y,x_3);\, y\in\omega,\, x_3\in(-\varepsilon,\varepsilon)\big\}\subseteq\mathbb{R}^3,$$

where ω is a domain in $\mathbb{R}^2$, i.e., a bounded and connected open subset of $\mathbb{R}^2$ that is locally on the same side of its boundary, and $\varepsilon>0$ is a real number. Such a plate thus have a constant thickness equal to 2ε, a middle surface given by

$$S:=\omega\times\{0\},$$

a lower face and an upper face respectively given by

$$\Gamma_- := \omega \times \{-\varepsilon\} \quad \text{and} \quad \Gamma_+ := \omega \times \{+\varepsilon\},$$

and a lateral face given by

$$\Gamma := \gamma \times [-\varepsilon, \varepsilon], \quad \text{where} \quad \gamma := \partial\omega.$$

The plate is assumed to be made of a homogeneous and isotropic elastic material, so that its two-dimensional elasticity tensor is given by

$$a_{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda+2\mu}\delta_{\sigma\tau}\delta_{\alpha\beta} + 2\mu(\delta_{\alpha\sigma}\delta_{\beta\tau} + \delta_{\alpha\tau}\delta_{\beta\sigma}),$$

where $\lambda \geq 0$ and $\mu > 0$ denote the Lamé constants of the elastic material, and $\delta_{\alpha\beta}$ designates the Kronecker symbol, i.e., $\delta_{\alpha\beta} := 1$ if $\alpha = \beta$ and $\delta_{\alpha\beta} := 0$ if $\alpha \neq \beta$. This tensor is invertible, in the sense that two symmetric tensors $(T_{\alpha\beta})$ and $(E_{\alpha\beta})$ satisfy

$$T_{\alpha\beta} = a_{\alpha\beta\sigma\tau} E_{\sigma\tau},$$

if and only if

$$E_{\alpha\beta} = b_{\alpha\beta\sigma\tau} T_{\sigma\tau},$$

where

$$b_{\alpha\beta\sigma\tau} := \frac{1}{8\mu}(\delta_{\alpha\sigma}\delta_{\beta\tau} + \delta_{\alpha\tau}\delta_{\beta\sigma}) - \frac{\lambda}{4\mu(3\lambda+2\mu)}\delta_{\alpha\beta}\delta_{\sigma\tau}.$$

The von Kármán equations for such a plate are two-dimensional partial differential equations obtained at the limit $\varepsilon \to 0$ from the three-dimensional partial differential equations of a nonlinearly elastic plate subjected to specific boundary conditions and applied forces, as shown in [8] by means of a formal asymptotic analysis. We assume in this paper that the plate satisfy these specific assumptions, which are described below.

First, the three-dimensional admissible displacement fields $u_i \boldsymbol{e}_i : \Omega \to \mathbb{E}^3$ of the plate satisfy the boundary conditions

$$u_3 = 0 \quad \text{and} \quad \partial_3 u_\alpha = 0 \quad \text{on } \Gamma.$$

Secondly, the plate is subjected to applied body forces parallel to the vector $\boldsymbol{e}_3$ of density

$$f_3 \boldsymbol{e}_3 \in L^2(\Omega; \mathbb{E}^3)$$

per unit volume in Ω, to applied surface forces parallel to $\boldsymbol{e}_3$ per unit area on the lower and upper faces Γ_- and Γ_+ of density

$$g_3 \boldsymbol{e}_3 \in L^2(\Gamma_- \cup \Gamma_+; \mathbb{E}^3)$$

per unit area on $\Gamma_- \cup \Gamma_+$, and to applied surface forces parallel to the plane spanned by the vectors $\boldsymbol{e}_\alpha$ on the lateral face Γ whose resultant density obtained by integration across the thickness is

$$h_\alpha \boldsymbol{e}_\alpha \in L^2(\gamma;\mathbb{E}^3)$$

per unit length along γ. Note that these functions h_α must satisfy the compatibility conditions (see [8])

$$\int_\gamma h_\alpha \mathrm{d}\gamma = 0, \quad \int_\gamma (y_1 h_2 - y_2 h_1)\mathrm{d}\gamma = 0.$$

Then the densities per unit area along ω, and per unit length along γ, of the resulting two-dimensional forces acting on the middle surface of the plate are

$$p_3 \boldsymbol{e}_3 : \omega \ \to \ \mathbb{E}^3,$$

where

$$p_3 := \int_{-\varepsilon}^{\varepsilon} f_3(\cdot,x_3)\mathrm{d}x_3 + g_3(\cdot,+\varepsilon) + g_3(\cdot,-\varepsilon) \quad \text{in } \omega,$$

and

$$h_\alpha \boldsymbol{e}_\alpha : \gamma \ \to \ \mathbb{E}^3.$$

These two densities are precisely those appearing in the von Kármán equations defined in the next section.

The assumptions on the Lamé constants and on the applied forces used in [8] in order to derive the von Kármán equations by letting $\varepsilon \to 0$ in the equations of three-dimensional elasticity of a nonlinearly elastic plate made of a St Venant-Kirchhoff material are the following:

$$\lambda = \mathcal{O}(1), \quad \mu = \mathcal{O}(1), \quad h_\alpha = \mathcal{O}(\varepsilon^2), \quad p_3 = \mathcal{O}(\varepsilon^4),$$

in which case the resulting displacement field $u_i \boldsymbol{e}_i : \Omega \to \mathbb{E}^3$ satisfies

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_\alpha(\cdot,x_3)\mathrm{d}x_3 = \mathcal{O}(\varepsilon^2), \quad \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_3(\cdot,x_3)\mathrm{d}x_3 = \mathcal{O}(\varepsilon).$$

3 The classical von Kármán's equations

We give a brief, but self-contained, account of the classical formulation of the von Kármán equations modeling the behavior of a nonlinearly elastic plate. For a more detailed introduction to these equations, we refer the reader to, e.g., [2,9].

The von Kármán's equations allow to determine the displacement field

$$\boldsymbol{\zeta} := \zeta_i \boldsymbol{e}_i : \overline{\omega} \to \mathbb{E}^3$$

of the middle surface of the plate satisfying the assumptions of Section 2. In turn, $\boldsymbol{\zeta}$ allows to determine the bending moments

$$M_{\alpha\beta} := a_{\alpha\beta\sigma\tau} \partial_{\sigma\tau} \zeta_3$$

and the stress resultants

$$N_{\alpha\beta} := \frac{1}{2} a_{\alpha\beta\sigma\tau} (\partial_\alpha \zeta_\beta + \partial_\beta \zeta_\alpha + \partial_\alpha \zeta_3 \partial_\beta \zeta_3)$$

of the middle surface of the plate, then the stress tensor fields of the entire plate (so not only of its middle surface) by

$$\boldsymbol{\sigma} := \sigma_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j : \overline{\Omega} \to \mathbb{R}^{3\times 3},$$

where, for all $(y, x_3) \in \Omega$,

$$\begin{aligned}
&\sigma_{\alpha\beta}(y,x_3) := \frac{1}{2}\big(N_{\alpha\beta}(y) - x_3 M_{\alpha\beta}(y)\big),\\
&\sigma_{\alpha 3}(y,x_3) = \sigma_{3\alpha}(y,x_3) := 0,\\
&\sigma_{33}(y,x_3) := 0,
\end{aligned}$$

and finally the displacement field of the entire plate by

$$\boldsymbol{u} := u_i \boldsymbol{e}_i : \Omega \to \mathbb{E}^3,$$

where, for all $(y, x_3) \in \Omega$,

$$\begin{aligned}
&u_\alpha(y,x_3) := \zeta_\alpha(y) - x_3 \partial_\alpha \zeta_3(y),\\
&u_3(y,x_3) := \zeta_3(y).
\end{aligned}$$

Thus determining the displacement field and the strain and stress fields inside the plate reduces to computing the displacement field $\boldsymbol{\zeta} = (\zeta_i)$ of the middle surface of the plate. The von Kármán equations provide a way to compute $\boldsymbol{\zeta} = (\zeta_i)$ in the particular case where, in addition to the assumptions made in Section 2, it is assumed that ω is simply-connected, by first determining the two functions

$$\zeta_3 : \omega \to \mathbb{R}, \quad \phi : \omega \to \mathbb{R},$$

where ζ_3 is the vertical component of the displacement field $\boldsymbol{\zeta}$ and ϕ is the Airy stress function.

