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UNCONDITIONAL CONVERGENCE AND ERROR ESTIMATES OF A FULLY DISCRETE FINITE ELEMENT METHOD FOR THE MICROPOLAR NAVIER-STOKES EQUATIONS*

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Abstract

In this paper, we consider the initial-boundary value problem (IBVP) for the micropolar Naviers-Stokes equations (MNSE) and analyze a first order fully discrete mixed finite element scheme. We first establish some regularity results for the solution of MNSE, which seem to be not available in the literature. Next, we study a semi-implicit time-discrete scheme for the MNSE and prove $\boldsymbol{L}^2$-$\boldsymbol{H}^1$ error estimates for the time discrete solution. Furthermore, certain regularity results for the time discrete solution are establishes rigorously. Based on these regularity results, we prove the unconditional $\boldsymbol{L}^2$-$\boldsymbol{H}^1$ error estimates for the finite element solution of MNSE. Finally, some numerical examples are carried out to demonstrate both accuracy and efficiency of the fully discrete finite element scheme.

Mathematics subject classification: 76M10, 65M12, 65M15, 65M60.
Key words: Micropolar fluids, Regularity estimates, Euler semi-implicit scheme, Mixed finite element methods, Unconditional convergence.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded convex polyhedron domain. We consider the homogeneous incompressible viscous Newtonian Micropolar fluids. The microstructure systems consists of the incompressible Navier-Stokes equations for velocity with pressure and angular momentum equations for angular velocity, which are described by (see, e.g., [9, 10])

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - (\mu+\mu_r)\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} + 2\mu_r \mathbf{curl}\,\boldsymbol{w}, \tag{1.1}$$

$$\operatorname{div}\boldsymbol{u} = 0, \tag{1.2}$$

$$\partial_t \boldsymbol{w} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{w} - c_1\Delta\boldsymbol{w} - c_2\nabla\operatorname{div}\boldsymbol{w} + 4\mu_r\boldsymbol{w} = \boldsymbol{g} + 2\mu_r\mathbf{curl}\,\boldsymbol{u} \tag{1.3}$$

for $(x,t)\in\Omega\times(0,T)$, where $\boldsymbol{u}$ is the linear velocity; $\boldsymbol{w}$ is the angular velocity; p is the fluid pressure, $\boldsymbol{f}$ is the density of external body forces per unit mass, $\boldsymbol{g}$ is the body source of moments. c_1, c_2, μ_r and μ are material coefficients which are all constant coefficients greater than zero. The system (1.1)-(1.3) is supplemented with initial conditions for the linear velocity and the angular velocity

$$\boldsymbol{u}(x,0)=\boldsymbol{u}_0, \quad \boldsymbol{w}(x,0)=\boldsymbol{w}_0, \tag{1.4}$$

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together with the following the no-slip boundary condition for the linear velocity and the angular velocity:

$$\boldsymbol{u} = \mathbf{0}, \quad \boldsymbol{w} = \mathbf{0} \quad \text{on } \partial\Omega \times (0,T). \tag{1.5}$$

The system (1.1)-(1.5) describes the fluid particles may translate and rotate independently of the fluid (see, e.g., [4, 9, 10]). The Micropolar fluid models degenerates into the classical Navier-Stokes equations when microrotation is neglected ($\boldsymbol{w} = \mathbf{0}$) and $\mu_r = 0$. In the literature, many numerical methods have been developed for solving the nonstationary Navier-Stokes equations, see, e.g., [14,16,18,22,24] and the references therein. The microstructure fluid models is an important generalization of the incompressible Navier-Stokes equations. The derivation and physical discussion of the MNSE are referred to [3, 10, 21]. It is well konwn that the microstructure systems plays an important role in smart fluids and polarizable media [28,31]. In recent years, the micromachining technology has been used to develop a number of microfluidic systems, for example, silicon, glass, quartz and plastics [31]. Microchannels and chambers are the essential part of micropolar fluid system. Microchannels are also used for reactant transport, such as biochemical reaction chambers, physical particle separation, inkjet print heads, or as heat exchangers to cool computer chips. For more information about microchannels, we refer to [31] and the references therein.

The existence of solutions for the Micropolar fluid equations was established by Łukaszewicz [21]. In recent years, some attention has been paid to the numerical methods for this microstructure system. The penalty finite element method for the Micropolar fluid equations was proposed by Ortega *et al.* [30]. An important progress was made by Nochetto *et al.* [29], they proposed and analyzed an unconditionally stable semi-implicit fully discrete finite element scheme for MNSE with the nice features that the computation of the linear and angular velocities are decoupled. More recently, the convergence analysis of fractional time-stepping techniques for the Micropolar fluid equations had been obtained by Salgado [32]. The fractional time-stepping schemes or projection methods are known as an efficient decoupled schemes for incompressible flows, some extensions and applications of these methods to MNSE can be found in [20] with semi-discrete schemes and [35, 40] with fully discrete schemes. However, in our opinion, there are still some important problems in these references that remain to be solved. Firstly, all the convergence results for the finite element methods of the above works are obtained under proper regularity assumptions on the exact solution of the Micropolar fluid equations, which seems to be lacking in the literature. Secondly, due to the highly nonlinear structures of MNSE, the analysis of fully discrete finite element methods often requires that the time step τ satisfies an CFL like condition. For example, the condition $\tau h^{-1/2} \leq C$ are proposed for the error estimates in [29], where h is the spacial mesh width. Similar conditions are also imposed in the analysis in [32, 40]. Such smallness assumption on the time step will affect the applications of the finite element methods for MNSE.

The aim of this work is threefold. Firstly, we show certain regularity results for the solution of MNSE with the help of the energy method (see, e.g., [16, 36]). These regularity results are essential for the error estimates of numerical methods for the Micropolar fluid equations. Secondly, we will give the $\boldsymbol{L}^2$-$\boldsymbol{H}^1$ error estimates for the Euler semi-implicit time discrete solution of MNSE. Furthermore, certain regularity results for the time discrete solution are also proved rigorously, which play a key role in the unconditional convergence analysis of fully discrete finite element method. However, for some of them, we have not seen a strict proof in relevant literature. Lastly, we consider the Euler semi-implicit fully discrete scheme based on mixed finite element method for the MNSE and prove the unconditional $\boldsymbol{L}^2$-$\boldsymbol{H}^1$ error estimates.

For this purpose, the error between the exact solution and the solution of the fully discrete scheme are divided into three parts: the error between the exact solution and the time discrete solution, the time discrete solution and certain Galerkin projection of the time discrete solution, Galerkin projection of the time discrete solution and the fully discrete solution. Concerning more unconditional convergence results of nonlinear partial differential equation, interested readers are referred to [11,14,15,23,25,34] and the references cited therein. Note that although the three-dimensional case is analyzed in this paper, all the results of this paper are certainly valid for two-dimensional domains.

The rest of this work is organized as follows. In Section 2, we introduce some notations and basic assumptions for this work. In Section 3, we establish some regularity results for the solution of the MNSE. In Section 4, we propose an Euler semi-implicit time discrete schemes to the MNSE and show the $\boldsymbol{L}^2$-$\boldsymbol{H}^1$ error estimate for the time-discrete solution. We developed some regularity results for the time-discrete solution in Section 5. In Section 6, we show $\boldsymbol{L}^2$-$\boldsymbol{H}^1$ unconditional convergence of the fully discrete solution. Finally, we provide some numerical experiments validation of this error estimates in Section 7.

2. Notation and Preliminaries

In this section, we present some notations and preliminary results for this work. For $1 \leq p \leq \infty$, let $L^p(\Omega)$ be the usual Lebesgue space on Ω. For all non-negative integers k and r, $W^{k,r}(\Omega)$ stands for the standard Sobolev spaces on Ω. Moreover, we write $H^k(\Omega) = W^{k,2}(\Omega)$. We introduce the Hilbert spaces like $H^1(\Omega)$, $L^2(\Omega)$ and their subspaces $H_0^1(\Omega)$, $L_n^2(\Omega)$, which is zero on $\partial\Omega$. Finally, we also introduce the following classical divergence-free spaces (see, e.g., [36]):

$$\begin{aligned}
\boldsymbol{L}^2_{\mathrm{div}}(\Omega) &= \left\{ \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) |\, \operatorname{div} \boldsymbol{v} = 0,\ \boldsymbol{v}|_{\partial\Omega} = \boldsymbol{0} \right\}, \\
\boldsymbol{H}^1_{\mathrm{div}}(\Omega) &= \left\{ \boldsymbol{v} \in \boldsymbol{H}^1(\Omega) |\, \operatorname{div} \boldsymbol{v} = 0,\ \boldsymbol{v}|_{\partial\Omega} = \boldsymbol{0} \right\}.
\end{aligned}$$

The space $L_0^2(\Omega)$ stands for the average of the space $L^2(\Omega)$ equal to zero on Ω. The $L^2(\Omega)$ inner product is denoted by $(\cdot,\cdot)$. The vector-valued quantities will be denoted in boldface notations, such as $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{L}^2(\Omega) = (L^2(\Omega))^3$.

By applying the Green's formula for the curl operator, we have the following the identity:

$$(\mathbf{curl}\, \boldsymbol{u}, \boldsymbol{v}) = (\mathbf{curl}\, \boldsymbol{v}, \boldsymbol{u}), \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega). \tag{2.1}$$

Under the assumptions that Ω is bounded and simply connected, there exists the orthogonal decomposition of $\boldsymbol{H}_0^1(\Omega)$ (see, e.g., [13])

$$\|\nabla \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} = \|\operatorname{div} \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\, \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}, \quad \forall\, \boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega). \tag{2.2}$$

For convenience, we introduce some bilinear and trilinear forms as follows:

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} dx, \quad d(\boldsymbol{v}, q) = \int_\Omega \operatorname{div} \boldsymbol{v} q dx, \quad b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \int_\Omega (\boldsymbol{u} \cdot \nabla) \boldsymbol{v} \cdot \boldsymbol{w} dx$$

for any $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, q) \in (\boldsymbol{H}_0^1(\Omega))^3 \times L_0^2(\Omega)$. If $\operatorname{div} \boldsymbol{u} = 0$, the trilinear forms have following relations (see, e.g., [36]):

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = 0, \quad b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}). \tag{2.3}$$

Multiplying the system (1.1)-(1.5) by test functions $(\boldsymbol{v}, q, \boldsymbol{\psi}) \in \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega) \times \boldsymbol{H}_0^1(\Omega)$ and integrating by parts, respectively, then the weak formulation of the system (1.1)-(1.5) can be written as: Find $(\boldsymbol{u}, p, \boldsymbol{w}) \in \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega) \times \boldsymbol{H}_0^1(\Omega)$ such that

$$\begin{aligned}
&(\boldsymbol{u}_t, \boldsymbol{v}) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + (\mu + \mu_r) a(\boldsymbol{u}, \boldsymbol{v}) - d(\boldsymbol{v}, p) + d(\boldsymbol{u}, q) \\
=& (\boldsymbol{f}, \boldsymbol{v}) + 2\mu_r (\mathbf{curl}\, \boldsymbol{w}, \boldsymbol{v}), \qquad (2.4) \\
&(\boldsymbol{w}_t, \boldsymbol{\psi}) + b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{\psi}) + c_1 a(\boldsymbol{w}, \boldsymbol{\psi}) + c_2 (\operatorname{div} \boldsymbol{w}, \operatorname{div} \boldsymbol{\psi}) + 4\mu_r (\boldsymbol{w}, \boldsymbol{\psi}) \\
=& (\boldsymbol{g}, \boldsymbol{\psi}) + 2\mu_r (\mathbf{curl}\, \boldsymbol{u}, \boldsymbol{\psi}). \qquad (2.5)
\end{aligned}$$

The weak formulation (2.4)-(2.5) of pressure p can be eliminated in the usual way for the test function $\boldsymbol{v} \in \boldsymbol{H}_{\mathrm{div}}^1(\Omega)$. Thus we can show the solution $(\boldsymbol{u}, \boldsymbol{w})$ of the system (2.4)-(2.5) satisfies the following weak formulation: for any $(\boldsymbol{v}, \boldsymbol{\psi}) \in \boldsymbol{H}_{\mathrm{div}}^1(\Omega) \times \boldsymbol{H}_0^1(\Omega)$, find $(\boldsymbol{u}, \boldsymbol{w}) \in \boldsymbol{H}_{\mathrm{div}}^1(\Omega) \times \boldsymbol{H}_0^1(\Omega)$ such that

$$\begin{aligned}
&(\boldsymbol{u}_t, \boldsymbol{v}) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + (\mu + \mu_r) a(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) + 2\mu_r (\mathbf{curl}\, \boldsymbol{w}, \boldsymbol{v}), \qquad (2.6) \\
&(\boldsymbol{w}_t, \boldsymbol{\psi}) + b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{\psi}) + c_1 a(\boldsymbol{w}, \boldsymbol{\psi}) + c_2 (\operatorname{div} \boldsymbol{w}, \operatorname{div} \boldsymbol{\psi}) + 4\mu_r (\boldsymbol{w}, \boldsymbol{\psi}) \\
=& (\boldsymbol{g}, \boldsymbol{\psi}) + 2\mu_r (\mathbf{curl}\, \boldsymbol{u}, \boldsymbol{\psi}). \qquad (2.7)
\end{aligned}$$

The existence of weak solution for the problem (2.4)-(2.5) is proved by Theorem 1.6.1 of [21] (or see, e.g., [29]). However, these results are not enough to the subsequent convergence analysis of numerical methods. In next section, we will prove some regularity results on the solution of the problem (2.4)-(2.5). To this end, we make the following assumptions for the problem (2.4)-(2.5).

Hypothesis 2.1. *The initial datas* $\boldsymbol{u}_0 \in \boldsymbol{H}_{\mathrm{div}}^1(\Omega) \cap \boldsymbol{H}^2(\Omega)$, $\boldsymbol{w}_0 \in \boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{H}^2(\Omega)$, $p_0 \in L_0^2(\Omega) \cap \boldsymbol{H}^1(\Omega)$, *the external force* $\boldsymbol{f}$ *and the body source* $\boldsymbol{g}$ *satisfies the following boundedness:*

$$\begin{aligned}
&\sup_{0 \le t \le T} (\|\boldsymbol{f}(t)\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{g}(t)\|_{\boldsymbol{L}^2(\Omega)} + \|\partial_t \boldsymbol{f}(t)\|_{\boldsymbol{L}^2(\Omega)} + \|\partial_t \boldsymbol{g}(t)\|_{\boldsymbol{L}^2(\Omega)}) \\
&+ \|\boldsymbol{u}_0\|_{\boldsymbol{H}^2(\Omega)} + \|\boldsymbol{w}_0\|_{\boldsymbol{H}^2(\Omega)} \le C.
\end{aligned}$$

Here and after, we denote by $C > 0$ a general constant which depends on the domain Ω, the fixed time T, the material coefficients (μ_r, μ, c_1, c_2), the initial data $(\boldsymbol{u}_0, \boldsymbol{w}_0)$ and the external forces $(\boldsymbol{f}, \boldsymbol{g})$.