To define this Airy stress function, note that the assumptions that the applied body forces and the applied surface forces on $\Gamma_- \cup \Gamma_+$ are parallel to the vector $\boldsymbol{e}_3$ (Section 2) implies that the classical equations of equilibrium satisfied by the stress resultants and bending moments of the middle surface of the plate take the form

$$\begin{aligned} &\partial_\beta N_{\alpha\beta} = 0 && \text{in } \omega, \\ &\partial_{\alpha\beta} M_{\alpha\beta}(\boldsymbol{\zeta}) - \partial_\alpha (N_{\alpha\beta} \partial_\beta \zeta_3) = p_3 && \text{in } \omega. \end{aligned}$$

Then the first equation of the above system, the assumption that ω is simply-connected, and the symmetry relations

$$N_{\alpha\beta} = N_{\beta\alpha} = 0 \quad \text{in } \omega,$$

together imply that there exists a function, called the Airy stress function,

$$\phi : \omega \ \to \ \mathbb{R},$$

which is unique up to the addition of an affine function such that

$$N_{11} = \partial_{22}\phi, \quad N_{22} = \partial_{11}\phi, \quad N_{12} = N_{21} = -\partial_{12}\phi.$$

Consequently, the horizontal components ζ_α of the displacement field $\boldsymbol{\zeta} = (\zeta_i)$ of the middle surface of the plate are completely determined by the vertical component ζ_3 and the Airy function ϕ, via the equations (see, e.g., [8,9])

$$\frac{1}{2}(\partial_\alpha \zeta_\beta + \partial_\beta \zeta_\alpha) = b_{\alpha\beta\sigma\tau}(\Delta\phi - \partial_{\sigma\tau}\phi) - \frac{1}{2}\partial_\alpha \zeta_3 \partial_\beta \zeta_3.$$

Since the functions

$$e_{\alpha\beta} := \frac{1}{2}(\partial_\alpha \zeta_\beta + \partial_\beta \zeta_\alpha)$$

satisfy the compatibility conditions of Saint Venant, that is

$$\partial_{11} e_{22} + \partial_{22} e_{11} = 2\partial_{12} e_{12} \quad \text{in } \omega,$$

the previous equation shows that the pair of functions (ϕ, ζ_3) must satisfy the compatibility condition

$$\Delta^2 \phi + \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}[\zeta_3, \zeta_3] = 0 \quad \text{in } \omega,$$

where $[\cdot,\cdot]$ denotes the Monge-Ampère bilinear form, defined for all $(u,v)\in H^2(\omega)\times H^2(\omega)$ by

$$[u,v]:=\partial_{11}u\partial_{22}v+\partial_{22}u\partial_{11}v-2\partial_{12}u\partial_{12}v.$$

In this fashion, the Airy stress function ϕ and the vertical component ζ_3 of the displacement field become the two unknowns of the von Kármán equations, instead of the three unknowns ζ_i representing the Cartesian components of the displacement field $\boldsymbol{\zeta}=(\zeta_i)$.

Finally, in order to explicitly define the boundary conditions satisfied by the Airy stress function ϕ, assume without loss of generality that the origin of the plane spanned by the vectors $\boldsymbol{e}_\alpha$ belongs to the boundary γ of ω. Then the Airy stress function ϕ satisfies the boundary conditions

$$\phi=\phi_0 \quad \text{and} \quad \partial_\nu\phi=\phi_1 \quad \text{on } \gamma,$$

where the functions $\phi_0:\gamma\to\mathbb{R}$ and $\phi_1:\gamma\to\mathbb{R}$ are defined at each $y\in\omega$ by

$$\begin{aligned}
\phi_0(y)&:=\int_{\gamma(y)}(x_1-y_1)h_2(x)\mathrm{d}\gamma(x)-\int_{\gamma(y)}(x_2-y_2)h_1(x)\mathrm{d}\gamma(x),\\
\phi_1(y)&:=-\nu_1(y)\int_{\gamma(y)}h_2(x)\mathrm{d}\gamma(x)+\nu_2(y)\int_{\gamma(y)}h_1(x)\mathrm{d}\gamma(x)
\end{aligned}$$

in terms of the functions $h_\alpha:\gamma\to\mathbb{R}$ introduced in Section 2. In these two formulas, $\gamma(y)$ designates the portion of the curve $\gamma:=\partial\omega$ joining the origin $0\in\gamma$ to $y\in\gamma$, while $\nu_1(y)$ and $\nu_2(y)$ denote the Cartesian components of the unit outer normal vector to the curve γ at y.

In conclusion, the two-dimensional nonlinear von Kármán equations assert that the vertical displacement ζ_3 and the Airy stress function ϕ associated with the unknown displacement field of the middle surface of a nonlinearly elastic plate should satisfy the following boundary value problem:

$$\begin{aligned}
&\varepsilon^3\frac{8\mu(\lambda+\mu)}{3(\lambda+2\mu)}\Delta^2\zeta_3-\varepsilon[\phi,\zeta_3]=p_3 && \text{in } \omega,\\
&\Delta^2\phi+\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}[\zeta_3,\zeta_3]=0 && \text{in } \omega,\\
&\zeta_3=0 \quad \text{and} \quad \partial_\nu\zeta_3=0 && \text{on } \gamma,\\
&\phi=\phi_0 \quad \text{and} \quad \partial_\nu\phi=\phi_1 && \text{on } \gamma,
\end{aligned}\tag{3.1}$$

where the constants $\lambda\geqslant 0$ and $\mu>0$ are the Lamé constants of the elastic material constituting the plate and the functions $p_3:\omega\to\mathbb{R}$ and $\phi_0,\phi_1:\gamma\to\mathbb{R}$ are defined in terms of the densities f_3,g_3 and h_α of the forces applied to the plate (see Section 2).

4 The intrinsic von Kármán equations

An intrinsic approach to plate theory consists in providing intrinsic equations for computing the strains and stresses inside the plate, as opposed to the classical approach which provides equations depending on the displacement field.

The sole unknowns appearing in the intrinsic von Kármán equations are the bending moments

$$(M_{\alpha\beta}):\overline{\omega} \ \to \ \mathbb{S}^2,$$

and the stress resultants

$$(N_{\alpha\beta}):\overline{\omega} \ \to \ \mathbb{S}^2,$$

which are both symmetric tensor fields defined on the middle surface S of the plate.

The advantage of choosing $M_{\alpha\beta}$ and $N_{\alpha\beta}$ instead of ζ_3 and ϕ as the new unknowns is that they are the most relevant unknowns in computational mechanics. In particular, the stress tensor field $\boldsymbol{\sigma}:=(\sigma_{ij})$ inside the entire plate Ω can be computed by purely algebraic operations (i.e., without differentiation) in terms of $M_{\alpha\beta}$ and $N_{\alpha\beta}$ by

$$\begin{aligned}
&\sigma_{\alpha\beta}(y,x_3):=\frac{1}{2}\big(N_{\alpha\beta}(y)-x_3M_{\alpha\beta}(y)\big),\\
&\sigma_{\alpha 3}(y,x_3)=\sigma_{3\alpha}(y,x_3)=\sigma_{33}(y,x_3):=0
\end{aligned}$$

for all $(y,x_3)\in\overline{\Omega}$, see [9].

In order to find the intrinsic equations satisfied by the bending moments $M_{\alpha\beta}$ and stress resultants $N_{\alpha\beta}$, we proceed in two steps.

First, we show that the bending moments and stress resultants associated with a solution (ζ_3,ϕ) of von Kármán equations necessarily satisfy a boundary value problem where ζ_3 and ϕ no longer appear, cf. Theorem 4.1.

Second, we show that if $(M_{\alpha\beta},N_{\alpha\beta})$ is a solution of the boundary value problem defined in Theorem 4.1, then they are the bending moments and stress resultants associated with a displacement field $\boldsymbol{\zeta}=(\zeta_i)$ and the pair of functions (ζ_3,ϕ), where ϕ is the Airy function associated with $\boldsymbol{\zeta}$, satisfies the von Kármán equations (3.1), cf. Theorem 4.2.

Given the Lamé constants $\lambda\geqslant 0$ and $\mu>0$ of the elastic material constituting the plate (see Section 2), the tensors $a_{\alpha\beta\sigma\tau}$ and its inverse $b_{\alpha\beta\sigma\tau}$ are those defined in Section 2. Then we define the new tensor

$$c_{\alpha\beta\sigma\tau}:=\frac{1+\nu^2}{4E}\delta_{\alpha\beta}\delta_{\sigma\tau}-\frac{E}{32\mu^2}(\delta_{\alpha\sigma}\delta_{\beta\tau}+\delta_{\alpha\tau}\delta_{\beta\sigma}),$$

where

$$E := \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} \quad \text{and} \quad \nu := \frac{\lambda}{2(\lambda+\mu)}$$

denote respectively Young's modulus and Poisson's ratio of the elastic material.

Theorem 4.1. *Assume that ω is a simply-connected open subset of $\mathbb{R}^2$ with a boundary of class $\mathcal{C}^2$.*

Let $(\zeta_3,\phi) \in \mathcal{C}^4(\overline{\omega};\mathbb{R}^2)$ be a solution to von Kármán equations (3.1) *corresponding to the data*

$$p_3 \in \mathcal{C}^0(\overline{\omega}), \quad \phi_0 \in \mathcal{C}^2(\gamma), \quad \phi_1 \in \mathcal{C}^1(\gamma).$$

Define the functions

$$k_\alpha := \partial_\tau(\nu_\alpha \partial_\tau \phi_0 - \tau_\alpha \phi_1) \in \mathcal{C}^0(\gamma),$$

where $\nu = (\nu_\alpha) : \gamma \to \mathbb{R}^2$, denote the components of the unit outer normal vector field to the boundary γ of ω, and $\tau = (\tau_\alpha) : \gamma \to \mathbb{R}^2$, where

$$\tau_1 = \nu_2, \quad \tau_2 = -\nu_1$$

denote the components of the positively-oriented unit tangent vector field along the boundary γ of ω.

Then the functions

$$M_{\alpha\beta} := a_{\alpha\beta\sigma\tau} \partial_{\sigma\tau} \zeta_3, \quad N_{\alpha\beta} := \Delta\phi \delta_{\alpha\beta} - \partial_{\alpha\beta}\phi$$

satisfy the boundary value problem

$$\begin{aligned}
&\frac{\varepsilon^3}{3}\partial_{\alpha\beta} M_{\alpha\beta} - \varepsilon b_{\alpha\beta\sigma\tau} M_{\sigma\tau} N_{\alpha\beta} = p_3 && \text{in } \omega,\\
&\partial_{\alpha\alpha} N_{\beta\beta} + c_{\alpha\beta\sigma\tau} M_{\sigma\tau} M_{\alpha\beta} = 0 && \text{in } \omega,\\
&\partial_\beta N_{\alpha\beta} = 0 && \text{in } \omega,\\
&\partial_\alpha (b_{\beta\gamma\sigma\tau} M_{\sigma\tau}) - \partial_\beta (b_{\alpha\gamma\sigma\tau} M_{\sigma\tau}) = 0 && \text{in } \omega,\\
&b_{\alpha\beta\sigma\tau} M_{\sigma\tau} \tau_\beta = 0 && \text{on } \gamma,\\
&N_{\alpha\beta}\nu_\beta = k_\alpha && \text{on } \gamma.
\end{aligned} \tag{4.1}$$

The boundary value problem (4.1) *constitute the intrinsic von Kármán equations corresponding to the same data p_3, ϕ_0, ϕ_1, this time by means of the functions p_3 and k_α.*

Proof. The definitions of $a_{\alpha\beta\sigma\tau}$ and $M_{\alpha\beta}$ imply that

$$M_{\alpha\beta} := a_{\alpha\beta\sigma\tau}\partial_{\sigma\tau}\zeta_3 = \frac{4\lambda\mu}{\lambda+2\mu}(\partial_{\sigma\sigma}\zeta_3)\delta_{\alpha\beta} + 4\mu\partial_{\alpha\beta}\zeta_3 \quad \text{in } \omega,$$

so that

$$\partial_{\alpha\beta}M_{\alpha\beta} = \frac{8\mu(\lambda+\mu)}{\lambda+2\mu}\Delta^2\zeta_3 \quad \text{in } \omega.$$

The definitions of $b_{\alpha\beta\sigma\tau}, M_{\alpha\beta}, N_{\alpha\beta}$, and of the Monge-Ampère bilinear form $[\cdot,\cdot]$ (Section 3) imply that

$$b_{\alpha\beta\sigma\tau}M_{\sigma\tau}N_{\alpha\beta} = \partial_{\alpha\beta}\zeta_3(\Delta\phi\delta_{\alpha\beta} - \partial_{\alpha\beta}\phi) = [\phi,\zeta_3] \quad \text{in } \omega.$$

Thus the first equation of the boundary value problem (4.1) follows from the first equation of (3.1).

The definition of $N_{\alpha\beta}$ implies that

$$\partial_{\alpha\alpha}N_{\beta\beta} = \Delta^2\phi \quad \text{in } \omega.$$

The definitions of $c_{\alpha\beta\sigma\tau}$ and $a_{\alpha\beta\sigma\tau}$ imply that (after a series of long, but straightforward, calculations)

$$c_{\alpha\beta\sigma\tau}a_{\sigma\tau\varphi\psi}a_{\alpha\beta\gamma\delta} = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\left(\delta_{\varphi\psi}\delta_{\gamma\delta} - \frac{1}{2}(\delta_{\varphi\gamma}\delta_{\psi\delta} + \delta_{\varphi\delta}\delta_{\psi\gamma})\right).$$

Consequently,

$$\begin{aligned}
c_{\alpha\beta\sigma\tau}M_{\sigma\tau}M_{\alpha\beta} &= (c_{\alpha\beta\sigma\tau}a_{\sigma\tau\gamma\delta}a_{\alpha\beta\varphi\psi})\partial_{\varphi\psi}\zeta_3\partial_{\gamma\delta}\zeta_3 \\
&= \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}(\partial_{\varphi\varphi}\zeta_3\partial_{\gamma\gamma}\zeta_3 - \partial_{\varphi\psi}\zeta_3\partial_{\varphi\psi}\zeta_3) \\
&= \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}[\zeta_3,\zeta_3] \quad \text{in } \omega.
\end{aligned}$$

Thus the second equation of the boundary value problem (4.1) follows from the second equation of (3.1).

The third equation of (4.1) follows from the definition of $N_{\alpha\beta}$

$$\partial_\beta N_{\alpha\beta} = \partial_\beta(\Delta\phi\delta_{\alpha\beta} - \partial_{\alpha\beta}\phi) = 0 \quad \text{in } \omega.$$

The fourth equation of (4.1) follows from the definitions of $M_{\alpha\beta}$, $a_{\alpha\beta\sigma\tau}$ and $b_{\alpha\beta\sigma\tau}$

$$\partial_\alpha(b_{\beta\gamma\sigma\tau}M_{\sigma\tau}) = \partial_\alpha(\partial_{\beta\gamma}\zeta_3) = \partial_\beta(\partial_{\alpha\gamma}\zeta_3) = \partial_\beta(b_{\alpha\gamma\sigma\tau}M_{\sigma\tau}) \quad \text{in } \omega.$$

It remains to prove that the functions $M_{\alpha\beta}$ and $N_{\alpha\beta}$ satisfy the boundary conditions given by the last two equations of (4.1).