Hypothesis 2.2. *The system* (2.4)-(2.5) *has a weak solution* $\boldsymbol{u} \in L^2(0, T; \boldsymbol{H}_{\mathrm{div}}^1(\Omega))$, $\boldsymbol{w} \in L^2(0, T; \boldsymbol{H}_0^1(\Omega))$ *satisfying*

$$\int_0^T \|\nabla \boldsymbol{u}(t)\|_{\boldsymbol{L}^2(\Omega)}^4 dt \le C.$$

Hypothesis 2.3. *We assume that the boundary of Ω is smooth so that the unique solution $(\boldsymbol{v}, q) \in \boldsymbol{H}_{\mathrm{div}}^1(\Omega) \times L_0^2(\Omega)$ of the steady Stokes problem*

$$-\Delta \boldsymbol{v} + \nabla p = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{v}|_\Omega = 0, \quad \boldsymbol{v}|_{\partial\Omega} = \mathbf{0},$$

and the unique solution $\boldsymbol{\psi} \in \boldsymbol{H}_0^1(\Omega)$ of the pure displacement problem

$$-\Delta \boldsymbol{\psi} - \nabla \operatorname{div} \boldsymbol{\psi} = \boldsymbol{g}, \quad \boldsymbol{\psi}|_{\partial\Omega} = 0,$$

for the functions $\boldsymbol{f}, \boldsymbol{g} \in \boldsymbol{L}^2(\Omega)$ satisfies the following regularity estimates:

$$\begin{aligned}
&\|\boldsymbol{v}\|_{\boldsymbol{H}^2(\Omega)} + \|q\|_{H^1(\Omega)} \le C \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}, \\
&\|\boldsymbol{\psi}\|_{\boldsymbol{H}^2(\Omega)} + \|\operatorname{div} \boldsymbol{\psi}\|_{H^1(\Omega)} \le C \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}.
\end{aligned}$$

Remark 2.1. It is well known that the Hypothesis 2.3 holds under the assumption that the boundary $\partial\Omega$ is convex polyhedron or of class $C^{1,1}$ (see, e.g., [2, 16]).

Finally, we introduce the stokes operator $\mathscr{A}_1$ and the Lamé operator $\mathscr{A}_2$ such that

$$\begin{aligned} &\mathscr{A}_1 : D(\mathscr{A}_1) = \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_{\rm div}(\Omega) \to \boldsymbol{L}^2_{\rm div}(\Omega), \quad \mathscr{A}_1 = -\mathcal{P}_1\Delta, \\ &\mathscr{A}_2 : D(\mathscr{A}_2) = \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_0(\Omega) \to \boldsymbol{L}^2_n(\Omega), \qquad \mathscr{A}_2 = -\mathcal{P}_2(c_1\Delta + c_2\nabla\,{\rm div}), \end{aligned}$$

where $\mathcal{P}_1$ is an $\boldsymbol{L}^2$-projection from $\boldsymbol{L}^2(\Omega)$ to $\boldsymbol{L}^2_{\rm div}(\Omega)$ and $\mathcal{P}_2$ is an $\boldsymbol{L}^2$-projection from $\boldsymbol{L}^2(\Omega)$ to $\boldsymbol{L}^2_n(\Omega)$. The strongly elliptic operator (or Lamé operator) $\mathscr{A}_2$ is indeed a positive operator. Clearly, $\|\mathscr{A}_2\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}$ and $\|\boldsymbol{v}\|_{\boldsymbol{H}^2(\Omega)}$ are equivalent norms in $D(\mathscr{A}_2)$. It is seen that the following inequalities hold (see, e.g., [12, 16]):

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^\infty(\Omega)} + \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,3}(\Omega)} \le C\|\boldsymbol{v}\|^{\frac{1}{2}}_{\boldsymbol{H}^1(\Omega)}\|\mathscr{A}_i\boldsymbol{v}\|^{\frac{1}{2}}_{\boldsymbol{L}^2(\Omega)}, \quad \forall\,\boldsymbol{v} \in D(\mathscr{A}_i), \tag{2.8}$$

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^2(\Omega)} \le C\|\mathscr{A}_i\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}, \qquad \forall\,\boldsymbol{v} \in D(\mathscr{A}_i). \tag{2.9}$$

3. Some a Priori Estimates of the Solution

In this section, we establish some a priori estimates for the solution of Micropolar fluids by energy technique. These a priori estimates imply the regularity estimates of the exact solution, which are necessary for the subsequent error estimates of the fully discrete scheme. Before proceeding further, we need the following Gronwall Lemma (see, e.g., remark to Lemma 1 in [17]):

Lemma 3.1. *Let $\phi(t)$, $\varphi(t)$ and $\eta(t)$ be nonnegative continuous functions in $[0,T]$ such that*

$$\phi(t) + \int_0^t \varphi(s)ds \le \phi(0) + \int_0^t \eta(s)\phi(s)ds + C_0,$$

where the constant C_0 is nonnegative, then

$$\phi(t) + \int_0^t \varphi(t)ds \le (\phi(0) + C_0)\exp\left(\int_0^t \eta(s)ds\right), \quad \forall\, 0 \le t \le T.$$

We first present the following energy estimate for the solution of the problem (2.4)-(2.5) which has been given in [29].

Theorem 3.1. *Suppose that Hypothesis* 2.1 *holds. The solution $(\boldsymbol{u}, \boldsymbol{w}, p)$ of the problem* (2.4)-(2.5) *satisfies*

$$\begin{aligned} &\sup_{0\le t\le T}\left(\|\boldsymbol{u}(t)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{w}(t)\|^2_{\boldsymbol{L}^2(\Omega)}\right) \\ &\quad + \int_0^T (\|\nabla\boldsymbol{u}(t)\|^2_{\boldsymbol{L}^2(\Omega)} + \|p(t)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla\boldsymbol{w}(t)\|^2_{\boldsymbol{L}^2(\Omega)})dt \le C. \end{aligned} \tag{3.1}$$

This energy estimate implies that the solution of the problem (2.4)-(2.5) satisfies

$$\begin{aligned} &\boldsymbol{u} \in L^2(0,T;\boldsymbol{H}^1_{\rm div}(\Omega)) \cap L^\infty(0,T;\boldsymbol{L}^2_{\rm div}(\Omega)), \\ &\boldsymbol{w} \in L^2(0,T;\boldsymbol{H}^1_0(\Omega)) \cap L^\infty(0,T;\boldsymbol{L}^2(\Omega)). \end{aligned}$$

In the next step, we will show that

$$\begin{aligned} &\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{H}^1_{\rm div}(\Omega)) \cap L^2(0,T;\boldsymbol{H}^2(\Omega)), \\ &\boldsymbol{w} \in L^\infty(0,T;\boldsymbol{H}^1_0(\Omega)) \cap L^2(0,T;\boldsymbol{H}^2(\Omega)). \end{aligned}$$

Theorem 3.2. *Suppose that Hypothesis* 2.1-2.2 *hold. Then the solution* $(\boldsymbol{u}, \boldsymbol{w})$ *of the problem* (2.4)-(2.5) *satisfies*

$$\sup_{0\le t\le T}\Big(\|\boldsymbol{u}(t)\|^2_{\boldsymbol{H}^1(\Omega)}+\|\boldsymbol{w}(t)\|^2_{\boldsymbol{H}^1(\Omega)}\Big)+\int_0^T\Big(\|\boldsymbol{u}(t)\|^2_{\boldsymbol{H}^2(\Omega)}+\|\boldsymbol{w}(t)\|^2_{\boldsymbol{H}^2(\Omega)}\Big)dt\le C. \tag{3.2}$$

Proof. Let $(\boldsymbol{v}, q) = (\mathscr{A}_1\boldsymbol{u}, 0)$ in (2.4) and $\boldsymbol{\psi} = \mathscr{A}_2\boldsymbol{w}$ in (2.5) and adding together, we can show

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}+c_1\|\nabla\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)}+c_2\|\operatorname{div}\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)})\\
&\quad+(\mu+\mu_r)\|\mathscr{A}_1\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}+\|\mathscr{A}_2\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)}+4\mu_rc_1\|\nabla\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)}\\
&\quad+4\mu_rc_2\|\operatorname{div}\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)}+b(\boldsymbol{u},\boldsymbol{u},\mathscr{A}_1\boldsymbol{u})+b(\boldsymbol{u},\boldsymbol{w},\mathscr{A}_2\boldsymbol{w})\\
&=(\boldsymbol{f},\mathscr{A}_1\boldsymbol{u})+(\boldsymbol{g},\mathscr{A}_2\boldsymbol{w})+2\mu_r(\mathbf{curl}\,\boldsymbol{w},\mathscr{A}_1\boldsymbol{u})+2\mu_r(\mathbf{curl}\,\boldsymbol{u},\mathscr{A}_2\boldsymbol{w}).
\end{aligned}\tag{3.3}$$

By virtue of the Hölder inequality, the Young inequality, (2.2) and (2.8), we have

$$\begin{aligned}
|b(\boldsymbol{u},\boldsymbol{u},\mathscr{A}_1\boldsymbol{u})|&\le\|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)}\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^3(\Omega)}\|\mathscr{A}_1\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}\\
&\le\frac{\mu}{2}\|\mathscr{A}_1\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}+C\|\nabla\boldsymbol{u}\|^6_{\boldsymbol{L}^2(\Omega)},\\
|b(\boldsymbol{u},\boldsymbol{w},\mathscr{A}_2\boldsymbol{w})|&\le\|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)}\|\nabla\boldsymbol{w}\|_{\boldsymbol{L}^3(\Omega)}\|\mathscr{A}_2\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)}\\
&\le\frac{1}{4}\|\mathscr{A}_2\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)}+C\|\nabla\boldsymbol{u}\|^4_{\boldsymbol{L}^2(\Omega)}\|\nabla\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)},\\
|(\boldsymbol{f},\mathscr{A}_1\boldsymbol{u})|&\le\frac{\mu_r}{4}\|\mathscr{A}_1\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}+C\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)},\\
|(\boldsymbol{g},\mathscr{A}_2\boldsymbol{w})|&\le\frac{1}{8}\|\mathscr{A}_2\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)}+C\|\boldsymbol{g}\|^2_{\boldsymbol{L}^2(\Omega)},\\
2\mu_r\,|(\mathbf{curl}\,\boldsymbol{w},\mathscr{A}_1\boldsymbol{u})|&\le C\|\nabla\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)}+\frac{\mu_r}{4}\|\mathscr{A}_1\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)},\\
2\mu_r\,|(\mathbf{curl}\,\boldsymbol{u},\mathscr{A}_2\boldsymbol{w})|&\le C\|\nabla\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}+\frac{1}{8}\|\mathscr{A}_2\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)}.
\end{aligned}$$

Combining the above inequalities and (3.3), we can show

$$\begin{aligned}
&\frac{d}{dt}\Big(\|\nabla\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}+c_1\|\nabla\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)}+c_2\|\operatorname{div}\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)}\Big)\\
&\quad+(\mu+\mu_r)\|\mathscr{A}_1\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}+\|\mathscr{A}_2\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)}\\
&\le C\Big(\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)}+\|\boldsymbol{g}\|^2_{\boldsymbol{L}^2(\Omega)}\Big)+C\Big(1+\|\nabla\boldsymbol{u}\|^4_{\boldsymbol{L}^2(\Omega)}\Big)\Big(\|\nabla\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}+\|\nabla\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)}\Big).
\end{aligned}\tag{3.4}$$

Integrating time from 0 to t for the inequality (3.4), we conclude that

$$\begin{aligned}
&\|\nabla\boldsymbol{u}(t)\|^2_{\boldsymbol{L}^2(\Omega)}+c_1\|\nabla\boldsymbol{w}(t)\|^2_{\boldsymbol{L}^2(\Omega)}+\int_0^t\Big(\|\mathscr{A}_1\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}+\|\mathscr{A}_2\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)}\Big)dt\\
&\le\|\nabla\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2(\Omega)}+(c_1+c_2)\|\nabla\boldsymbol{w}_0\|^2_{\boldsymbol{L}^2(\Omega)}+C\int_0^T\Big(\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)}+\|\boldsymbol{g}\|^2_{\boldsymbol{L}^2(\Omega)}\Big)dt\\
&\quad+C\int_0^t\Big(1+\|\nabla\boldsymbol{u}\|^4_{\boldsymbol{L}^2(\Omega)}\Big)\Big(\|\nabla\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}+\|\nabla\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)}\Big)dt.
\end{aligned}\tag{3.5}$$

By applying Hypothesis 2.2, we obtain

$$\int_0^T \Big(1 + \|\nabla \boldsymbol{u}\|^4_{\boldsymbol{L}^2(\Omega)}\Big) dt \leq C. \tag{3.6}$$

By applying Lemma 3.1, Hypothesis 2.1 and (3.6) for the inequality (3.6), we can get (3.2). □

We also need to give some a priori estimates on the time derivative of the solution.

Theorem 3.3. *Suppose that Hypothesis* 2.1-2.2 *hold. Then the solution* $(\boldsymbol{u}, \boldsymbol{w})$ *of the problem* (2.4)-(2.5) *satisfies the estimates*

$$\sup_{0\leq t\leq T} \Big(\|\boldsymbol{u}_t(t)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{w}_t(t)\|^2_{\boldsymbol{L}^2(\Omega)}\Big) + \int_0^T \Big(\|\nabla \boldsymbol{u}_t\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla \boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)}\Big) dt \leq C. \tag{3.7}$$

Proof. By differentiating both sides of (2.4) and (2.5) with respect to t, we conclude that

$$\begin{aligned} &(\boldsymbol{u}_{tt}, \boldsymbol{v}) + b(\boldsymbol{u}_t, \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{u}, \boldsymbol{u}_t, \boldsymbol{v}) + (\mu + \mu_r) a(\boldsymbol{u}_t, \boldsymbol{v}) - d(\boldsymbol{v}, p_t) + d(\boldsymbol{u}_t, q) \\ &= (\boldsymbol{f}_t, \boldsymbol{v}) + 2\mu_r (\mathbf{curl}\, \boldsymbol{w}_t, \boldsymbol{v}), \end{aligned} \tag{3.8}$$

$$\begin{aligned} &(\boldsymbol{w}_{tt}, \boldsymbol{\psi}) + b(\boldsymbol{u}_t, \boldsymbol{w}, \boldsymbol{\psi}) + b(\boldsymbol{u}, \boldsymbol{w}_t, \boldsymbol{\psi}) + c_1 a(\boldsymbol{w}_t, \boldsymbol{\psi}) + c_2 (\operatorname{div} \boldsymbol{w}_t, \operatorname{div} \boldsymbol{\psi}) + 4\mu_r (\boldsymbol{w}_t, \boldsymbol{\psi}) \\ &= (\boldsymbol{g}_t, \boldsymbol{\psi}) + 2\mu_r (\mathbf{curl}\, \boldsymbol{u}_t, \boldsymbol{\psi}) \end{aligned} \tag{3.9}$$

for any $(\boldsymbol{v}, q, \boldsymbol{\psi}) \in \boldsymbol{H}^1_0(\Omega) \times L^2_0(\Omega) \times \boldsymbol{H}^1_0(\Omega)$ and for all $t \in (0, T)$.