Let $\varepsilon_{\alpha\beta}$ denote the Levi-Civita permutation symbol, defined by $\varepsilon_{\alpha\beta}=-\varepsilon_{\beta\alpha}$ and $\varepsilon_{12}=1$. Since

$$N_{\alpha\beta}=\varepsilon_{\alpha\sigma}\varepsilon_{\beta\gamma}\partial_{\sigma\gamma}\phi, \quad \tau_\alpha=\varepsilon_{\alpha\beta}\nu_\beta, \quad \nu_\alpha=-\varepsilon_{\alpha\beta}\tau_\beta \quad \text{on } \gamma,$$

we deduce from the boundary conditions $\phi=\phi_0$ and $\partial_\nu\phi=\phi_1$ on γ (see the boundary value problem (3.1)) and from the definition of $N_{\alpha\beta}$ that

$$\begin{aligned} N_{\alpha\beta}\nu_\beta &= \varepsilon_{\alpha\sigma}\varepsilon_{\beta\gamma}\nu_\beta\partial_{\sigma\gamma}\phi = -\varepsilon_{\alpha\sigma}\tau_\gamma\partial_{\sigma\gamma}\phi = -\varepsilon_{\alpha\sigma}\partial_\tau(\partial_\sigma\phi) \\ &= -\varepsilon_{\alpha\sigma}\partial_\tau(\tau_\sigma\partial_\tau\phi+\nu_\sigma\partial_\nu\phi) = -\partial_\tau(\varepsilon_{\alpha\sigma}\tau_\sigma\partial_\tau\phi+\varepsilon_{\alpha\sigma}\nu_\sigma\partial_\nu\phi) \\ &= -\partial_\tau(-\nu_\alpha\partial_\tau\phi+\tau_\alpha\partial_\nu\phi) = \partial_\tau(\nu_\alpha\partial_\tau\phi_0-\tau_\alpha\phi_1) = k_\alpha \quad \text{on } \gamma. \end{aligned}$$

We also deduce from the boundary conditions $\zeta_3=\partial_\nu\zeta_3=0$ on γ (see (3.1)) and from the definition of $M_{\alpha\beta}$ that

$$b_{\alpha\beta\sigma\tau}M_{\sigma\tau}\tau_\beta = \tau_\beta\partial_{\alpha\beta}\zeta_3 = \partial_\tau(\partial_\alpha\zeta_3) = \partial_\tau(\tau_\alpha\partial_\tau\zeta_3+\nu_\alpha\partial_\nu\zeta_3) = 0 \quad \text{on } \gamma.$$

Hence, the proof is complete. $\square$

The next theorem shows that the system of equations (4.1) found in Theorem 4.1 is well-posed, in the sense that its equations allow to determine uniquely the bending moments $M_{\alpha\beta}$ and the stress resultants $N_{\alpha\beta}$ arising in an elastic plate. This justifies calling this system the intrinsic von Kármán equations.

Theorem 4.2. *Assume that ω is simply-connected open subset of $\mathbb{R}^2$ with a boundary $\gamma:=\partial\omega$ of class $\mathcal{C}^2$.*

Let $(M_{\alpha\beta})\in\mathcal{C}^2(\overline{\omega};\mathbb{S}^2)$ and $(N_{\alpha\beta})\in\mathcal{C}^2(\overline{\omega};\mathbb{S}^2)$ be symmetric matrix fields that satisfy the intrinsic von Kármán equations (4.1) *corresponding to the functions*

$$p_3\in\mathcal{C}^0(\overline{\omega}), \quad k_\alpha:=\partial_\tau(\nu_\alpha\partial_\tau\phi_0-\tau_\alpha\phi_1)\in\mathcal{C}^0(\gamma),$$

where $\phi_0\in\mathcal{C}^2(\gamma)$ and $\phi_1\in\mathcal{C}^1(\gamma)$ are two given functions.

Then there exists a unique function $\zeta_3\in\mathcal{C}^4(\overline{\omega})$ such that

$$\begin{aligned} &a_{\alpha\beta\sigma\tau}\partial_{\sigma\tau}\zeta_3 = M_{\alpha\beta} \quad \textit{in } \omega, \\ &\zeta_3=0, \quad \partial_\nu\zeta_3=0 \quad \textit{on } \gamma, \end{aligned} \tag{4.2}$$

and there exists a unique function $\phi\in\mathcal{C}^4(\overline{\omega})$ such that

$$\begin{aligned} &\Delta\phi\,\delta_{\alpha\beta}-\partial_{\alpha\beta}\phi = N_{\alpha\beta} \quad \textit{in } \omega, \\ &\phi=\phi_0, \quad \partial_\nu\phi=\phi_1 \quad \textit{on } \gamma. \end{aligned} \tag{4.3}$$

Moreover, the pair (ϕ,ζ_3) *satisfies the von Kármán equations* (3.1) *corresponding to the data*

$$p_3\in\mathcal{C}^0(\overline{\omega}),\quad \phi_0\in\mathcal{C}^2(\gamma),\quad \phi_1\in\mathcal{C}^1(\gamma).$$

Proof. Since the set ω is simply-connected, the equations

$$\begin{aligned}&\partial_\alpha(b_{\beta\gamma\sigma\tau}M_{\sigma\tau})-\partial_\beta(b_{\alpha\gamma\sigma\tau}M_{\sigma\tau})=0 && \text{in } \omega,\\ &\partial_\beta N_{\alpha\beta}=0 && \text{in } \omega,\end{aligned}$$

which are respectively equivalent to the equations

$$\begin{aligned}&\partial_1(b_{2\gamma\sigma\tau}M_{\sigma\tau})=\partial_2(b_{1\gamma\sigma\tau}M_{\sigma\tau}) && \text{in } \omega,\\ &\partial_1 N_{\alpha 1}=\partial_2(-N_{\alpha 2}) && \text{in } \omega,\end{aligned}$$

imply that there exist functions $\xi_\gamma\in\mathcal{C}^3(\overline{\omega})$ and $\eta_\alpha\in\mathcal{C}^3(\overline{\omega})$ (the regularity of these functions follow from the regularity assumptions on $M_{\sigma\tau}$ and $N_{\alpha\beta}$) such that

$$\begin{aligned}&b_{\alpha\gamma\sigma\tau}M_{\sigma\tau}=\partial_\alpha\xi_\gamma && \text{in } \omega,\\ &N_{\alpha\beta}=\varepsilon_{\beta\sigma}\partial_\sigma\eta_\alpha && \text{in } \omega,\end{aligned}$$

where $\varepsilon_{\beta\sigma}$ is the Levi-Civita permutation symbol.

Next, the symmetries $b_{\alpha\gamma\sigma\tau}=b_{\gamma\alpha\sigma\tau}$ and $N_{\alpha\beta}=N_{\beta\alpha}$ imply that

$$\begin{aligned}&\partial_1\xi_2=\partial_2\xi_1 && \text{in } \omega,\\ &\partial_1(-\eta_1)=\partial_2\eta_2 && \text{in } \omega.\end{aligned}$$

Therefore, using again the assumption that ω is simply-connected, there exist functions $\widetilde{\zeta}_3\in\mathcal{C}^4(\overline{\omega})$ and $\widetilde{\phi}\in\mathcal{C}^4(\overline{\omega})$ (the regularity of these functions follow from the regularity of the functions ξ_γ and η_α) such that

$$\begin{aligned}&\xi_\gamma=\partial_\gamma\widetilde{\zeta}_3 && \text{in } \omega,\\ &\eta_\alpha=\varepsilon_{\alpha\tau}\partial_\tau\widetilde{\phi} && \text{in } \omega.\end{aligned}$$

Then we infer from the above relations that

$$b_{\alpha\gamma\sigma\tau}M_{\sigma\tau}=\partial_{\alpha\gamma}\widetilde{\zeta}_3\quad \text{in } \omega,$$

or equivalently,

$$a_{\alpha\beta\sigma\tau}\partial_{\sigma\tau}\widetilde{\zeta}_3=M_{\alpha\beta}\quad \text{in } \omega,$$

and that

$$N_{\alpha\beta}=\varepsilon_{\beta\sigma}\varepsilon_{\alpha\tau}\partial_{\sigma\tau}\widetilde{\phi} \quad \text{in } \omega,$$

or equivalently,

$$\Delta\widetilde{\phi}\delta_{\alpha\beta}-\partial_{\alpha\beta}\widetilde{\phi}=N_{\alpha\beta} \quad \text{in } \omega.$$

Consequently, for any affine functions $f,g:\overline{\omega}\to\mathbb{R}$, the functions

$$\phi:=\widetilde{\phi}-f, \quad \zeta_3:=\widetilde{\zeta}_3-g$$

satisfy the equations

$$\begin{aligned} a_{\alpha\beta\sigma\tau}\partial_{\sigma\tau}\zeta_3&=M_{\alpha\beta} && \text{in } \omega,\\ \Delta\phi\,\delta_{\alpha\beta}-\partial_{\alpha\beta}\phi&=N_{\alpha\beta} && \text{in } \omega. \end{aligned}$$

It remains to prove that the functions ϕ and ζ_3 satisfy the boundary conditions announced in the theorem for a unique choice of the affine functions f and g. To this end, we will use the boundary conditions of (4.1) and that γ is connected.