Taking $(\boldsymbol{v}, q) = (\boldsymbol{u}_t, p_t)$ in (3.8) and $\boldsymbol{\psi} = \boldsymbol{w}_t$ in (3.9) and adding together, by applying (2.1), (2.3), we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \Big(\|\boldsymbol{u}_t\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)}\Big) + \mu \|\nabla \boldsymbol{u}_t\|^2_{\boldsymbol{L}^2(\Omega)} \\ &\quad + c_1 \|\nabla \boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)} + c_2 \|\operatorname{div} \boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)} \\ &\quad + \mu_r \|\mathbf{curl}\, \boldsymbol{u}_t - 2\boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)} \\ &\quad + b(\boldsymbol{u}_t, \boldsymbol{u}, \boldsymbol{u}_t) + b(\boldsymbol{u}_t, \boldsymbol{w}, \boldsymbol{w}_t) \\ &= (\boldsymbol{f}_t, \boldsymbol{u}_t) + (\boldsymbol{g}_t, \boldsymbol{w}_t). \end{aligned} \tag{3.10}$$

We apply the Hölder inequality and the Young inequality to obtain

$$\begin{aligned} |b(\boldsymbol{u}_t, \boldsymbol{u}, \boldsymbol{u}_t)| &\leq \frac{\mu}{6} \|\nabla \boldsymbol{u}_t\|^2_{\boldsymbol{L}^2(\Omega)} + C \|\boldsymbol{u}\|^2_{\boldsymbol{H}^2(\Omega)} \|\boldsymbol{u}_t\|^2_{\boldsymbol{L}^2(\Omega)}, \\ |b(\boldsymbol{u}_t, \boldsymbol{w}, \boldsymbol{w}_t)| &\leq \frac{\mu}{6} \|\nabla \boldsymbol{u}_t\|^2_{\boldsymbol{L}^2(\Omega)} + C \|\boldsymbol{w}\|^2_{\boldsymbol{H}^2(\Omega)} \|\boldsymbol{u}_t\|^2_{\boldsymbol{L}^2(\Omega)}, \\ |(\boldsymbol{f}_t, \boldsymbol{u}_t)| &\leq \frac{\mu}{6} \|\nabla \boldsymbol{u}_t\|^2_{\boldsymbol{L}^2(\Omega)} + C \|\boldsymbol{f}_t\|^2_{\boldsymbol{L}^2(\Omega)}, \\ |(\boldsymbol{g}_t, \boldsymbol{w}_t)| &\leq \frac{c_1}{2} \|\nabla \boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)} + C \|\boldsymbol{g}_t\|^2_{\boldsymbol{L}^2(\Omega)}. \end{aligned}$$

Combining the above inequalities and (3.10), we can show

$$\begin{aligned} &\frac{d}{dt} \Big(\|\boldsymbol{u}_t\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)}\Big) + \mu \|\nabla \boldsymbol{u}_t\|^2_{\boldsymbol{L}^2(\Omega)} + c_1 \|\nabla \boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)} \\ &\leq C \Big(\|\boldsymbol{f}_t\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{g}_t\|^2_{\boldsymbol{L}^2(\Omega)}\Big) + C \Big(\|\boldsymbol{u}\|^2_{\boldsymbol{H}^2(\Omega)} + \|\boldsymbol{w}\|^2_{\boldsymbol{H}^2(\Omega)}\Big) \Big(\|\boldsymbol{u}_t\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)}\Big). \end{aligned} \tag{3.11}$$

By applying the $\boldsymbol{L}^2$-orthogonal decomposition $\boldsymbol{L}^2(\Omega) = \boldsymbol{L}^2_{\mathrm{div}}(\Omega) \oplus \nabla H^1(\Omega)$ (see, e.g., [13]) and the divergence-free property of $\boldsymbol{u}_t(0)$, we have

$$\|\boldsymbol{u}_t(0)\|_{\boldsymbol{L}^2(\Omega)} = \sup_{\boldsymbol{v}\in \boldsymbol{L}^2_{\mathrm{div}}(\Omega)} \frac{(\boldsymbol{u}_t(0), \boldsymbol{v})}{\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}}, \qquad \|\boldsymbol{w}_t(0)\|_{\boldsymbol{L}^2(\Omega)} = \sup_{\boldsymbol{\psi}\in \boldsymbol{L}^2(\Omega)} \frac{(\boldsymbol{w}_t(0), \boldsymbol{\psi})}{\|\boldsymbol{\psi}\|_{\boldsymbol{L}^2(\Omega)}}.$$

Employing (2.2), (2.8) and the Hölder inequality, we conclude that

$$\begin{aligned}
\|\boldsymbol{u}_t(0)\|_{\boldsymbol{L}^2(\Omega)} &\le C\|(\boldsymbol{u}_0\cdot\nabla)\boldsymbol{u}_0\|_{\boldsymbol{L}^2(\Omega)} + C\|\boldsymbol{u}_0\|_{\boldsymbol{H}^2(\Omega)} \\
&\quad + C\|\boldsymbol{f}(0)\|_{\boldsymbol{L}^2(\Omega)} + C\|\mathbf{curl}\,\boldsymbol{w}_0\|_{\boldsymbol{L}^2(\Omega)} \\
&\le C\|\boldsymbol{u}_0\|_{\boldsymbol{H}^2(\Omega)}\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^2(\Omega)} + C\|\boldsymbol{u}_0\|_{\boldsymbol{H}^2(\Omega)} \\
&\quad + C\|\boldsymbol{f}(0)\|_{\boldsymbol{L}^2(\Omega)} + C\|\nabla\boldsymbol{w}_0\|_{\boldsymbol{L}^2(\Omega)}, &(3.12)\\
\|\boldsymbol{w}_t(0)\|_{\boldsymbol{L}^2(\Omega)} &\le C\|(\boldsymbol{u}_0\cdot\nabla)\boldsymbol{w}_0\|_{\boldsymbol{L}^2(\Omega)} + C\|\boldsymbol{w}_0\|_{\boldsymbol{H}^2(\Omega)} \\
&\quad + C\|\boldsymbol{g}(0)\|_{\boldsymbol{L}^2(\Omega)} + C\|\mathbf{curl}\,\boldsymbol{u}_0\|_{\boldsymbol{L}^2(\Omega)} \\
&\le C\|\boldsymbol{u}_0\|_{\boldsymbol{H}^2(\Omega)}\|\nabla\boldsymbol{w}_0\|_{\boldsymbol{L}^2(\Omega)} + C\|\boldsymbol{w}_0\|_{\boldsymbol{H}^2(\Omega)} \\
&\quad + C\|\boldsymbol{g}(0)\|_{\boldsymbol{L}^2(\Omega)} + C\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^2(\Omega)}. &(3.13)
\end{aligned}$$

Combining (3.12), (3.13) and Hypothesis 2.1, we have

$$\|\boldsymbol{u}_t(0)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{w}_t(0)\|^2_{\boldsymbol{L}^2(\Omega)} + C\int_0^T \Big(\|\boldsymbol{f}_t\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{g}_t\|^2_{\boldsymbol{L}^2(\Omega)}\Big)dt \le C. \tag{3.14}$$

Integrating time from 0 to t for the inequality (3.11), we obtain

$$\begin{aligned}
&\|\boldsymbol{u}_t\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)} + \int_0^t \Big(\|\nabla\boldsymbol{u}_t\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla\boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)}\Big)dt \\
\le\ &\|\boldsymbol{u}_t(0)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{w}_t(0)\|^2_{\boldsymbol{L}^2(\Omega)} + C\int_0^T \Big(\|\boldsymbol{f}_t\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{g}_t\|^2_{\boldsymbol{L}^2(\Omega)}\Big)dt \\
&+ C\int_0^t \Big(\|\boldsymbol{u}\|^2_{\boldsymbol{H}^2(\Omega)} + \|\boldsymbol{w}\|^2_{\boldsymbol{H}^2(\Omega)}\Big)\Big(\|\boldsymbol{u}_t\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)}\Big)dt.
\end{aligned} \tag{3.15}$$

By applying Theorem 3.2, we conclude that

$$\int_0^T \Big(\|\boldsymbol{u}\|^2_{\boldsymbol{H}^2(\Omega)} + \|\boldsymbol{w}\|^2_{\boldsymbol{H}^2(\Omega)}\Big)dt \le C. \tag{3.16}$$

By applying Lemma 3.1, (3.14) and (3.16) for the inequality (3.15), we show that the required estimate (3.7). □

By Theorem 3.3, we know that

$$\boldsymbol{u}_t, \boldsymbol{w}_t \in L^\infty(0,T;\boldsymbol{L}^2(\Omega)) \cap L^2(0,T;\boldsymbol{H}^1(\Omega)).$$

In the next step, we will show a priori estimate on the second order spatial derivatives and time derivatives of the solution, respectively, i.e.,

$$\boldsymbol{u}, \boldsymbol{w} \in L^\infty(0,T;\boldsymbol{H}^2(\Omega)), \quad \boldsymbol{u}_{tt}, \boldsymbol{w}_{tt} \in L^2(0,T;\boldsymbol{H}^{-1}(\Omega)).$$

Theorem 3.4. *Suppose that Hypothesis* 2.1-2.3 *hold. The solution* $(\boldsymbol{u}, \boldsymbol{w}, p)$ *of the problem* (2.4)-(2.5) *satisfies the estimates:*

$$\begin{aligned}&\sup_{0\le t\le T}\left(\|\boldsymbol{u}(t)\|^2_{\boldsymbol{H}^2(\Omega)}+\|\boldsymbol{w}(t)\|^2_{\boldsymbol{H}^2(\Omega)}+\|p(t)\|^2_{H^1(\Omega)}\right)\\&\quad+\int_0^T\left(\|\boldsymbol{u}_{tt}\|^2_{(\boldsymbol{H}^1_{\rm div}(\Omega))'}+\|\boldsymbol{w}_{tt}\|^2_{(\boldsymbol{H}^{-1}(\Omega))}\right)dt\le C.\end{aligned}\tag{3.17}$$

Proof. According to (2.4)-(2.5), we have $(\boldsymbol{u}, p, \boldsymbol{w})$ satisfy the following system:

$$\begin{cases}-(\mu+\mu_r)\Delta\boldsymbol{u}+\nabla p=-\boldsymbol{u}_t-(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\boldsymbol{f}+2\mu_r\,\mathbf{curl}\,\boldsymbol{w},\\ \operatorname{div}\boldsymbol{u}=0,\\ -c_1\Delta\boldsymbol{w}-c_2\nabla\operatorname{div}\boldsymbol{w}=-\boldsymbol{w}_t-(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}-4\mu_r\boldsymbol{w}+\boldsymbol{g}+2\mu_r\,\mathbf{curl}\,\boldsymbol{u}\end{cases}$$

in $\mathcal{D}'((0,T)\times\Omega)$.

We apply the Hölder inequality, the Young inequality, Hypothesis 2.3, (2.2) and (2.8) to obtain

$$\begin{aligned}&(\mu+\mu_r)\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)}+\|p\|_{H^1(\Omega)}\\&\le C\|\boldsymbol{u}_t\|_{\boldsymbol{L}^2(\Omega)}+C\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}+C\|\nabla\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)}+C\|\nabla\boldsymbol{u}\|^3_{\boldsymbol{L}^2(\Omega)}\\&\quad+\frac{\mu+\mu_r}{4}\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)},\end{aligned}\tag{3.18}$$

$$\begin{aligned}&c_1\|\boldsymbol{w}\|_{\boldsymbol{H}^2(\Omega)}+c_2\|\operatorname{div}\boldsymbol{w}\|_{H^1(\Omega)}\\&\le C\|\boldsymbol{w}_t\|_{\boldsymbol{L}^2(\Omega)}+C\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}+C\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}+C\|\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)}\\&\quad+C\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}\|\nabla\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)}+\frac{\mu+\mu_r}{4}\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)}.\end{aligned}\tag{3.19}$$

Then we have

$$\begin{aligned}&\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)}+\|p\|_{H^1(\Omega)}+\|\boldsymbol{w}\|_{\boldsymbol{H}^2(\Omega)}\\&\le C\|\boldsymbol{u}_t\|_{\boldsymbol{L}^2(\Omega)}+C\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}+C\|\nabla\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)}+C\|\boldsymbol{w}_t\|_{\boldsymbol{L}^2(\Omega)}+C\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}\\&\quad+C\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}+C\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}\|\nabla\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)}+C\|\nabla\boldsymbol{u}\|^3_{\boldsymbol{L}^2(\Omega)}.\end{aligned}\tag{3.20}$$

By applying (3.8)-(3.9) and (2.2), Hölder inequality and Poincaré inequality, we get

$$\begin{aligned}\|\boldsymbol{u}_{tt}\|_{(\boldsymbol{H}^1_{\rm div}(\Omega))'}&\le C\|\nabla\boldsymbol{u}_t\|_{\boldsymbol{L}^2(\Omega)}+C\|\nabla\boldsymbol{u}_t\|_{\boldsymbol{L}^2(\Omega)}\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}\\&\quad+C\|\boldsymbol{f}_t\|_{\boldsymbol{L}^2(\Omega)}+C\|\boldsymbol{w}_t\|_{\boldsymbol{L}^2(\Omega)},\end{aligned}\tag{3.21}$$

$$\begin{aligned}\|\boldsymbol{w}_{tt}\|_{(\boldsymbol{H}^{-1}(\Omega))}&\le C\|\nabla\boldsymbol{w}_t\|_{\boldsymbol{L}^2(\Omega)}+C\|\boldsymbol{g}_t\|_{\boldsymbol{L}^2(\Omega)}+C\|\boldsymbol{u}_t\|_{\boldsymbol{L}^2(\Omega)}\\&\quad+C\|\nabla\boldsymbol{u}_t\|_{\boldsymbol{L}^2(\Omega)}\|\nabla\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)}+C\|\nabla\boldsymbol{w}_t\|_{\boldsymbol{L}^2(\Omega)}\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}.\end{aligned}\tag{3.22}$$

Combining the inequalities (3.21) and (3.22), we have

$$\begin{aligned}&\|\boldsymbol{u}_{tt}\|^2_{(\boldsymbol{H}^1_{\rm div}(\Omega))'}+\|\boldsymbol{w}_{tt}\|^2_{(\boldsymbol{H}^{-1}(\Omega))}\\&\le C\left(\|\boldsymbol{f}_t\|^2_{\boldsymbol{L}^2(\Omega)}+\|\boldsymbol{g}_t\|^2_{\boldsymbol{L}^2(\Omega)}\right)+C\left(1+\|\nabla\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}+\|\nabla\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)}\right)\\&\quad\times\left(\|\nabla\boldsymbol{u}_t\|^2_{\boldsymbol{L}^2(\Omega)}+\|\nabla\boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)}\right).\end{aligned}\tag{3.23}$$

Integrating time from 0 to t for the inequality (3.23), we conclude that

$$\begin{aligned}
&\int_0^T \Big(\|\boldsymbol{u}_{tt}\|^2_{(\boldsymbol{H}^1_{\rm div}(\Omega))'} + \|\boldsymbol{w}_{tt}\|^2_{(\boldsymbol{H}^{-1}(\Omega))}\Big)dt \\
&\leq C\int_0^T \Big(\|\boldsymbol{f}_t\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{g}_t\|^2_{\boldsymbol{L}^2(\Omega)}\Big)dt \\
&\quad + C\int_0^T \Big(1+\|\nabla\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)}\Big)\Big(\|\nabla\boldsymbol{u}_t\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla\boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)}\Big)dt. \qquad (3.24)
\end{aligned}$$

A combination of (3.20) and (3.24), together with Theorems 3.2-3.3 and Hypothesis 2.1, implies the desired result. $\square$

Next, we will continue to bound the second order time derivatives of the solution, which are useful for the subsequent error analysis.

Theorem 3.5. *Suppose that Hypothesis* 2.1-2.3 *hold and let* $\nu(t) = \min(1,t)$. *The solution* $(\boldsymbol{u},\boldsymbol{w},p)$ *of the problem* (2.4)-(2.5) *satisfies the estimates*

$$\begin{aligned}
&\sup_{0\leq t\leq T}\nu(t)\Big(\mu\|\nabla\boldsymbol{u}_t\|^2_{\boldsymbol{L}^2(\Omega)} + c_1\|\nabla\boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)}\Big) + \int_0^T\nu(t)\Big(\|\boldsymbol{u}_{tt}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{w}_{tt}\|^2_{\boldsymbol{L}^2(\Omega)}\Big)dt \\
&\quad + \int_0^T \nu(t)\Big(\|\boldsymbol{u}_t\|^2_{\boldsymbol{H}^2(\Omega)} + \|\boldsymbol{w}_t\|^2_{\boldsymbol{H}^2(\Omega)} + \|p_t\|^2_{H^1(\Omega)}\Big)dt \leq C. \qquad (3.25)
\end{aligned}$$

Proof. Taking $(\boldsymbol{v},q) = (\boldsymbol{u}_{tt},0)$ in (3.10) and $\boldsymbol{\psi} = \boldsymbol{w}_{tt}$ in (3.11) and adding these two equations together, we can get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big((\mu+\mu_r)\|\nabla\boldsymbol{u}_t\|^2_{\boldsymbol{L}^2(\Omega)} + c_1\|\nabla\boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)}\Big) + \|\boldsymbol{u}_{tt}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{w}_{tt}\|^2_{\boldsymbol{L}^2(\Omega)} \\
&\quad + \frac{1}{2}\frac{d}{dt}\Big(c_2\|\operatorname{div}\boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)} + 4\mu_r\|\boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)}\Big) \\
&\quad + b(\boldsymbol{u}_t,\boldsymbol{u},\boldsymbol{u}_{tt}) + b(\boldsymbol{u},\boldsymbol{u}_t,\boldsymbol{u}_{tt}) + b(\boldsymbol{u}_t,\boldsymbol{w},\boldsymbol{w}_{tt}) + b(\boldsymbol{u},\boldsymbol{w}_t,\boldsymbol{w}_{tt}) \\
&= (\boldsymbol{f}_t,\boldsymbol{u}_{tt}) + (\boldsymbol{g}_t,\boldsymbol{w}_{tt}) + 2\mu_r(\mathbf{curl}\,\boldsymbol{w}_t,\boldsymbol{u}_{tt}) + 2\mu_r(\mathbf{curl}\,\boldsymbol{u}_t,\boldsymbol{w}_{tt}). \qquad (3.26)
\end{aligned}$$

By applying the Hölder inequality and the Young inequality and noticing (2.2), we obtain

$$\begin{aligned}
|b(\boldsymbol{u}_t,\boldsymbol{u},\boldsymbol{u}_{tt})| &\leq \|\boldsymbol{u}_t\|_{\boldsymbol{L}^6(\Omega)}\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^3(\Omega)}\|\boldsymbol{u}_{tt}\|_{\boldsymbol{L}^2(\Omega)} \\
&\leq C\|\nabla\boldsymbol{u}_t\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)}\|\boldsymbol{u}_{tt}\|_{\boldsymbol{L}^2(\Omega)} \\
&\leq \frac{1}{8}\|\boldsymbol{u}_{tt}\|^2_{\boldsymbol{L}^2(\Omega)} + C\|\boldsymbol{u}\|^2_{\boldsymbol{H}^2(\Omega)}\|\nabla\boldsymbol{u}_t\|^2_{\boldsymbol{L}^2(\Omega)}, \\
|b(\boldsymbol{u},\boldsymbol{u}_t,\boldsymbol{u}_{tt})| &\leq \|\boldsymbol{u}\|_{\boldsymbol{L}^\infty(\Omega)}\|\nabla\boldsymbol{u}_t\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{u}_{tt}\|_{\boldsymbol{L}^2(\Omega)} \\
&\leq C\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)}\|\nabla\boldsymbol{u}_t\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{u}_{tt}\|_{\boldsymbol{L}^2(\Omega)} \\
&\leq \frac{1}{8}\|\boldsymbol{u}_{tt}\|^2_{\boldsymbol{L}^2(\Omega)} + C\|\boldsymbol{u}\|^2_{\boldsymbol{H}^2(\Omega)}\|\nabla\boldsymbol{u}_t\|^2_{\boldsymbol{L}^2(\Omega)}, \\
|b(\boldsymbol{u}_t,\boldsymbol{w},\boldsymbol{w}_{tt})| &\leq \|\boldsymbol{u}_t\|_{\boldsymbol{L}^6(\Omega)}\|\nabla\boldsymbol{w}\|_{\boldsymbol{L}^3(\Omega)}\|\boldsymbol{w}_{tt}\|_{\boldsymbol{L}^2(\Omega)} \\
&\leq C\|\nabla\boldsymbol{u}_t\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{w}\|_{\boldsymbol{H}^2(\Omega)}\|\boldsymbol{w}_{tt}\|_{\boldsymbol{L}^2(\Omega)} \\
&\leq \frac{1}{8}\|\boldsymbol{w}_{tt}\|^2_{\boldsymbol{L}^2(\Omega)} + C\|\boldsymbol{w}\|^2_{\boldsymbol{H}^2(\Omega)}\|\nabla\boldsymbol{u}_t\|^2_{\boldsymbol{L}^2(\Omega)},
\end{aligned}$$

$$
\begin{aligned}
|b(\boldsymbol{u}, \boldsymbol{w}_t, \boldsymbol{w}_{tt})| &\le \|\boldsymbol{u}\|_{\boldsymbol{L}^\infty(\Omega)} \|\nabla \boldsymbol{w}_t\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{w}_{tt}\|_{\boldsymbol{L}^2(\Omega)} \\
&\le C\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} \|\nabla \boldsymbol{w}_t\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{w}_{tt}\|_{\boldsymbol{L}^2(\Omega)} \\
&\le \frac{1}{8}\|\boldsymbol{w}_{tt}\|^2_{\boldsymbol{L}^2(\Omega)} + C\|\boldsymbol{u}\|^2_{\boldsymbol{H}^2(\Omega)} \|\nabla \boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)}, \\
|(\boldsymbol{f}_t, \boldsymbol{u}_{tt}) + (\boldsymbol{g}_t, \boldsymbol{w}_{tt})| &\le \frac{1}{8}\left(\|\boldsymbol{u}_{tt}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{w}_{tt}\|^2_{\boldsymbol{L}^2(\Omega)}\right) \\
&\quad + C\left(\|\boldsymbol{f}_t\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{g}_t\|^2_{\boldsymbol{L}^2(\Omega)}\right), \\
2\mu_r\,|(\mathbf{curl}\,\boldsymbol{w}_t, \boldsymbol{u}_{tt})| &\le \frac{1}{8}\|\boldsymbol{u}_{tt}\|^2_{\boldsymbol{L}^2(\Omega)} + C\|\nabla \boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)}, \\
2\mu_r\,|(\mathbf{curl}\boldsymbol{u}_t, \boldsymbol{w}_{tt})| &\le \frac{1}{8}\|\boldsymbol{w}_{tt}\|^2_{\boldsymbol{L}^2(\Omega)} + C\|\nabla \boldsymbol{u}_t\|^2_{\boldsymbol{L}^2(\Omega)}.
\end{aligned}
$$

Combining the above inequalities and (3.26), we conclude that

$$
\begin{aligned}
&\frac{d}{dt}\left((\mu+\mu_r)\|\nabla \boldsymbol{u}_t\|^2_{\boldsymbol{L}^2(\Omega)} + c_1\|\nabla \boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)}\right) + \|\boldsymbol{u}_{tt}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{w}_{tt}\|^2_{\boldsymbol{L}^2(\Omega)} \\
&\quad + \frac{d}{dt}\left(c_2\|\operatorname{div}\boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)} + 4\mu_r\|\boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)}\right) \\
&\le C\left(\|\boldsymbol{f}_t\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{g}_t\|^2_{\boldsymbol{L}^2(\Omega)}\right) + C\left(1 + \|\boldsymbol{u}\|^2_{\boldsymbol{H}^2(\Omega)} + \|\boldsymbol{w}\|^2_{\boldsymbol{H}^2(\Omega)}\right) \\
&\quad \times \left(\|\nabla \boldsymbol{u}_t\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla \boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)}\right).
\end{aligned}
\tag{3.27}
$$

Multiplying (3.27) by $\nu(t)$ and integrating time from 0 to t for the inequality, we have

$$
\begin{aligned}
&\nu(t)\left(\mu\|\nabla \boldsymbol{u}_t\|^2_{\boldsymbol{L}^2(\Omega)} + c_1\|\nabla \boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)}\right) + \int_0^t \nu(t)\left(\|\boldsymbol{u}_{tt}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{w}_{tt}\|^2_{\boldsymbol{L}^2(\Omega)}\right)dt \\
&\le C\int_0^T \left(\|\boldsymbol{f}_t\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{g}_t\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla \boldsymbol{u}_t\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla \boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)}\right)dt \\
&\quad + C\int_0^t \nu(t)\left(1 + \|\boldsymbol{u}\|^2_{\boldsymbol{H}^2(\Omega)} + \|\boldsymbol{w}\|^2_{\boldsymbol{H}^2(\Omega)}\right)\left(\mu\|\nabla \boldsymbol{u}_t\|^2_{\boldsymbol{L}^2(\Omega)} + c_1\|\nabla \boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)}\right)dt.
\end{aligned}
\tag{3.28}
$$

By employing Theorem 3.2, we conclude that

$$
\int_0^T \left(1 + \|\boldsymbol{u}\|^2_{\boldsymbol{H}^2(\Omega)} + \|\boldsymbol{w}\|^2_{\boldsymbol{H}^2(\Omega)}\right)dt \le C. \tag{3.29}
$$

By applying Lemma 3.1, the inequality (3.29), Hypothesis 2.1 and Theorem 3.3 for the inequality (3.28), we obtain

$$
\begin{aligned}
&\sup_{0\le t\le T} \nu(t)\left(\mu\|\nabla \boldsymbol{u}_t\|^2_{\boldsymbol{L}^2(\Omega)} + c_1\|\nabla \boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)}\right) \\
&\quad + \int_0^T \nu(t)\left(\|\boldsymbol{u}_{tt}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{w}_{tt}\|^2_{\boldsymbol{L}^2(\Omega)}\right)dt \le C.
\end{aligned}
\tag{3.30}
$$

According to (3.10)-(3.12), we get that $(\boldsymbol{u}_t, p_t, \boldsymbol{w}_t)$ satisfy the following system:

$$
\begin{cases}
-(\mu+\mu_r)\Delta \boldsymbol{u}_t + \nabla p_t = -\boldsymbol{u}_{tt} - (\boldsymbol{u}_t\cdot\nabla)\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}_t \\
\quad + \boldsymbol{f}_t + 2\mu_r\mathbf{curl}\,\boldsymbol{w}_t, \quad \operatorname{div}\boldsymbol{u}_t = 0, \\
-c_1\Delta \boldsymbol{w}_t - c_2\nabla\operatorname{div}\boldsymbol{w}_t = -\boldsymbol{w}_{tt} - (\boldsymbol{u}_t\cdot\nabla)\boldsymbol{w} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{w}_t \\
\quad - 4\mu_r\boldsymbol{w}_t + \boldsymbol{g}_t + 2\mu_r\mathbf{curl}\,\boldsymbol{u}_t
\end{cases}
$$

in $\mathcal{D}'((0,T)\times\Omega)$. By applying the Hölder inequality, Hypothesis 2.3 and (2.2), we obtain

$$\begin{aligned}\|\boldsymbol{u}_t\|_{\boldsymbol{H}^2(\Omega)}+\|p_t\|_{H^1(\Omega)}\le{}& C\left(\|\boldsymbol{u}_{tt}\|_{\boldsymbol{L}^2(\Omega)}+\|\boldsymbol{f}_t\|_{\boldsymbol{L}^2(\Omega)}\right)+C\left(1+\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)}\right)\\&\times\left(\|\nabla\boldsymbol{u}_t\|_{\boldsymbol{L}^2(\Omega)}+\|\nabla\boldsymbol{w}_t\|_{\boldsymbol{L}^2(\Omega)}\right),\end{aligned}\tag{3.31}$$

$$\begin{aligned}\|\boldsymbol{w}_t\|_{\boldsymbol{H}^2(\Omega)}+\|\operatorname{div}\boldsymbol{w}_t\|_{H^1(\Omega)}\le{}& C\left(\|\boldsymbol{w}_{tt}\|_{\boldsymbol{L}^2(\Omega)}+\|\boldsymbol{g}_t\|_{\boldsymbol{L}^2(\Omega)}\right)+C\left(1+\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)}+\|\boldsymbol{w}\|_{\boldsymbol{H}^2(\Omega)}\right)\\&\times\left(\|\nabla\boldsymbol{u}_t\|_{\boldsymbol{L}^2(\Omega)}+\|\nabla\boldsymbol{w}_t\|_{\boldsymbol{L}^2(\Omega)}\right).\end{aligned}\tag{3.32}$$

Combining (3.31) and (3.32), we can show

$$\begin{aligned}&\|\boldsymbol{u}_t\|^2_{\boldsymbol{H}^2(\Omega)}+\|p_t\|^2_{H^1(\Omega)}+\|\boldsymbol{w}_t\|^2_{\boldsymbol{H}^2(\Omega)}\\&\le C\left(\|\boldsymbol{f}_t\|^2_{\boldsymbol{L}^2(\Omega)}+\|\boldsymbol{g}_t\|^2_{\boldsymbol{L}^2(\Omega)}+\|\boldsymbol{u}_{tt}\|^2_{\boldsymbol{L}^2(\Omega)}+\|\boldsymbol{w}_{tt}\|^2_{\boldsymbol{L}^2(\Omega)}\right)\\&\quad+C(1+\|\boldsymbol{u}\|^2_{\boldsymbol{H}^2(\Omega)}+\|\boldsymbol{w}\|^2_{\boldsymbol{H}^2(\Omega)})\left(\|\nabla\boldsymbol{u}_t\|^2_{\boldsymbol{L}^2(\Omega)}+\|\nabla\boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)}\right).\end{aligned}\tag{3.33}$$

Multiplying (3.33) by $\nu(t)$ and integrating time from 0 to T for the inequality, we have

$$\begin{aligned}&\int_0^T\nu(t)\left(\|\boldsymbol{u}_t\|^2_{\boldsymbol{H}^2(\Omega)}+\|p_t\|^2_{H^1(\Omega)}+\|\boldsymbol{w}_t\|^2_{\boldsymbol{H}^2(\Omega)}\right)dt\\&\le C\int_0^T\left(\|\boldsymbol{f}_t\|^2_{\boldsymbol{L}^2(\Omega)}+\|\boldsymbol{g}_t\|^2_{\boldsymbol{L}^2(\Omega)}\right)dt+C\int_0^T\nu(t)\left(\|\boldsymbol{u}_{tt}\|^2_{\boldsymbol{L}^2(\Omega)}+\|\boldsymbol{w}_{tt}\|^2_{\boldsymbol{L}^2(\Omega)}\right)dt\\&\quad+C\int_0^T\left(1+\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)}+\|\boldsymbol{w}\|_{\boldsymbol{H}^2(\Omega)}\right)\left(\|\nabla\boldsymbol{u}_t\|_{\boldsymbol{L}^2(\Omega)}+\|\nabla\boldsymbol{w}_t\|_{\boldsymbol{L}^2(\Omega)}\right)dt.\end{aligned}\tag{3.34}$$

By applying Hypothesis 2.1, (3.32), Theorems 3.3 and 3.4 for (3.34), we get

$$\int_0^T\nu(t)\left(\|\boldsymbol{u}_t\|^2_{\boldsymbol{H}^2(\Omega)}+\|p_t\|^2_{H^1(\Omega)}+\|\boldsymbol{w}_t\|^2_{\boldsymbol{H}^2(\Omega)}\right)dt\le C.\tag{3.35}$$

Combining (3.31) and (3.35), we can show the inequality (3.25). This completes the proof of Theorem 3.5. $\square$

We will study numerical discretizations for the problem (2.4)-(2.5) in next sections. To this end, we need to prove a uniqueness result of the system.