The boundary conditions satisfied by $N_{\alpha\beta}$, viz. (see Eqs. (4.1)),

$$N_{\alpha\beta}\nu_\beta=k_\alpha \quad \text{on } \gamma,$$

and the relations

$$N_{\alpha\beta}=\varepsilon_{\alpha\sigma}\varepsilon_{\beta\gamma}\partial_{\sigma\gamma}\widetilde{\phi}, \quad \tau_\alpha=\varepsilon_{\alpha\beta}\nu_\beta, \quad \nu_\alpha=-\varepsilon_{\alpha\beta}\tau_\beta,$$

together imply that

$$k_\alpha=N_{\alpha\beta}\nu_\beta=\varepsilon_{\alpha\sigma}\varepsilon_{\beta\gamma}\nu_\beta\partial_{\sigma\gamma}\widetilde{\phi}=-\partial_\tau(\varepsilon_{\alpha\sigma}\partial_\sigma\widetilde{\phi}) \quad \text{on } \gamma.$$

Hence, using the definition of the functions k_α, as given in the statement of the theorem, we have

$$\partial_\tau(\nu_\alpha\partial_\tau\phi_0-\tau_\alpha\phi_1)=-\partial_\tau(\varepsilon_{\alpha\sigma}\partial_\sigma\widetilde{\phi}) \quad \text{on } \gamma,$$

so that there exists constants $c_\alpha\in\mathbb{R}$ such that (we use here the assumption that γ is connected)

$$\nu_\alpha\partial_\tau\phi_0-\tau_\alpha\phi_1+\varepsilon_{\alpha\sigma}\partial_\sigma\widetilde{\phi}=c_\alpha \quad \text{on } \gamma.$$

Consequently,

$$\begin{aligned} \nu_\alpha\partial_\tau\phi_0-\tau_\alpha\phi_1-c_\alpha&=-\varepsilon_{\alpha\sigma}\partial_\sigma\widetilde{\phi}=-\varepsilon_{\alpha\sigma}\left(\tau_\sigma\partial_\tau\widetilde{\phi}+\nu_\sigma\partial_\nu\widetilde{\phi}\right)\\ &=-\varepsilon_{\alpha\sigma}\tau_\sigma\partial_\tau\widetilde{\phi}-\varepsilon_{\alpha\sigma}\nu_\sigma\partial_\nu\widetilde{\phi}\\ &=\nu_\alpha\partial_\tau\widetilde{\phi}-\tau_\alpha\partial_\nu\widetilde{\phi} \quad \text{on } \gamma. \end{aligned}$$

Then

$$\nu_\alpha \partial_\tau \big(\widetilde{\phi} - \phi_0\big) + \tau_\alpha \big(\phi_1 - \partial_\nu \widetilde{\phi}\big) = c_\alpha \quad \text{on } \gamma,$$

or equivalently

$$\begin{aligned} &\partial_\tau \big(\widetilde{\phi} - \phi_0\big) = c_\sigma \nu_\sigma \quad && \text{on } \gamma, \\ &\phi_1 - \partial_\nu \widetilde{\phi} = c_\sigma \tau_\sigma && \text{on } \gamma. \end{aligned}$$

Let $L : \mathbb{R}^2 \to \mathbb{R}$ be the linear function defined by

$$L(y) := (\varepsilon_{\alpha\sigma} c_\sigma) y_\alpha \quad \text{for all} \quad y = (y_\alpha) \in \mathbb{R}^2.$$

Then

$$\begin{aligned} c_\sigma \nu_\sigma = (\varepsilon_{\alpha\sigma} c_\sigma) \tau_\alpha = \tau_\alpha \partial_\alpha L = \partial_\tau L \quad \text{on } \gamma, \\ -c_\sigma \tau_\sigma = (\varepsilon_{\alpha\sigma} c_\sigma) \nu_\alpha = \nu_\alpha \partial_\alpha L = \partial_\nu L \quad \text{on } \gamma, \end{aligned}$$

so that the previous system is equivalent to

$$\begin{aligned} &\partial_\tau \big(\widetilde{\phi} - \phi_0 - L\big) = 0 \quad && \text{on } \gamma, \\ &\partial_\nu (\widetilde{\phi} - L) = \phi_1 && \text{on } \gamma. \end{aligned}$$

Since γ is connected, the first equation implies that there exists a constant c_3 such that

$$\widetilde{\phi} - \phi_0 - L = c_3 \quad \text{on } \gamma.$$

Thus the function

$$\phi := \widetilde{\phi} - f,$$

where $f : \overline{\omega} \to \mathbb{R}$ is the affine function defined by $f(y) := L(y) + c_3$ for all $y \in \overline{\omega}$, satisfies the boundary conditions

$$\begin{aligned} &\phi = \phi_0 \quad && \text{on } \gamma, \\ &\partial_\nu \phi = \phi_1 && \text{on } \gamma. \end{aligned}$$

The uniqueness of such an affine function f follows from the fact that if $\widetilde{f}$ has the same properties as f, then the difference $f_0 := (\widetilde{f} - f)$ is an affine function from $\overline{\omega}$ into $\mathbb{R}$ such that $f_0 = 0$ on γ and $\partial_\nu f_0 = 0$ on γ, this implies that $f_0 = 0$ in $\overline{\omega}$.

The existence and uniqueness of the affine function g appearing above in the definition of the function ζ_3 are established in a similar manner, with some minor differences due to the absence of the permutation tensor $\varepsilon_{\alpha\beta}$ in the definition of

$M_{\alpha\beta}$, as compared with that of $N_{\alpha\beta}$. We sketch this proof below for completeness, so as to provide an explicit way to compute the vertical component of the displacement field $\boldsymbol{\zeta}=(\zeta_i)$ of the middle surface of the plate from the knowledge of the bending moments $M_{\alpha\beta}$ of this surface.