Theorem 3.6. *Suppose that Hypothesis* 2.1-2.2 *hold and* $\boldsymbol{w}\in L^4(0,T;\boldsymbol{H}^1_0(\Omega))$*, then there exists a unique solution*

$$\begin{aligned}&\boldsymbol{u}\in L^2(0,T;\boldsymbol{H}^1_{\mathrm{div}}(\Omega))\cap L^\infty(0,T;\boldsymbol{L}^2_{\mathrm{div}}(\Omega)),\\&p\in L^2(0,T;\boldsymbol{L}^2(\Omega)),\\&\boldsymbol{w}\in L^2(0,T;\boldsymbol{H}^1_0(\Omega))\cap L^\infty(0,T;\boldsymbol{L}^2(\Omega))\end{aligned}$$

for the problem (2.4)-(2.5).

Proof. According to Theorem 3.1, we have the solution of the problem (2.4)-(2.5) satisfy the energy estimate (3.1). Next, we establish the uniqueness of solution for the problem (2.4)-(2.5). Let $(\boldsymbol{u}_1,p_1,\boldsymbol{w}_1)$ and $(\boldsymbol{u}_2,p_2,\boldsymbol{w}_2)$ be two solutions of the system (2.4)-(2.5) with the same initial data $(\boldsymbol{u}_0,\boldsymbol{w}_0)$. Let $(\boldsymbol{u}_{\mathrm{e}},p_{\mathrm{e}},\boldsymbol{w}_{\mathrm{e}})=(\boldsymbol{u}_2-\boldsymbol{u}_1,p_2-p_1,\boldsymbol{w}_2-\boldsymbol{w}_1)$, we conclude that

$$\begin{aligned}&(\partial_t\boldsymbol{u}_{\mathrm{e}},\boldsymbol{v})+b(\boldsymbol{u}_2,\boldsymbol{u}_{\mathrm{e}},\boldsymbol{v})+b(\boldsymbol{u}_{\mathrm{e}},\boldsymbol{u}_1,\boldsymbol{v})+(\mu+\mu_r)a(\boldsymbol{u}_{\mathrm{e}},\boldsymbol{v})\\&\quad-d(\boldsymbol{v},p_{\mathrm{e}})+d(\boldsymbol{u}_{\mathrm{e}},q)=2\mu_r(\mathbf{curl}\,\boldsymbol{w}_{\mathrm{e}},\boldsymbol{v}),\end{aligned}\tag{3.36}$$

$$\begin{aligned}&(\partial_t\boldsymbol{w}_{\mathrm{e}},\boldsymbol{\psi})+b(\boldsymbol{u}_2,\boldsymbol{w}_{\mathrm{e}},\boldsymbol{\psi})+b(\boldsymbol{u}_{\mathrm{e}},\boldsymbol{w}_1,\boldsymbol{\psi})+c_1a(\boldsymbol{w}_{\mathrm{e}},\boldsymbol{\psi})\\&\quad+c_2(\operatorname{div}\boldsymbol{w}_{\mathrm{e}},\operatorname{div}\boldsymbol{\psi})+4\mu_r(\boldsymbol{w}_{\mathrm{e}},\boldsymbol{\psi})=2\mu_r(\mathbf{curl}\,\boldsymbol{u}_{\mathrm{e}},\boldsymbol{\psi})\end{aligned}\tag{3.37}$$

for any $(\boldsymbol{v},q,\boldsymbol{\psi})\in\boldsymbol{H}^1_0(\Omega)\times L^2_0(\Omega)\times\boldsymbol{H}^1_0(\Omega)$.

Taking $(\boldsymbol{v}, q) = (\boldsymbol{u}_{\mathrm{e}}, p_{\mathrm{e}})$ in (3.36) and $\boldsymbol{\psi} = \boldsymbol{w}_{\mathrm{e}}$ in (3.37) and adding together, we can show

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\boldsymbol{u}_{\mathrm{e}}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{w}_{\mathrm{e}}\|^2_{\boldsymbol{L}^2(\Omega)}\right) + \mu\|\nabla\boldsymbol{u}_{\mathrm{e}}\|^2_{\boldsymbol{L}^2(\Omega)} \\
&\quad + c_1\|\nabla\boldsymbol{w}_{\mathrm{e}}\|^2_{\boldsymbol{L}^2(\Omega)} + c_2\|\operatorname{div}\boldsymbol{w}_{\mathrm{e}}\|^2_{\boldsymbol{L}^2(\Omega)} \\
&\quad + \mu_r\|\mathbf{curl}\,\boldsymbol{u}_{\mathrm{e}} - 2\boldsymbol{w}_{\mathrm{e}}\|^2_{\boldsymbol{L}^2(\Omega)} + b(\boldsymbol{u}_{\mathrm{e}}, \boldsymbol{u}_1, \boldsymbol{u}_{\mathrm{e}}) + b(\boldsymbol{u}_{\mathrm{e}}, \boldsymbol{w}_1, \boldsymbol{w}_{\mathrm{e}}) = 0. \qquad (3.38)
\end{aligned}$$

By applying the Hölder inequality, the interpolation inequality and the Young inequality, we have

$$\begin{aligned}
&|b(\boldsymbol{u}_{\mathrm{e}}, \boldsymbol{u}_1, \boldsymbol{u}_{\mathrm{e}})| \le \frac{\mu}{4}\|\nabla\boldsymbol{u}_{\mathrm{e}}\|^2_{\boldsymbol{L}^2(\Omega)} + C\|\nabla\boldsymbol{u}_1\|^4_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{u}_{\mathrm{e}}\|^2_{\boldsymbol{L}^2(\Omega)}, \\
&|b(\boldsymbol{u}_{\mathrm{e}}, \boldsymbol{w}_1, \boldsymbol{w}_{\mathrm{e}})| \le \frac{c_1}{2}\|\nabla\boldsymbol{w}_{\mathrm{e}}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\mu}{4}\|\nabla\boldsymbol{u}_{\mathrm{e}}\|^2_{\boldsymbol{L}^2(\Omega)} + C\|\nabla\boldsymbol{w}_1\|^4_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{w}_{\mathrm{e}}\|^2_{\boldsymbol{L}^2(\Omega)}.
\end{aligned}$$

Combining the above inequalities and (3.38), we get

$$\begin{aligned}
&\frac{d}{dt}\left(\|\boldsymbol{u}_{\mathrm{e}}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{w}_{\mathrm{e}}\|^2_{\boldsymbol{L}^2(\Omega)}\right) + \mu\|\nabla\boldsymbol{u}_{\mathrm{e}}\|^2_{\boldsymbol{L}^2(\Omega)} + c_1\|\nabla\boldsymbol{w}_{\mathrm{e}}\|^2_{\boldsymbol{L}^2(\Omega)} \\
&\le C\left(\|\nabla\boldsymbol{u}_1\|^4_{\boldsymbol{L}^2(\Omega)} + \|\nabla\boldsymbol{w}_1\|^4_{\boldsymbol{L}^2(\Omega)}\right)\left(\|\boldsymbol{u}_{\mathrm{e}}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{w}_{\mathrm{e}}\|^2_{\boldsymbol{L}^2(\Omega)}\right). \qquad (3.39)
\end{aligned}$$

Integrating time from 0 to t for the inequality (3.39), we can show

$$\begin{aligned}
&\|\boldsymbol{u}_{\mathrm{e}}(t)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{w}_{\mathrm{e}}(t)\|^2_{\boldsymbol{L}^2(\Omega)} + \int_0^t\left(\mu\|\nabla\boldsymbol{u}_{\mathrm{e}}\|^2_{\boldsymbol{L}^2(\Omega)} + c_1\|\nabla\boldsymbol{w}_{\mathrm{e}}\|^2_{\boldsymbol{L}^2(\Omega)}\right)dt \\
&\le \|\boldsymbol{u}_{\mathrm{e}}(0)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{w}_{\mathrm{e}}(0)\|^2_{\boldsymbol{L}^2(\Omega)} \\
&\quad + C\int_0^t\left(\|\nabla\boldsymbol{u}_1\|^4_{\boldsymbol{L}^2(\Omega)} + \|\nabla\boldsymbol{w}_1\|^4_{\boldsymbol{L}^2(\Omega)}\right)\left(\|\boldsymbol{u}_{\mathrm{e}}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{w}_{\mathrm{e}}\|^2_{\boldsymbol{L}^2(\Omega)}\right)dt. \qquad (3.40)
\end{aligned}$$

By employing Lemma 3.1 for the inequality (3.40) and using the initial data $(\boldsymbol{u}_{\mathrm{e}}(0), \boldsymbol{w}_{\mathrm{e}}(0)) = (\mathbf{0}, \mathbf{0})$ and $\boldsymbol{u}_1, \boldsymbol{w}_1 \in L^4(0, T; \boldsymbol{W}^{1,2}(\Omega))$, we obtain

$$\|\boldsymbol{u}_{\mathrm{e}}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{w}_{\mathrm{e}}\|^2_{\boldsymbol{L}^2(\Omega)} = 0,$$

which implies that $(\boldsymbol{u}_{\mathrm{e}}, \boldsymbol{w}_{\mathrm{e}}) = (\mathbf{0}, \mathbf{0})$. Taking $(\boldsymbol{u}_{\mathrm{e}}, \boldsymbol{w}_{\mathrm{e}}) = (\mathbf{0}, \mathbf{0})$ in (3.36), we can derive $d(\boldsymbol{v}, p_{\mathrm{e}}) = 0$. By applying the inf-sup condition, we have $\|p_{\mathrm{e}}\|_{L^2(\Omega)} = 0$, which implies that $p_{\mathrm{e}} = 0$, we completes the proof. □

4. A First-Order Euler Semi-Implicit Time-Discrete Scheme

In this section, we propose the Euler semi-implicit time-discrete scheme for the problem (1.1)-(1.5). Based on the priori estimates proved in Section 3, we establish $\boldsymbol{L}^2$ and $\boldsymbol{H}^1$ error estimates for the time discrete of the microstructure systems (1.1)-(1.5).

4.1. Time discretization

Let N be a fixed integer number and $0 = t_0 < t_1 < \cdots < t_N = T$ be a uniform partition of $[0, T]$ with time-step size $\tau = T/N$ (in general, the time-step size $\tau < 1$). Moreover, $t_n = n\tau$ denoted the discrete time points and $\boldsymbol{v}^n$ is the approximation value of the function value $\boldsymbol{v}$ at time t_n for $0 \le n \le N$. For convenience, we write $\boldsymbol{v}^n = \boldsymbol{v}^0$ for any $n < 0$. Finally, we define the notation $d_t\boldsymbol{v}^n = (\boldsymbol{v}^n - \boldsymbol{v}^{n-1})/\tau$ for $n \geq 1$, $d_t^2\boldsymbol{v}^n = (\boldsymbol{v}^n - 2\boldsymbol{v}^{n-1} + \boldsymbol{v}^{n-2})/\tau^2$ for $n \geq 2$. It is easy check that $d_t\boldsymbol{v}^n = \mathbf{0}$ for $n \leq 0$.

We initialize the scheme by $(\boldsymbol{u}^0, \boldsymbol{w}^0, p^0) = (\boldsymbol{u}_0, \boldsymbol{w}_0, p_0)$, then the Euler semi-implicit scheme of the problem (1.1)-(1.5) is given by: Find $(\boldsymbol{u}^n, p^n, \boldsymbol{w}^n) \in \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega) \times \boldsymbol{H}_0^1(\Omega)$ such that

$$
\begin{aligned}
&(d_t\boldsymbol{u}^n, \boldsymbol{v}) + b(\boldsymbol{u}^{n-1}, \boldsymbol{u}^n, \boldsymbol{v}) + (\mu + \mu_r)a(\boldsymbol{u}^n, \boldsymbol{v}) - d(\boldsymbol{v}, p^n) + d(\boldsymbol{u}^n, q) \\
&= (\boldsymbol{f}^n, \boldsymbol{v}) + 2\mu_r(\mathbf{curl}\, \boldsymbol{w}^{n-1}, \boldsymbol{v}), \qquad (4.1) \\
&(d_t\boldsymbol{w}^n, \boldsymbol{\psi}) + b(\boldsymbol{u}^{n-1}, \boldsymbol{w}^n, \boldsymbol{\psi}) + c_1 a(\boldsymbol{w}^n, \boldsymbol{\psi}) + c_2(\operatorname{div} \boldsymbol{w}^n, \operatorname{div} \boldsymbol{\psi}) + 4\mu_r(\boldsymbol{w}^n, \boldsymbol{\psi}) \\
&= (\boldsymbol{g}^n, \boldsymbol{\psi}) + 2\mu_r(\mathbf{curl}\, \boldsymbol{u}^n, \boldsymbol{\psi}) \qquad (4.2)
\end{aligned}
$$

for any $(\boldsymbol{v}, q, \boldsymbol{\psi}) \in \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega) \times \boldsymbol{H}_0^1(\Omega)$ with $1 \leq n \leq N$, where

$$
\boldsymbol{f}^n := \frac{1}{\tau}\int_{t_{n-1}}^{t_n} \boldsymbol{f}(t)dt, \quad \boldsymbol{g}^n := \frac{1}{\tau}\int_{t_{n-1}}^{t_n} \boldsymbol{g}(t)dt.
$$

Similarly, we can eliminate the pressure p under the condition that $\boldsymbol{u}^n \in \boldsymbol{H}_{\mathrm{div}}^1(\Omega)$ and the test function $\boldsymbol{v} \in \boldsymbol{H}_{\mathrm{div}}^1(\Omega)$. Obviously, we can show the solution $(\boldsymbol{u}^n, \boldsymbol{w}^n)$ satisfies the following weak formulation: Find $(\boldsymbol{u}^n, \boldsymbol{w}^n) \in \boldsymbol{H}_{\mathrm{div}}^1(\Omega) \times \boldsymbol{H}_0^1(\Omega)$ such that

$$
\begin{aligned}
&(d_t\boldsymbol{u}^n, \boldsymbol{v}) + b(\boldsymbol{u}^{n-1}, \boldsymbol{u}^n, \boldsymbol{v}) + (\mu + \mu_r)a(\boldsymbol{u}^n, \boldsymbol{v}) = (\boldsymbol{f}^n, \boldsymbol{v}) + 2\mu_r(\mathbf{curl}\, \boldsymbol{w}^{n-1}, \boldsymbol{v}), \qquad (4.3) \\
&(d_t\boldsymbol{w}^n, \boldsymbol{\psi}) + b(\boldsymbol{u}^{n-1}, \boldsymbol{w}^n, \boldsymbol{\psi}) + c_1 a(\boldsymbol{w}^n, \boldsymbol{\psi}) + c_2(\operatorname{div} \boldsymbol{w}^n, \operatorname{div} \boldsymbol{\psi}) + 4\mu_r(\boldsymbol{w}^n, \boldsymbol{\psi}) \\
&= (\boldsymbol{g}^n, \boldsymbol{\psi}) + 2\mu_r(\mathbf{curl}\, \boldsymbol{u}^n, \boldsymbol{\psi}) \qquad (4.4)
\end{aligned}
$$

for any $(\boldsymbol{v}, \boldsymbol{\psi}) \in \boldsymbol{H}_{\mathrm{div}}^1(\Omega) \times \boldsymbol{H}_0^1(\Omega)$ with $1 \leq n \leq N$.

In this paper, we need the variation counterpart of the discrete Grönwall lemma (see, e.g., remark to Lemma 5.1 in [18]).

Lemma 4.1. *Let C_0, τ, a_n, b_n, c_n and d_n be non-negative numbers with $n \geq 0$ such that*

$$
a_m + \tau\sum_{n=0}^{m} b_n \leq \tau\sum_{n=0}^{m-1} d_n a_n + \tau\sum_{n=0}^{m} c_n + C_0, \quad \forall\, m \geq 0,
$$

then

$$
a_m + \tau\sum_{n=0}^{m} b_n \leq \exp\left(\tau\sum_{n=0}^{m-1} d_n\right)\left(\tau\sum_{n=0}^{m} c_n + C_0\right), \quad \forall\, m \geq 0.
$$

The following theorem give the discrete energy estimate for the time discrete scheme.

Theorem 4.1. *Suppose that Hypothesis 2.1 holds. For any $1 \leq m \leq N$, the solution $(\boldsymbol{u}^n, \boldsymbol{w}^n, p^n)$ of the problem (4.1)-(4.2) satisfies the discrete energy estimate*

$$
\begin{aligned}
&\|\boldsymbol{u}^m\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\boldsymbol{w}^m\|_{\boldsymbol{L}^2(\Omega)}^2 + \tau\sum_{n=1}^{m}\Big(\tau\|d_t\boldsymbol{u}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + \tau\|d_t\boldsymbol{w}^n\|_{\boldsymbol{L}^2(\Omega)}^2\Big) \\
&+ \tau\sum_{n=1}^{m}\Big(\mu\|\nabla\boldsymbol{u}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + c_1\|\nabla\boldsymbol{w}^n\|_{\boldsymbol{L}^2(\Omega)}^2\Big) \leq C. \qquad (4.5)
\end{aligned}
$$

Proof. Taking $(\boldsymbol{v}, q) = 2\tau(\boldsymbol{u}^n, p^n)$ in (4.1), $\boldsymbol{\psi} = 2\tau\boldsymbol{w}^n$ in (4.2) and adding the two equations, by applying (2.1) and (2.3), the equality $2a(a-b) = |a|^2 - |b|^2 + |a-b|^2$, we obtain

$$\begin{aligned}
&\|\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + (1+8\tau\mu_r)\|\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \tau^2\|d_t\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \tau^2\|d_t\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)} \\
&\quad + 2\tau(\mu+\mu_r)\|\nabla\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + 2c_1\tau\|\nabla\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)} + 2c_2\tau\|\operatorname{div}\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)} \\
={}& \|\boldsymbol{u}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{w}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)} + 2\tau(\boldsymbol{f}^n, \boldsymbol{u}^n) + 2\tau(\boldsymbol{g}^n, \boldsymbol{w}^n) \\
&\quad + 4\tau\mu_r(\boldsymbol{w}^{n-1}, \mathbf{curl}\,\boldsymbol{u}^n) + 4\tau\mu_r(\mathbf{curl}\,\boldsymbol{u}^n, \boldsymbol{w}^n).
\end{aligned} \tag{4.6}$$

Employing the Hölder inequality, the Young inequality, the Poincaré inequality and (2.2), we have

$$\begin{aligned}
&2\left|(\boldsymbol{f}^n, \boldsymbol{u}^n)\right| \le \mu\|\nabla\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{C}{\tau}\int_{t_{n-1}}^{t_n} \|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)}dt, \\
&2\left|(\boldsymbol{g}^n, \boldsymbol{w}^n)\right| \le c_1\|\nabla\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{C}{\tau}\int_{t_{n-1}}^{t_n} \|\boldsymbol{g}\|^2_{\boldsymbol{L}^2(\Omega)}dt, \\
&4\mu_r\left|(\boldsymbol{w}^{n-1}, \mathbf{curl}\,\boldsymbol{u}^n)\right| \le \mu_r\|\nabla\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + 4\mu_r\|\boldsymbol{w}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)}, \\
&4\mu_r\left|(\boldsymbol{w}^n, \mathbf{curl}\,\boldsymbol{u}^n)\right| \le \mu_r\|\nabla\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + 4\mu_r\|\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}.
\end{aligned}$$

Combining the above inequalities and (4.6), we can show

$$\begin{aligned}
&\|\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + (1+4\tau\mu_r)\|\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \tau^2\|d_t\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} \\
&\quad + \tau^2\|d_t\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \tau\mu\|\nabla\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + c_1\tau\|\nabla\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)} \\
\le{}& \|\boldsymbol{u}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)} + (1+4\tau\mu_r)\|\boldsymbol{w}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)} + C\int_{t_{n-1}}^{t_n}\left(\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{g}\|^2_{\boldsymbol{L}^2(\Omega)}\right)dt
\end{aligned} \tag{4.7}$$

for all $1 \le n \le N$. Summing (4.7) with respect to n from $n=1$ to $n=m$ gives

$$\begin{aligned}
&\|\boldsymbol{u}^m\|^2_{\boldsymbol{L}^2(\Omega)} + (1+4\tau\mu_r)\|\boldsymbol{w}^m\|^2_{\boldsymbol{L}^2(\Omega)} + \tau\sum_{n=1}^{m}\left(\tau\|d_t\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \tau\|d_t\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right) \\
&\quad + \tau\sum_{n=1}^{m}\left(\mu\|\nabla\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + c_1\|\nabla\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right) \\
\le{}& \|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2(\Omega)} + (1+4\tau\mu_r)\|\boldsymbol{w}_0\|^2_{\boldsymbol{L}^2(\Omega)} + C\left(\|\boldsymbol{f}\|^2_{L^2(0,T;\boldsymbol{L}^2(\Omega))} + \|\boldsymbol{g}\|^2_{L^{(}0,T;\boldsymbol{L}^2(\Omega))}\right)
\end{aligned} \tag{4.8}$$

for all $1 \le m \le N$.

By applying Hypothesis 2.1 for the inequality (4.8), we can show the energy estimate (4.5). The proof is complete. $\square$

4.2. Error estimates: time semi-discrete

The main aim of this section is to give the $\boldsymbol{L}^2 - \boldsymbol{H}^1$ error estimates for the time-discrete solution $(\boldsymbol{u}^n, \boldsymbol{w}^n, p^n)$. To this end, we first derive an error equation for the discrete scheme developed in last section. Taking $t = t_n$ in both (2.4) and (2.5) and by applying Taylor formula, we have

$$\begin{aligned}
&(d_t\boldsymbol{u}(t_n), \boldsymbol{v}) + b(\boldsymbol{u}(t_n), \boldsymbol{u}(t_n), \boldsymbol{v}) + (\mu+\mu_r)a(\boldsymbol{u}(t_n), \boldsymbol{v}) - d(\boldsymbol{v}, p(t_n)) + d(\boldsymbol{u}(t_n), q) \\
={}& (\boldsymbol{f}(t_n), \boldsymbol{v}) + 2\mu_r(\mathbf{curl}\,\boldsymbol{w}(t_n), \boldsymbol{v}) - (\partial_{tt}\boldsymbol{u}^n, \boldsymbol{v}),
\end{aligned} \tag{4.9}$$

$$
\begin{aligned}
&(d_t\boldsymbol{w}(t_n),\boldsymbol{\psi})+b(\boldsymbol{u}(t_n),\boldsymbol{w}(t_n),\boldsymbol{\psi})+c_1a(\boldsymbol{w}(t_n),\boldsymbol{\psi})+c_2(\operatorname{div}\boldsymbol{w}(t_n),\operatorname{div}\boldsymbol{\psi})+4\mu_r(\boldsymbol{w}(t_n),\boldsymbol{\psi})\\
&=(\boldsymbol{g}(t_n),\boldsymbol{\psi})+2\mu_r(\mathbf{curl}\,\boldsymbol{u}(t_n),\boldsymbol{\psi})-(\partial_{tt}\boldsymbol{w}^n,\boldsymbol{\psi}) \qquad (4.10)
\end{aligned}
$$

for all $(\boldsymbol{v},q,\boldsymbol{\psi})\in\boldsymbol{H}_0^1(\Omega)\times L_0^2(\Omega)\times\boldsymbol{H}_0^1(\Omega)$ with $1\le n\le N$, where $\partial_{tt}\boldsymbol{u}^n$ and $\partial_{tt}\boldsymbol{w}^n$ are defined by

$$
\partial_{tt}\boldsymbol{u}^n:=\frac{1}{\tau}\int_{t_{n-1}}^{t_n}(t-t_{n-1})\boldsymbol{u}_{tt}dt,\quad \partial_{tt}\boldsymbol{w}^n:=\frac{1}{\tau}\int_{t_{n-1}}^{t_n}(t-t_{n-1})\boldsymbol{w}_{tt}dt.
$$

Subtracting (4.1) and (4.2) from (4.9) and (4.10), respectively, and setting $\mathcal{R}_{\boldsymbol{u}}^n=\boldsymbol{u}(t_n)-\boldsymbol{u}^n$, $\mathcal{R}_{\boldsymbol{w}}^n=\boldsymbol{w}(t_n)-\boldsymbol{w}^n$ and $\mathcal{R}_p^n=p(t_n)-p^n$, we can show

$$
\begin{aligned}
&(d_t\mathcal{R}_{\boldsymbol{u}}^n,\boldsymbol{v})+b(\mathcal{R}_{\boldsymbol{u}}^{n-1},\boldsymbol{u}(t_n),\boldsymbol{v})+b(\boldsymbol{u}^{n-1},\mathcal{R}_{\boldsymbol{u}}^n,\boldsymbol{v})+(\mu+\mu_r)a(\mathcal{R}_{\boldsymbol{u}}^n,\boldsymbol{v})\\
&\quad-d(\boldsymbol{v},\mathcal{R}_p^n)+d(\mathcal{R}_{\boldsymbol{u}}^n,q)-2\mu_r(\mathbf{curl}\mathcal{R}_{\boldsymbol{w}}^{n-1},\boldsymbol{v})=(\mathcal{E}_1^n,\boldsymbol{v}), \qquad (4.11)\\
&(d_t\mathcal{R}_{\boldsymbol{w}}^n,\boldsymbol{\psi})+b(\mathcal{R}_{\boldsymbol{u}}^{n-1},\boldsymbol{w}(t_n),\boldsymbol{\psi})+b(\boldsymbol{u}^{n-1},\mathcal{R}_{\boldsymbol{w}}^n,\boldsymbol{\psi})+c_1a(\mathcal{R}_{\boldsymbol{w}}^n,\boldsymbol{\psi})\\
&\quad+c_2(\operatorname{div}\mathcal{R}_{\boldsymbol{w}}^n,\operatorname{div}\boldsymbol{\psi})+4\mu_r(\mathcal{R}_{\boldsymbol{w}}^n,\boldsymbol{\psi})-2\mu_r(\mathbf{curl}\mathcal{R}_{\boldsymbol{u}}^n,\boldsymbol{\psi})=(\mathcal{E}_2^n,\boldsymbol{\psi}) \qquad (4.12)
\end{aligned}
$$

for all $(\boldsymbol{v},q,\boldsymbol{\psi})\in\boldsymbol{H}_0^1(\Omega)\times L_0^2(\Omega)\times\boldsymbol{H}_0^1(\Omega)$, where $\mathcal{E}_1^n$ and $\mathcal{E}_2^n$ are given by

$$
\begin{aligned}
&(\mathcal{E}_1^n,\boldsymbol{v})=(\partial_t\boldsymbol{f}^n,\boldsymbol{v})-(\partial_{tt}\boldsymbol{u}^n,\boldsymbol{v})-b(\partial_t\boldsymbol{u}^n,\boldsymbol{u}(t_n),\boldsymbol{v})+2\mu_r(\mathbf{curl}\,\partial_t\boldsymbol{w}^n,\boldsymbol{v}), \qquad (4.13)\\
&(\mathcal{E}_2^n,\boldsymbol{\psi})=(\partial_t\boldsymbol{g}^n,\boldsymbol{\psi})-(\partial_{tt}\boldsymbol{w}^n,\boldsymbol{\psi})-b(\partial_t\boldsymbol{u}^n,\boldsymbol{w}(t_n),\boldsymbol{\psi}), \qquad (4.14)
\end{aligned}
$$

and $\partial_t\boldsymbol{u}^n$, $\partial_t\boldsymbol{w}^n$, $\partial_t\boldsymbol{f}$ and $\partial_t\boldsymbol{g}$ are given by

$$
\begin{aligned}
\partial_t\boldsymbol{u}^n&:=\int_{t_{n-1}}^{t_n}\boldsymbol{u}_tdt,\quad \partial_t\boldsymbol{f}^n:=\frac{1}{\tau}\int_{t_{n-1}}^{t_n}(t-t_{n-1})\boldsymbol{f}_tdt,\\
\partial_t\boldsymbol{w}^n&:=\int_{t_{n-1}}^{t_n}\boldsymbol{w}_tdt,\quad \partial_t\boldsymbol{g}^n:=\frac{1}{\tau}\int_{t_{n-1}}^{t_n}(t-t_{n-1})\boldsymbol{g}_tdt.
\end{aligned}
$$

To derive an error estimates for the time discrete solution $(\boldsymbol{u}^n,\boldsymbol{w}^n,p^n)$, we need to give an estimates for the residual errors $\mathcal{E}_1^n$ and $\mathcal{E}_2^n$.