The boundary conditions satisfied by $M_{\alpha\beta}$, viz. (see Eqs. (4.1)),

$$b_{\alpha\beta\sigma\tau}M_{\sigma\tau}\tau_\beta=0 \quad \text{on } \gamma,$$

and the relations

$$M_{\alpha\beta}=a_{\alpha\beta\sigma\tau}\partial_{\sigma\tau}\widetilde{\zeta}_3 \quad \text{in } \omega,$$

together imply that

$$\partial_\tau(\partial_\alpha\widetilde{\zeta}_3)=\tau_\beta\partial_{\alpha\beta}\widetilde{\zeta}_3=0 \quad \text{on } \gamma.$$

Since γ is connected, this implies the existence of constants $b_\alpha\in\mathbb{R}$ such that

$$\partial_\alpha\widetilde{\zeta}_3=b_\alpha \quad \text{on } \gamma.$$

Consequently,

$$\partial_\alpha(\widetilde{\zeta}_3-\widetilde{g})=0 \quad \text{on } \gamma,$$

where $\widetilde{g}:\mathbb{R}^2\to\mathbb{R}$ is the linear function defined by

$$\widetilde{g}(y):=b_\alpha y_\alpha \quad \text{for all} \quad y=(y_\alpha)\in\overline{\omega}.$$

Furthermore, since

$$\partial_\tau(\widetilde{\zeta}_3-\widetilde{g})=\tau_\alpha\partial_\alpha(\widetilde{\zeta}_3-\widetilde{g})=0 \quad \text{on } \gamma,$$

there exists a constant b_3 (we use here the assumption that γ is connected) such that

$$\widetilde{\zeta}_3-\widetilde{g}=b_3 \quad \text{on } \gamma.$$

Then we infer from the previous relations that the function

$$\zeta_3:=\widetilde{\zeta}_3-g,$$

where $g:\overline{\omega}\to\mathbb{R}$ is the affine function defined by $g(y):=\widetilde{g}(y)+b_3$ for all $y\in\overline{\omega}$, satisfies the boundary conditions

$$\begin{aligned}
\zeta_3&=0 && \text{on } \gamma,\\
\partial_\nu\zeta_3&=\nu_\alpha\partial_\alpha\zeta_3=0 && \text{on } \gamma.
\end{aligned}$$

The uniqueness of such an affine function g is proved by the same argument as the one used above to prove the uniqueness of such an affine function f.

We have therefore established the existence of unique functions $\zeta_3 \in \mathcal{C}^4(\overline{\omega})$ and $\phi \in \mathcal{C}^4(\overline{\omega})$, respectively satisfying the systems (4.2) and (4.3).

It remains to prove that the pair (ϕ,ζ_3) satisfies in addition the von Kármán equations (3.1) corresponding to the data

$$p_3 \in \mathcal{C}^0(\overline{\omega}), \quad \phi_0 \in \mathcal{C}^2(\gamma), \quad \phi_1 \in \mathcal{C}^1(\gamma).$$

To begin with, we infer from the relations (4.2) established above and from the definition of the tensor $a_{\alpha\beta\sigma\tau}$ given in Section 2 that, on the one hand,

$$\partial_{\alpha\beta} M_{\alpha\beta} = \partial_{\alpha\beta}(a_{\alpha\beta\sigma\tau}\partial_{\sigma\tau}\zeta_3) = \frac{8\mu(\lambda+\mu)}{\lambda+2\mu}\Delta^2\zeta_3 \quad \text{in } \omega.$$

Then we infer from the relations (4.2) and (4.3) and from the definition of the tensor $b_{\alpha\beta\sigma\tau}$ given in Section 2 that, on the other hand,

$$b_{\alpha\beta\sigma\tau} M_{\sigma\tau} N_{\alpha\beta} = \partial_{\alpha\beta}\zeta_3(\Delta\phi\delta_{\alpha\beta} - \partial_{\alpha\beta}\phi) = [\phi,\zeta_3] \quad \text{in } \omega,$$

where $[\cdot,\cdot]$ denotes the Monge-Ampère bilinear form. Using these two relations in the first equation of (4.1) gives

$$\varepsilon^3 \frac{8\mu(\lambda+\mu)}{3(\lambda+2\mu)}\Delta^2\zeta_3 - \varepsilon[\phi,\zeta_3] = \frac{\varepsilon^3}{3}\partial_{\alpha\beta}M_{\alpha\beta} - \varepsilon b_{\alpha\beta\sigma\tau}M_{\sigma\tau}N_{\alpha\beta} = p_3 \quad \text{in } \omega,$$

which is precisely the first equation of the boundary value problem (3.1).

Next we infer from the relations (4.3) that, on the one hand,

$$\partial_{\alpha\alpha} N_{\beta\beta} = \Delta^2\phi \quad \text{in } \omega.$$

Then we infer from the relations (4.2) and from the definitions of the tensors $a_{\alpha\beta\sigma\tau}$ (Section 2) and $c_{\alpha\beta\sigma\tau}$ (Section 4) that, on the other hand,

$$\begin{aligned} c_{\alpha\beta\sigma\tau} M_{\sigma\tau} M_{\alpha\beta} &= c_{\alpha\beta\sigma\tau}(a_{\sigma\tau\gamma\delta}\partial_{\gamma\delta}\zeta_3)(a_{\alpha\beta\varphi\psi}\partial_{\varphi\psi}\zeta_3) \\ &= \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\partial_{\alpha\alpha}\zeta_3\partial_{\beta\beta}\zeta_3 - \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\partial_{\alpha\beta}\zeta_3\partial_{\alpha\beta}\zeta_3 \\ &= \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}[\zeta_3,\zeta_3] \quad \text{in } \omega. \end{aligned}$$

Using these two relations in the second equation of (4.1) shows that

$$\Delta^2\phi+\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}[\zeta_3,\zeta_3]=\partial_{\alpha\alpha}N_{\beta\beta}+c_{\alpha\beta\sigma\tau}M_{\sigma\tau}M_{\alpha\beta}=0 \quad \text{in } \omega,$$

which is precisely the second equation of the boundary value problem (3.1).

Finally, we infer from the relations (4.2) and (4.3) that the functions ζ_3 and ϕ satisfy the last two equations of the boundary value problem (3.1). This completes the proof. $\square$

5 Weak solutions to the intrinsic von Kármán equations

As usual, $\mathcal{D}(\omega)$ denotes the space of all infinitely differentiable functions $\psi:\omega\to\mathbb{R}$ with compact support contained in ω and $\mathcal{D}'(\omega)$ denotes the space of distributions in ω.

We use the notation $L^p(\omega)$ for the Lebesgue space with exponent $p\geqslant 1$, $H^1(\omega)$ and $H^2(\omega)$ for the Sobolev spaces of functions in $L^2(\omega)$ with derivatives also in $L^2(\omega)$, and $H^2_0(\omega)$ for the closure in $H^2(\omega)$ of its subspace $\mathcal{D}(\omega)$.

The notation $H^{1/2}(\gamma)$ denotes the image of $H^1(\omega)$ by the trace operator and $H^{-1/2}(\gamma)$ denotes the dual space of $H^{1/2}(\gamma)$. The notation $H^{3/2}(\gamma)$ denotes the image of $H^2(\omega)$ by the trace operator and $H^{-3/2}(\gamma)$ denotes the dual space of $H^{3/2}(\gamma)$.

A weak solution to the intrinsic von Kármán equations (4.1) corresponding to the functions

$$p_3\in L^1(\omega), \quad k_\alpha\in H^{-\frac{1}{2}}(\gamma),$$

is a pair of symmetric tensor fields

$$(M_{\alpha\beta})\in L^2(\omega;\mathbb{S}^2), \quad (N_{\alpha\beta})\in L^2(\omega;\mathbb{S}^2),$$

that satisfies the compatibility conditions

$$\begin{aligned} &\partial_\beta N_{\alpha\beta}=0 && \text{in } \omega,\\ &\partial_\alpha(b_{\beta\gamma\sigma\tau}M_{\sigma\tau})-\partial_\beta(b_{\alpha\gamma\sigma\tau}M_{\sigma\tau})=0 && \text{in } \omega, \end{aligned} \tag{5.1}$$

the boundary conditions

$$\begin{aligned} &b_{\alpha\beta\sigma\tau}M_{\sigma\tau}\tau_\beta=0 && \text{on } \gamma,\\ &N_{\alpha\beta}\nu_\beta=k_\alpha && \text{on } \gamma, \end{aligned} \tag{5.2}$$

and the variational equations

$$\begin{aligned}&\frac{\varepsilon^3}{3}\int_\omega M_{\alpha\beta}\partial_{\alpha\beta}\eta\,\mathrm{d}y-\varepsilon\int_\omega(b_{\alpha\beta\sigma\tau}M_{\sigma\tau}N_{\alpha\beta})\eta\,\mathrm{d}y=\int_\omega p_3\eta\,\mathrm{d}y,\\&\int_\omega N_{\alpha\alpha}\partial_{\beta\beta}\psi\,\mathrm{d}y+\int_\omega(c_{\alpha\beta\sigma\tau}M_{\sigma\tau}M_{\alpha\beta})\psi\,\mathrm{d}y=0\end{aligned}\tag{5.3}$$

for all $\eta\in H_0^2(\omega)$ and all $\psi\in H_0^2(\omega)$.