Lemma 4.2. *Suppose that Hypothesis* 2.1-2.3 *hold. Then* $\mathcal{E}_1^n$ *and* $\mathcal{E}_2^n$ *satisfy the following estimates:*

$$
\begin{aligned}
&\tau\sum_{n=1}^m\left(\|\mathcal{E}_1^n\|^2_{(\boldsymbol{H}^1_{\rm div}(\Omega))'}+\|\mathcal{E}_2^n\|^2_{(\boldsymbol{H}^{-1}(\Omega))}\right)\le C\tau^2,\quad\forall\,1\le m\le N, \qquad (4.15)\\
&\tau\sum_{n=1}^m\nu(t_n)\left(\|\mathcal{E}_1^n\|^2_{\boldsymbol{L}^2(\Omega)}+\|\mathcal{E}_2^n\|^2_{\boldsymbol{L}^2(\Omega)}\right)\le C\tau^2,\quad\forall\,1\le m\le N. \qquad (4.16)
\end{aligned}
$$

Proof. By applying (4.13)-(4.14), $\boldsymbol{H}^1_{\rm div}(\Omega)\subset\boldsymbol{H}_0^1(\Omega)$, the Hölder inequality and (2.2), we obtain

$$
\begin{aligned}
\|\mathcal{E}_1^n\|^2_{(\boldsymbol{H}^1_{\rm div}(\Omega))'}\le{}& C\tau^{\frac12}\|\boldsymbol{f}_t\|_{L^2(t_{n-1},t_n;\boldsymbol{L}^2(\Omega))}+C\tau^{\frac12}\|\boldsymbol{u}_{tt}\|_{L^2(t_{n-1},t_n;(\boldsymbol{H}^1_{\rm div}(\Omega))')}\\
&+C\tau^{\frac12}\|\nabla\boldsymbol{u}(t_n)\|_{\boldsymbol{L}^2(\Omega)}\|\nabla\boldsymbol{u}_t\|_{L^2(t_{n-1},t_n;\boldsymbol{L}^2(\Omega))}\\
&+C\tau^{\frac12}\|\nabla\boldsymbol{w}_t\|_{L^2(t_{n-1},t_n;\boldsymbol{L}^2(\Omega))}, \qquad (4.17)\\
\|\mathcal{E}_2^n\|_{(\boldsymbol{H}^{-1}(\Omega))}\le{}& C\tau^{\frac12}\|\boldsymbol{g}_t\|_{L^2(t_{n-1},t_n;\boldsymbol{L}^2(\Omega))}+C\tau^{\frac12}\|\boldsymbol{w}_{tt}\|_{L^2(t_{n-1},t_n;(\boldsymbol{H}^1_0(\Omega))')}\\
&+C\tau^{\frac12}\|\nabla\boldsymbol{w}(t_n)\|_{\boldsymbol{L}^2(\Omega)}\|\nabla\boldsymbol{u}_t\|_{L^2(t_{n-1},t_n;\boldsymbol{L}^2(\Omega))}. \qquad (4.18)
\end{aligned}
$$

Combining (4.17)-(4.18) and using Theorem 3.2, we conclude that

$$\begin{aligned}
&\tau\left(\|\mathcal{E}_1^n\|^2_{(\boldsymbol{H}^1_{\mathrm{div}}(\Omega))'}+\|\mathcal{E}_2^n\|^2_{(\boldsymbol{H}^{-1}(\Omega))}\right)\\
&\le C\tau^2\int_{t_{n-1}}^{t_n}\left(\|\boldsymbol{f}_t\|^2_{\boldsymbol{L}^2(\Omega)}+\|\boldsymbol{g}_t\|^2_{\boldsymbol{L}^2(\Omega)}\right)dt\\
&\quad+C\tau^2\int_{t_{n-1}}^{t_n}\left(\|\boldsymbol{u}_{tt}\|^2_{(\boldsymbol{H}^1_{\mathrm{div}}(\Omega))'}+\|\boldsymbol{w}_{tt}\|^2_{(\boldsymbol{H}^{-1}(\Omega))}\right)dt\\
&\quad+C\tau^2\int_{t_{n-1}}^{t_n}\left(\|\nabla\boldsymbol{u}_t\|^2_{\boldsymbol{L}^2(\Omega)}+\|\nabla\boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)}\right)dt
\end{aligned}\tag{4.19}$$

for all $1\le n\le N$. Summing (4.19) with respect to n from $n=1$ to $n=m$ and by applying Hypothesis 2.1 and Theorems 3.2-3.4, we have the estimate (4.15). Similarly, by applying (4.13)-(4.14) and the Hölder inequality, we get

$$\begin{aligned}
\|\mathcal{E}_1^n\|_{\boldsymbol{L}^2(\Omega)}\le\ & C\tau^{\frac12}\|\boldsymbol{f}_t\|_{L^2(t_{n-1},t_n;\boldsymbol{L}^2(\Omega))}+C\tau^{\frac12}\|\boldsymbol{u}_{tt}\|_{L^2(t_{n-1},t_n;\boldsymbol{L}^2(\Omega))}\\
&+C\tau^{\frac12}\|\boldsymbol{u}(t_n)\|_{\boldsymbol{H}^2(\Omega)}\|\nabla\boldsymbol{u}_t\|_{L^2(t_{n-1},t_n;\boldsymbol{L}^2(\Omega))}\\
&+C\tau^{\frac12}\|\nabla\boldsymbol{w}_t\|_{L^2(t_{n-1},t_n;\boldsymbol{L}^2(\Omega))},
\end{aligned}\tag{4.20}$$

$$\begin{aligned}
\|\mathcal{E}_2^n\|_{\boldsymbol{L}^2(\Omega)}\le\ & C\tau^{\frac12}\|\boldsymbol{g}_t\|_{L^2(t_{n-1},t_n;\boldsymbol{L}^2(\Omega))}+C\tau^{\frac12}\|\boldsymbol{w}_{tt}\|_{L^2(t_{n-1},t_n;\boldsymbol{L}^2(\Omega))}\\
&+C\tau^{\frac12}\|\boldsymbol{w}(t_n)\|_{\boldsymbol{H}^2(\Omega)}\|\nabla\boldsymbol{u}_t\|_{L^2(t_{n-1},t_n;\boldsymbol{L}^2(\Omega))}.
\end{aligned}\tag{4.21}$$

Combining (4.20) and (4.21), we obtain

$$\begin{aligned}
&\tau\nu(t_n)\left(\|\mathcal{E}_1^n\|^2_{\boldsymbol{L}^2(\Omega)}+\|\mathcal{E}_2^n\|^2_{\boldsymbol{L}^2(\Omega)}\right)\\
&\le C\tau^2\left(1+\|\boldsymbol{u}(t_n)\|^2_{\boldsymbol{H}^2(\Omega)}+\|\boldsymbol{w}(t_n)\|^2_{\boldsymbol{H}^2(\Omega)}\right)\int_{t_{n-1}}^{t_n}\left(\|\nabla\boldsymbol{u}_t\|^2_{\boldsymbol{L}^2(\Omega)}+\|\nabla\boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)}\right)dt\\
&\quad+C\tau^2\int_{t_{n-1}}^{t_n}\left(\|\boldsymbol{f}_t\|^2_{\boldsymbol{L}^2(\Omega)}+\|\boldsymbol{g}_t\|^2_{\boldsymbol{L}^2(\Omega)}+\nu(t_n)\|\boldsymbol{u}_{tt}\|^2_{\boldsymbol{L}^2(\Omega)}+\nu(t_n)\|\boldsymbol{w}_{tt}\|^2_{\boldsymbol{L}^2(\Omega)}\right)dt
\end{aligned}\tag{4.22}$$

for all $1\le n\le N$. Summing (4.22) with respect to n from $n=1$ to $n=m$ and by applying Hypothesis 2.1 and Theorems 3.3-3.5, we get the required estimate (4.16). The proof is complete. $\square$

In the remainder of this section, we will prove the error estimates for the time-discrete scheme.

Theorem 4.2. *Suppose that Hypothesis* 2.1-2.3 *hold. For all* $1\le m\le N$*, the solution* $(\boldsymbol{u}^n,\boldsymbol{w}^n,p^n)$ *of the problem* (4.1)-(4.2) *satisfies the error estimate*

$$\begin{aligned}
&\sup_{1\le n\le m}\left(\|\boldsymbol{u}(t_n)-\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)}+\|\boldsymbol{w}(t_n)-\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right)\\
&\quad+\tau\sum_{n=1}^m\left(\|\nabla(\boldsymbol{u}(t_n)-\boldsymbol{u}^n)\|^2_{\boldsymbol{L}^2(\Omega)}+\|\nabla(\boldsymbol{w}(t_n)-\boldsymbol{w}^n)\|^2_{\boldsymbol{L}^2(\Omega)}\right)\le C\tau^2.
\end{aligned}\tag{4.23}$$

Proof. Taking $(\boldsymbol{v},q)=2\tau(\mathcal{R}^n_{\boldsymbol{u}},\mathcal{R}^n_p)$ in (4.11) and $\boldsymbol{\psi}=2\tau\mathcal{R}^n_{\boldsymbol{w}}$ in (4.12) and adding together, by applying (2.3) and the equality

$$2a(a-b)=|a|^2-|b|^2+|a-b|^2,$$

we have

$$\begin{aligned}
&\|\mathcal{R}_{\boldsymbol{u}}^n\|^2_{\boldsymbol{L}^2(\Omega)} + (1+8\mu_r\tau)\|\mathcal{R}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \tau^2\|d_t\mathcal{R}_{\boldsymbol{u}}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \tau^2\|d_t\mathcal{R}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)}\\
&\quad + 2\tau(\mu+\mu_r)\|\nabla\mathcal{R}_{\boldsymbol{u}}^n\|^2_{\boldsymbol{L}^2(\Omega)} + 2c_1\tau\|\nabla\mathcal{R}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)} + 2c_2\tau\|\operatorname{div}\mathcal{R}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)}\\
&\quad + 2\tau b(\mathcal{R}_{\boldsymbol{u}}^{n-1}, \boldsymbol{u}(t_n), \mathcal{R}_{\boldsymbol{u}}^n) + 2\tau b(\mathcal{R}_{\boldsymbol{u}}^{n-1}, \boldsymbol{w}(t_n), \mathcal{R}_{\boldsymbol{w}}^n)\\
&\quad - 4\tau\mu_r(\mathbf{curl}\mathcal{R}_{\boldsymbol{w}}^{n-1}, \mathcal{R}_{\boldsymbol{u}}^n) - 4\tau\mu_r(\mathbf{curl}\mathcal{R}_{\boldsymbol{u}}^n, \mathcal{R}_{\boldsymbol{w}}^n)\\
&= \|\mathcal{R}_{\boldsymbol{u}}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathcal{R}_{\boldsymbol{w}}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)} + 2\tau(\mathcal{E}_1^n, \mathcal{R}_{\boldsymbol{u}}^n) + 2\tau(\mathcal{E}_2^n, \mathcal{R}_{\boldsymbol{w}}^n).
\end{aligned} \tag{4.24}$$

By applying the Hölder and Young inequalities, $\mathcal{R}_{\boldsymbol{u}}^n \in \boldsymbol{H}^1_{\mathrm{div}}(\Omega)$ and (2.1)-(2.2), we obtain

$$\begin{aligned}
&2\left|b(\mathcal{R}_{\boldsymbol{u}}^{n-1}, \boldsymbol{u}(t_n), \mathcal{R}_{\boldsymbol{u}}^n)\right| \le \frac{\mu}{2}\|\nabla\mathcal{R}_{\boldsymbol{u}}^n\|^2_{\boldsymbol{L}^2(\Omega)} + C\|\boldsymbol{u}(t_n)\|^2_{\boldsymbol{H}^2(\Omega)}\|\mathcal{R}_{\boldsymbol{u}}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)},\\
&2\left|b(\mathcal{R}_{\boldsymbol{u}}^{n-1}, \boldsymbol{w}(t_n), \mathcal{R}_{\boldsymbol{w}}^n)\right| \le \frac{c_1}{2}\|\nabla\mathcal{R}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)} + C\|\boldsymbol{w}(t_n)\|^2_{\boldsymbol{H}^2(\Omega)}\|\mathcal{R}_{\boldsymbol{u}}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)},\\
&4\mu_r\left|(\mathbf{curl}\mathcal{R}_{\boldsymbol{w}}^{n-1}, \mathcal{R}_{\boldsymbol{u}}^n)\right| \le 4\mu_r\|\mathcal{R}_{\boldsymbol{w}}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)} + \mu_r\|\nabla\mathcal{R}_{\boldsymbol{u}}^n\|^2_{\boldsymbol{L}^2(\Omega)},\\
&4\mu_r\left|(\mathbf{curl}\mathcal{R}_{\boldsymbol{u}}^n, \mathcal{R}_{\boldsymbol{w}}^n)\right| \le 4\mu_r\|\mathcal{R}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \mu_r\|\nabla\mathcal{R}_{\boldsymbol{u}}^n\|^2_{\boldsymbol{L}^2(\Omega)},\\
&2\left|(\mathcal{E}_1^n, \mathcal{R}_{\boldsymbol{u}}^n)\right| \le \frac{\mu}{2}\|\nabla\mathcal{R}_{\boldsymbol{u}}^n\|^2_{\boldsymbol{L}^2(\Omega)} + C\|\mathcal{E}_1^n\|^2_{(\boldsymbol{H}^1_{\mathrm{div}}(\Omega))'},\\
&2\left|(\mathcal{E}_2^n, \mathcal{R}_{\boldsymbol{w}}^n)\right| \le \frac{c_1}{2}\|\nabla\mathcal{R}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)} + C\|\mathcal{E}_2^n\|^2_{(\boldsymbol{H}^{-1}(\Omega))}.
\end{aligned}$$

Combining the above inequalities and (4.24), we can show

$$\begin{aligned}
&\|\mathcal{R}_{\boldsymbol{u}}^n\|^2_{\boldsymbol{L}^2(\Omega)} + (1+4\mu_r\tau)\|\mathcal{R}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \mu\tau\|\nabla\mathcal{R}_{\boldsymbol{u}}^n\|^2_{\boldsymbol{L}^2(\Omega)} + c_1\tau\|\nabla\mathcal{R}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)}\\
&\le \|\mathcal{R}_{\boldsymbol{u}}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)} + (1+4\mu_r\tau)\|\mathcal{R}_{\boldsymbol{w}}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)} + C\tau\left(\|\mathcal{E}_1^n\|^2_{(\boldsymbol{H}^1_{\mathrm{div}}(\Omega))'} + \|\mathcal{E}_2^n\|^2_{(\boldsymbol{H}^{-1}(\Omega))}\right)\\
&\quad + C\tau\left(\|\boldsymbol{u}(t_n)\|^2_{\boldsymbol{H}^2(\Omega)} + \|\boldsymbol{w}(t_n)\|^2_{\boldsymbol{H}^2(\Omega)}\right)\left(\|\mathcal{R}_{\boldsymbol{u}}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathcal{R}_{\boldsymbol{w}}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)}\right)
\end{aligned} \tag{4.25}$$

for all $1 \le n \le N$. Summing (4.25) with respect to n from $n = 1$ to $n = m$ and by applying Lemma 4.2 and noticing $(\mathcal{R}_{\boldsymbol{u}}^0, \mathcal{R}_{\boldsymbol{w}}^0) = (\mathbf{0}, \mathbf{0})$, we have

$$\begin{aligned}
&\|\mathcal{R}_{\boldsymbol{u}}^m\|^2_{\boldsymbol{L}^2(\Omega)} + (1+4\mu_r\tau)\|\mathcal{R}_{\boldsymbol{w}}^m\|^2_{\boldsymbol{L}^2(\Omega)} + \tau\sum_{n=1}^{m}\left(\mu\|\nabla\mathcal{R}_{\boldsymbol{u}}^n\|^2_{\boldsymbol{L}^2(\Omega)} + c_1\|\nabla\mathcal{R}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right)\\
&\le C\tau\sum_{n=0}^{m-1}\left(\|\boldsymbol{u}(t_{n+1})\|^2_{\boldsymbol{H}^2(\Omega)} + \|\boldsymbol{w}(t_{n+1})\|^2_{\boldsymbol{H}^2(\Omega)}\right)\left(\|\mathcal{R}_{\boldsymbol{u}}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathcal{R}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right)\\
&\quad + C\tau\sum_{n=1}^{m}\left(\|\mathcal{E}_1^n\|^2_{(\boldsymbol{H}^1_{\mathrm{div}}(\Omega))'} + \|\mathcal{E}_2^n\|^2_{(\boldsymbol{H}^{-1}(\Omega))}\right)\\
&\le \tau\sum_{n=0}^{m-1} d_n\left(\|\mathcal{R}_{\boldsymbol{u}}^n\|^2_{\boldsymbol{L}^2(\Omega)} + (1+4\mu_r\tau)\|\mathcal{R}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right) + C\tau^2,
\end{aligned} \tag{4.26}$$

where $d_n = C(\|\boldsymbol{u}(t_{n+1})\|^2_{\boldsymbol{H}^2(\Omega)} + \|\boldsymbol{w}(t_{n+1})\|^2_{\boldsymbol{H}^2(\Omega)})$.