The boundary conditions (5.2) make sense thanks to the compatibility conditions (5.1) satisfied by $M_{\alpha\beta}$ and $N_{\alpha\beta}$, which in effect provide these functions with sufficient regularity in order to define their traces on γ appearing in (5.2). More specifically, $(N_{\alpha\beta}\nu_\beta)$ and $(b_{\alpha\beta\sigma\tau}M_{\sigma\tau}\tau_\beta)$ are well-defined in $H^{-1/2}(\gamma)$ by

$$\begin{aligned}&\langle N_{\alpha\beta}\nu_\beta,\eta\rangle:=\int_\omega(N_{\alpha\beta}\partial_\beta\eta)\mathrm{d}y,\\&\langle b_{\alpha\beta\sigma\tau}M_{\sigma\tau}\tau_\beta,\psi\rangle:=\int_\omega(b_{\alpha1\sigma\tau}M_{\sigma\tau}\partial_2\psi-b_{\alpha2\sigma\tau}M_{\sigma\tau}\partial_1\psi)\mathrm{d}y\end{aligned}\tag{5.4}$$

for all functions $\eta\in H^1(\omega)$ and $\psi\in H^1(\omega)$, where $\langle\cdot,\cdot\rangle$ denotes the duality bracket between $H^{-1/2}(\gamma)$ and $H^{1/2}(\gamma)$.

Note that these trace operators extend the corresponding trace operators for smooth tensor fields $(M_{\alpha\beta})\in H^1(\omega;\mathbb{S}^2)$ and $(N_{\alpha\beta})\in H^1(\omega;\mathbb{S}^2)$, since

$$\int_\gamma(N_{\alpha\beta}\nu_\beta)\eta\,\mathrm{d}\gamma=\int_\omega\big(N_{\alpha\beta}\partial_\beta\eta+(\partial_\beta N_{\alpha\beta})\eta\big)\,\mathrm{d}y$$

for all functions $\eta\in H^1(\omega)$, and

$$\begin{aligned}\int_\gamma(b_{\alpha\beta\sigma\tau}M_{\sigma\tau}\tau_\beta)\psi\,\mathrm{d}\gamma=&\int_\omega(b_{\alpha1\sigma\tau}M_{\sigma\tau}\partial_2\psi-b_{\alpha2\sigma\tau}M_{\sigma\tau}\partial_1\psi)\mathrm{d}y\\&+\int_\omega\big(\partial_2(b_{\alpha1\sigma\tau}M_{\sigma\tau})-\partial_1(b_{\alpha2\sigma\tau}M_{\sigma\tau})\big)\psi\,\mathrm{d}y\end{aligned}$$

for all functions $\psi\in H^1(\omega)$.

The above definition of weak solution to the intrinsic von Kármán equations (4.1) is justified by the following theorem.

Theorem 5.1. *Two symmetric tensor fields* $(M_{\alpha\beta})\in\mathcal{C}^2(\overline{\omega};\mathbb{S}^2)$ *and* $(N_{\alpha\beta})\in\mathcal{C}^2(\overline{\omega};\mathbb{S}^2)$ *satisfy the intrinsic von Kármán equations* (4.1) *if and only if they satisfy the compatibility conditions* (5.1)*, the boundary conditions* (5.2)*, and the variational equations* (5.3).

Proof. First, let $(M_{\alpha\beta})\in\mathcal{C}^2(\overline{\omega};\mathbb{S}^2)$ and $(N_{\alpha\beta})\in\mathcal{C}^2(\overline{\omega};\mathbb{S}^2)$ be tensor fields that satisfy the Eqs. (4.1). Then, for all $\eta\in H_0^2(\omega)$ and all $\psi\in H_0^2(\omega)$, we have

$$\begin{aligned}\int_\omega p_3\eta\,\mathrm{d}y&=\frac{\varepsilon^3}{3}\int_\omega(\partial_{\alpha\beta}M_{\alpha\beta})\eta\,\mathrm{d}y-\varepsilon\int_\omega(b_{\alpha\beta\sigma\tau}M_{\sigma\tau}N_{\alpha\beta})\eta\,\mathrm{d}y\\&=\frac{\varepsilon^3}{3}\int_\omega M_{\alpha\beta}\partial_{\alpha\beta}\eta\,\mathrm{d}y-\varepsilon\int_\omega(b_{\alpha\beta\sigma\tau}M_{\sigma\tau}N_{\alpha\beta})\eta\,\mathrm{d}y,\end{aligned}$$

and

$$\begin{aligned}0&=\int_\omega(\partial_{\alpha\alpha}N_{\beta\beta})\psi\,\mathrm{d}y+\int_\omega(c_{\alpha\beta\sigma\tau}M_{\sigma\tau}M_{\alpha\beta})\psi\,\mathrm{d}y\\&=\int_\omega N_{\beta\beta}\partial_{\alpha\alpha}\psi\,\mathrm{d}y+\int_\omega(c_{\alpha\beta\sigma\tau}M_{\sigma\tau}M_{\alpha\beta})\psi\,\mathrm{d}y.\end{aligned}$$

This proves that the functions $M_{\alpha\beta}$ and $N_{\alpha\beta}$ satisfy the variational equations (5.3). That they also satisfy the compatibility conditions (5.1) and the boundary conditions (5.2) is clear.

Secondly, let $(M_{\alpha\beta})\in\mathcal{C}^2(\overline{\omega};\mathbb{S}^2)$ and $(N_{\alpha\beta})\in\mathcal{C}^2(\overline{\omega};\mathbb{S}^2)$ be tensor fields that satisfy the compatibility conditions (5.1), the boundary conditions (5.2), and the variational equations (5.3). In particular then,

$$\frac{\varepsilon^3}{3}\int_\omega M_{\alpha\beta}\partial_{\alpha\beta}\eta\,\mathrm{d}y-\varepsilon\int_\omega(b_{\alpha\beta\sigma\tau}M_{\sigma\tau}N_{\alpha\beta})\eta\,\mathrm{d}y=\int_\omega p_3\eta\,\mathrm{d}y$$

for all $\eta\in\mathcal{D}(\omega)$ and

$$\int_\omega N_{\beta\beta}\partial_{\alpha\alpha}\psi\,\mathrm{d}y+\int_\omega(c_{\alpha\beta\sigma\tau}M_{\sigma\tau}M_{\alpha\beta})\psi\,\mathrm{d}y=0$$

for all $\psi\in\mathcal{D}(\omega)$.

Consequently,

$$\int_\omega\left(\frac{\varepsilon^3}{3}(\partial_{\alpha\beta}M_{\alpha\beta})-\varepsilon(b_{\alpha\beta\sigma\tau}M_{\sigma\tau}N_{\alpha\beta})-p_3\right)\eta\,\mathrm{d}y=0$$

for all $\eta\in\mathcal{D}(\omega)$, and

$$\int_\omega\big((\partial_{\alpha\alpha}N_{\beta\beta})+(c_{\alpha\beta\sigma\tau}M_{\sigma\tau}M_{\alpha\beta})\big)\psi\,\mathrm{d}y=0$$

for all $\psi\in\mathcal{D}(\omega)$.

Since η and ψ are arbitrary functions in the space $\mathcal{D}(\omega)$, the two relations above show that the functions $M_{\alpha\beta}$ and $N_{\alpha\beta}$ satisfy the first two equations of the system (4.1) in the distributional sense, hence in the classical sense since the functions $M_{\alpha\beta}$ and $N_{\alpha\beta}$ belong to the space $\mathcal{C}^2(\overline{\omega})$. That the remaining equations of the system (4.1) are satisfied is clear. The proof is complete. $\square$

We conclude this section by establishing the existence of weak solutions to the intrinsic von Kármán equations.

Theorem 5.2. *Assume that ω is a simply-connected open subset of $\mathbb{R}^2$ with a boundary $\gamma := \partial\omega$ of class $\mathcal{C}^2$. Let*

$$p_3 \in L^1(\omega), \quad k_\alpha := \partial_\tau(\nu_\alpha \partial_\tau \phi_0 - \tau_\alpha \phi_1) \in H^{-\frac{1}{2}}(\gamma),$$

where $\phi_0 \in H^{3/2}(\gamma)$ and $\phi_1 \in H^{1/2}(\gamma)$ are two given functions.

Then the intrinsic von Kármán equations (4.1) *corresponding to the functions p_3 and k_α possess a weak solution $(M_{\alpha\beta}) \in L^2(\omega;\mathbb{S}^2)$ and $(N_{\alpha\beta}) \in L^2(\omega;\mathbb{S}^2)$.*

Proof. The assumptions of the theorem on p_3, ϕ_0 and ϕ_1 imply that there exists a pair of functions $\phi \in H^2(\omega)$ and $\zeta_3 \in H^2(\omega)$ that satisfies the von Kármán equations (3.1) corresponding to the data p_3, ϕ_0 and ϕ_1, cf. [3–5,16].

Define the functions

$$M_{\alpha\beta} := a_{\alpha\beta\sigma\tau} \partial_{\sigma\tau} \zeta_3, \quad N_{\alpha\beta} := \Delta\phi \delta_{\alpha\beta} - \partial_{\alpha\beta}\phi.$$

Then $M_{\alpha\beta} = M_{\beta\alpha} \in L^2(\omega)$ and $N_{\alpha\beta} = N_{\beta\alpha} \in L^2(\omega)$. Besides, we infer from the symmetry of the second derivatives of ϕ and ζ_3 and from the definition of the tensors $a_{\beta\gamma\sigma\tau}$ and $b_{\beta\gamma\sigma\tau}$ (see Section 2) that

$$\partial_\beta N_{\alpha\beta} = \partial_\alpha(\Delta\phi) - \partial_\beta(\partial_{\alpha\beta}\phi) = 0 \quad \text{in } \mathcal{D}'(\omega),$$

and

$$\partial_\alpha(b_{\beta\gamma\sigma\tau} M_{\sigma\tau}) - \partial_\beta(b_{\alpha\gamma\sigma\tau} M_{\sigma\tau}) = \partial_\alpha(\partial_{\beta\gamma}\zeta_3) - \partial_\beta(\partial_{\alpha\gamma}\zeta_3) = 0 \quad \text{in } \mathcal{D}'(\omega).$$

Next, using the boundary conditions satisfied by the function ζ_3 (see (3.1)) and the definition (5.4) of the trace on γ of $(b_{\alpha\beta\sigma\tau} M_{\sigma\tau} \tau_\beta)$, we deduce that

$$b_{\alpha\beta\sigma\tau} M_{\sigma\tau} \tau_\beta = \tau_\beta \partial_{\alpha\beta}\zeta_3 = \partial_\tau(\tau_\alpha \partial_\tau \zeta_3 + \nu_\alpha \partial_\nu \zeta_3) = 0 \quad \text{in } H^{-\frac{1}{2}}(\gamma).$$

Using the boundary conditions satisfied by the function ϕ (see (3.1)), the definition (5.4) of the traces of $N_{\alpha\beta}\nu_\beta$ on γ, the definition of the Levi-Civita permutation symbol $\varepsilon_{\alpha\beta}$, and the chosen orientation of the Cartesian bases along γ formed by the tangent and normal vector fields (τ_α) and (ν_α), we deduce that

$$\begin{aligned} N_{\alpha\beta}\nu_\beta &= \nu_\beta \varepsilon_{\alpha\sigma}\varepsilon_{\beta\delta}\partial_{\sigma\delta}\phi = \varepsilon_{\sigma\alpha}\tau_\delta\partial_{\sigma\delta}\phi = \varepsilon_{\sigma\alpha}\partial_\tau(\partial_\sigma\phi) \\ &= \varepsilon_{\sigma\alpha}\partial_\tau(\tau_\sigma\partial_\tau\phi + \nu_\sigma\partial_\nu\phi) = \partial_\tau(\nu_\alpha\partial_\tau\phi - \tau_\alpha\partial_\nu\phi) \\ &= \partial_\tau(\nu_\alpha\partial_\tau\phi_0 - \tau_\alpha\phi_1) = k_\alpha \quad \text{in } H^{-\frac{1}{2}}(\gamma). \end{aligned}$$

We established in the proof of Theorem 4.1 that

$$\begin{aligned} \partial_{\alpha\beta}M_{\alpha\beta} &= \frac{8\mu(\lambda+\mu)}{\lambda+2\mu}\Delta^2\zeta_3 \quad \text{in } \omega, \\ b_{\alpha\beta\sigma\tau}M_{\sigma\tau}N_{\alpha\beta} &= [\phi,\zeta_3] \qquad \text{in } \omega \end{aligned}$$

under the assumption that $\zeta_3,\phi \in \mathcal{C}^4(\overline{\omega})$. A similar argument shows that the above relations still hold in the distributional sense under the weaker assumption that $\zeta_3,\phi \in H^2(\omega)$. Then the first equation of (3.1), viz.,

$$\varepsilon^3\frac{8\mu(\lambda+\mu)}{3(\lambda+2\mu)}\Delta^2\zeta_3 - \varepsilon[\phi,\zeta_3] = p_3 \quad \text{in } \mathcal{D}'(\omega),$$

implies that

$$\frac{\varepsilon^3}{3}\partial_{\alpha\beta}M_{\alpha\beta} - \varepsilon b_{\alpha\beta\sigma\tau}M_{\sigma\tau}N_{\alpha\beta} = p_3 \quad \text{in } \mathcal{D}'(\omega),$$

or equivalently, that

$$\frac{\varepsilon^3}{3}\int_\omega M_{\alpha\beta}\partial_{\alpha\beta}\eta\,\mathrm{d}y - \varepsilon\int_\omega (b_{\alpha\beta\sigma\tau}M_{\sigma\tau}N_{\alpha\beta})\eta\,\mathrm{d}y = \int_\omega p_3\eta\,\mathrm{d}y$$

for all $\eta \in H_0^2(\omega)$. Thus the tensor fields $M_{\alpha\beta} \in L^2(\omega)$ and $N_{\alpha\beta} \in L^2(\omega)$ satisfy the first equation of the variational equation (5.3).

We also established in the proof of Theorem 4.1 that

$$\partial_{\alpha\alpha}N_{\beta\beta} = \Delta^2\phi \quad \text{in } \omega,$$

and

$$c_{\alpha\beta\sigma\tau}M_{\sigma\tau}M_{\alpha\beta} = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}[\zeta_3,\zeta_3] \quad \text{in } \omega$$

under the assumption that $\phi,\zeta_3 \in \mathcal{C}^4(\overline{\omega})$. A similar argument shows that the above relations still hold, albeit only in the distributional sense, under the weaker assumption that $\phi,\zeta_3 \in H^2(\omega)$. Then the second equation of (3.1), viz.,

$$\Delta^2\phi + \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}[\zeta_3,\zeta_3] = 0 = p_3 \quad \text{in } \mathcal{D}'(\omega),$$

implies that

$$\partial_{\alpha\alpha} N_{\beta\beta} + c_{\alpha\beta\sigma\tau} M_{\sigma\tau} M_{\alpha\beta} = 0 \quad \text{in } \mathcal{D}'(\omega),$$

or equivalently,

$$\int_\omega N_{\alpha\alpha} \partial_{\beta\beta} \psi \, \mathrm{d}y + \int_\omega (c_{\alpha\beta\sigma\tau} M_{\sigma\tau} M_{\alpha\beta}) \psi \, \mathrm{d}y = 0$$

for all $\psi \in H_0^2(\omega)$. Thus the tensor fields $M_{\alpha\beta} \in L^2(\omega)$ and $N_{\alpha\beta} \in L^2(\omega)$ also satisfy the second equation of the variational equation (5.3). This completes the proof of the theorem. □

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