By applying Lemma 4.1 and Theorem 3.4 for the inequality (4.26), we obtain

$$\begin{aligned}
&\|\mathcal{R}_{\boldsymbol{u}}^m\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathcal{R}_{\boldsymbol{w}}^m\|^2_{\boldsymbol{L}^2(\Omega)} + \tau\sum_{n=1}^{m}\left(\mu\|\nabla\mathcal{R}_{\boldsymbol{u}}^n\|^2_{\boldsymbol{L}^2(\Omega)} + c_1\|\nabla\mathcal{R}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right)\\
&\le C\tau^2\exp\left(\tau\sum_{n=0}^{m-1} d_n\right) \le C\tau^2
\end{aligned}$$

for all $1 \le m \le N$. The proof is complete. $\square$

Theorem 4.3. *Suppose that Hypothesis* 2.1-2.3 *hold. For all* $1 \le m \le N$*, the solution* $(\boldsymbol{u}^n, \boldsymbol{w}^n, p^n)$ *of the problem* (4.1)-(4.2) *satisfies the error estimate*

$$\begin{aligned}
&\sup_{1\le n\le m} \nu(t_n)\left(\|\nabla(\boldsymbol{u}(t_n)-\boldsymbol{u}^n)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla(\boldsymbol{w}(t_n)-\boldsymbol{w}^n)\|^2_{\boldsymbol{L}^2(\Omega)}\right)\\
&\quad + \tau\sum_{n=1}^m \nu(t_n)\left(\|d_t(\boldsymbol{u}(t_n)-\boldsymbol{u}^n)\|^2_{\boldsymbol{L}^2(\Omega)} + \|d_t(\boldsymbol{w}(t_n)-\boldsymbol{w}^n)\|^2_{\boldsymbol{L}^2(\Omega)}\right) \le C\tau^2. \qquad (4.27)
\end{aligned}$$

Proof. Taking $(\boldsymbol{v}, q) = 2\tau(d_t\mathcal{R}^n_{\boldsymbol{u}}, 0)$ in (4.11) and $\boldsymbol{\psi} = 2\tau d_t\mathcal{R}^n_{\boldsymbol{u}}$ in (4.12) and adding together, by applying the equality

$$2a(a-b) = |a|^2 - |b|^2 + |a-b|^2,$$

we have

$$\begin{aligned}
&(\mu+\mu_r)\left(\|\nabla\mathcal{R}^n_{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2(\Omega)} - \|\nabla\mathcal{R}^{n-1}_{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla(\mathcal{R}^n_{\boldsymbol{u}} - \mathcal{R}^{n-1}_{\boldsymbol{u}})\|^2_{\boldsymbol{L}^2(\Omega)}\right)\\
&\quad + c_1\left(\|\nabla\mathcal{R}^n_{\boldsymbol{w}}\|^2_{\boldsymbol{L}^2(\Omega)} - \|\nabla\mathcal{R}^{n-1}_{\boldsymbol{w}}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla(\mathcal{R}^n_{\boldsymbol{w}} - \mathcal{R}^{n-1}_{\boldsymbol{w}})\|^2_{\boldsymbol{L}^2(\Omega)}\right)\\
&\quad + c_2\left(\|\operatorname{div}\mathcal{R}^n_{\boldsymbol{w}}\|^2_{\boldsymbol{L}^2(\Omega)} - \|\operatorname{div}\mathcal{R}^{n-1}_{\boldsymbol{w}}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\operatorname{div}(\mathcal{R}^n_{\boldsymbol{w}} - \mathcal{R}^{n-1}_{\boldsymbol{w}})\|^2_{\boldsymbol{L}^2(\Omega)}\right)\\
&\quad + 8\mu_r\left(\|\mathcal{R}^n_{\boldsymbol{w}}\|^2_{\boldsymbol{L}^2(\Omega)} - \|\mathcal{R}^{n-1}_{\boldsymbol{w}}\|^2_{\boldsymbol{L}^2(\Omega)} + \|(\mathcal{R}^n_{\boldsymbol{w}} - \mathcal{R}^{n-1}_{\boldsymbol{w}})\|^2_{\boldsymbol{L}^2(\Omega)}\right)\\
&\quad + 2\tau b\left(\mathcal{R}^{n-1}_{\boldsymbol{u}}, \boldsymbol{u}(t_n), d_t\mathcal{R}^n_{\boldsymbol{u}}\right) + 2\tau b\left(\boldsymbol{u}(t_{n-1}), \mathcal{R}^{n-1}_{\boldsymbol{u}}, d_t\mathcal{R}^n_{\boldsymbol{u}}\right)\\
&\quad + 2\tau b\left(\mathcal{R}^{n-1}_{\boldsymbol{u}}, \boldsymbol{w}(t_n), d_t\mathcal{R}^n_{\boldsymbol{w}}\right) + 2\tau b\left(\boldsymbol{u}(t_{n-1}), \mathcal{R}^{n-1}_{\boldsymbol{w}}, d_t\mathcal{R}^n_{\boldsymbol{w}}\right)\\
&\quad - 2\tau b\left(\mathcal{R}^{n-1}_{\boldsymbol{u}}, \mathcal{R}^n_{\boldsymbol{u}}, d_t\mathcal{R}^n_{\boldsymbol{u}}\right) - 2\tau b\left(\mathcal{R}^{n-1}_{\boldsymbol{u}}, \mathcal{R}^n_{\boldsymbol{w}}, d_t\mathcal{R}^n_{\boldsymbol{u}}\right)\\
&\quad + 2\tau\|d_t\mathcal{R}^n_{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2(\Omega)} + 2\tau\|d_t\mathcal{R}^n_{\boldsymbol{w}}\|^2_{\boldsymbol{L}^2(\Omega)}\\
&\quad - 4\mu_r\tau\left(\mathbf{curl}\mathcal{R}^{n-1}_{\boldsymbol{w}}, d_t\mathcal{R}^n_{\boldsymbol{u}}\right) - 4\mu_r\tau\left(\mathbf{curl}\mathcal{R}^n_{\boldsymbol{u}}, d_t\mathcal{R}^n_{\boldsymbol{w}}\right)\\
&= 2\tau(\mathcal{E}^n_1, d_t\mathcal{R}^n_{\boldsymbol{u}}) + 2\tau(\mathcal{E}^n_2, d_t\mathcal{R}^n_{\boldsymbol{w}}). \qquad (4.28)
\end{aligned}$$

By applying the Hölder inequality, the Young inequality and (2.2), we can show

$$\begin{aligned}
&4\mu_r\left|(\mathbf{curl}\mathcal{R}^{n-1}_{\boldsymbol{w}}, d_t\mathcal{R}^n_{\boldsymbol{u}})\right| \le \frac{1}{4}\|d_t\mathcal{R}^n_{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2(\Omega)} + C\|\nabla\mathcal{R}^{n-1}_{\boldsymbol{w}}\|^2_{\boldsymbol{L}^2(\Omega)},\\
&4\mu_r\left|(\mathbf{curl}\mathcal{R}^n_{\boldsymbol{u}}, d_t\mathcal{R}^n_{\boldsymbol{w}})\right| \le \frac{1}{4}\|d_t\mathcal{R}^n_{\boldsymbol{w}}\|^2_{\boldsymbol{L}^2(\Omega)} + C\|\nabla\mathcal{R}^n_{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2(\Omega)},\\
&2\left|(\mathcal{E}^n_1, d_t\mathcal{R}^n_{\boldsymbol{u}})\right| \le \frac{1}{4}\|d_t\mathcal{R}^n_{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2(\Omega)} + C\|\mathcal{E}^n_1\|^2_{\boldsymbol{L}^2(\Omega)},\\
&2\left|(\mathcal{E}^n_2, d_t\mathcal{R}^n_{\boldsymbol{w}})\right| \le \frac{1}{4}\|d_t\mathcal{R}^n_{\boldsymbol{w}}\|^2_{\boldsymbol{L}^2(\Omega)} + C\|\mathcal{E}^n_2\|^2_{\boldsymbol{L}^2(\Omega)},\\
&2\left|b(\mathcal{R}^{n-1}_{\boldsymbol{u}}, \boldsymbol{u}(t_n), d_t\mathcal{R}^n_{\boldsymbol{u}})\right| \le 2\|\mathcal{R}^{n-1}_{\boldsymbol{u}}\|_{\boldsymbol{L}^6(\Omega)}\|\nabla\boldsymbol{u}(t_n)\|_{\boldsymbol{L}^3(\Omega)}\|d_t\mathcal{R}^n_{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega)}\\
&\qquad\qquad \le \frac{1}{4}\|d_t\mathcal{R}^n_{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2(\Omega)} + C\|\boldsymbol{u}(t_n)\|^2_{\boldsymbol{H}^2(\Omega)}\|\nabla\mathcal{R}^{n-1}_{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2(\Omega)},\\
&2\left|b(\boldsymbol{u}(t_{n-1}), \mathcal{R}^{n-1}_{\boldsymbol{u}}, d_t\mathcal{R}^n_{\boldsymbol{u}})\right| \le 2\|\boldsymbol{u}(t_{n-1})\|_{\boldsymbol{L}^\infty(\Omega)}\|\nabla\mathcal{R}^{n-1}_{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega)}\|d_t\mathcal{R}^n_{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega)}\\
&\qquad\qquad \le \frac{1}{4}\|d_t\mathcal{R}^n_{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2(\Omega)} + C\|\boldsymbol{u}(t_{n-1})\|^2_{\boldsymbol{H}^2(\Omega)}\|\nabla\mathcal{R}^{n-1}_{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2(\Omega)},\\
&2\left|b(\mathcal{R}^{n-1}_{\boldsymbol{u}}, \mathcal{R}^n_{\boldsymbol{u}}, d_t\mathcal{R}^n_{\boldsymbol{u}})\right| \le 2\tau^{-1}\|\nabla\mathcal{R}^{n-1}_{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega)}\|\nabla\mathcal{R}^n_{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega)}\|\nabla(\mathcal{R}^n_{\boldsymbol{u}} - \mathcal{R}^{n-1}_{\boldsymbol{u}})\|_{\boldsymbol{L}^2(\Omega)}\\
&\qquad\qquad \le \frac{\mu+\mu_r}{2\tau}\|\nabla(\mathcal{R}^n_{\boldsymbol{u}} - \mathcal{R}^{n-1}_{\boldsymbol{u}})\|^2_{\boldsymbol{L}^2(\Omega)} + C\tau^{-1}\|\nabla\mathcal{R}^{n-1}_{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2(\Omega)}\|\nabla\mathcal{R}^n_{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2(\Omega)},
\end{aligned}$$

$$\begin{aligned}
2\left|b(\mathcal{R}_{\boldsymbol{u}}^{n-1},\boldsymbol{w}(t_n),d_t\mathcal{R}_{\boldsymbol{w}}^n)\right| &\le 2\|\mathcal{R}_{\boldsymbol{u}}^{n-1}\|_{\boldsymbol{L}^6(\Omega)}\|\nabla\boldsymbol{w}(t_n)\|_{\boldsymbol{L}^3(\Omega)}\|d_t\mathcal{R}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}\\
&\le \frac{1}{4}\|d_t\mathcal{R}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + C\|\boldsymbol{w}(t_n)\|_{\boldsymbol{H}^2(\Omega)}^2\|\nabla\mathcal{R}_{\boldsymbol{u}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2,\\
2\left|b(\boldsymbol{u}(t_{n-1}),\mathcal{R}_{\boldsymbol{w}}^{n-1},d_t\mathcal{R}_{\boldsymbol{w}}^n)\right| &\le 2\|\boldsymbol{u}(t_{n-1})\|_{\boldsymbol{L}^\infty(\Omega)}\|\nabla\mathcal{R}_{\boldsymbol{w}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}\|d_t\mathcal{R}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}\\
&\le \frac{1}{4}\|d_t\mathcal{R}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + C\|\boldsymbol{u}(t_{n-1})\|_{\boldsymbol{H}^2(\Omega)}^2\|\nabla\mathcal{R}_{\boldsymbol{w}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2,\\
2\left|b(\mathcal{R}_{\boldsymbol{u}}^{n-1},\mathcal{R}_{\boldsymbol{w}}^n,d_t\mathcal{R}_{\boldsymbol{w}}^n)\right| &\le 2\tau^{-1}\|\nabla\mathcal{R}_{\boldsymbol{u}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}\|\nabla\mathcal{R}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}\|\nabla(\mathcal{R}_{\boldsymbol{w}}^n-\mathcal{R}_{\boldsymbol{w}}^{n-1})\|_{\boldsymbol{L}^2(\Omega)}\\
&\le \frac{c_1}{2\tau}\|\nabla(\mathcal{R}_{\boldsymbol{w}}^n-\mathcal{R}_{\boldsymbol{w}}^{n-1})\|_{\boldsymbol{L}^2(\Omega)}^2 + C\tau^{-1}\|\nabla\mathcal{R}_{\boldsymbol{u}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2\|\nabla\mathcal{R}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2.
\end{aligned}$$

Combining the above inequalities and (4.28) and by applying Theorem 3.4, we get

$$\begin{aligned}
&\tau\|d_t\mathcal{R}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + \tau\|d_t\mathcal{R}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + (\mu+\mu_r)\left(\|\nabla\mathcal{R}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 - \|\nabla\mathcal{R}_{\boldsymbol{u}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2\right)\\
&\quad + c_1\left(\|\nabla\mathcal{R}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 - \|\nabla\mathcal{R}_{\boldsymbol{w}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2\right) + 8\mu_r\left(\|\mathcal{R}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 - \|\mathcal{R}_{\boldsymbol{w}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2\right)\\
&\quad + 2c_2\left(\|\operatorname{div}\mathcal{R}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 - \|\operatorname{div}\mathcal{R}_{\boldsymbol{w}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2\right)\\
&\le C\tau\left(\|\mathcal{E}_1^n\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\mathcal{E}_2^n\|_{\boldsymbol{L}^2(\Omega)}^2\right) + C\left(\|\nabla\mathcal{R}_{\boldsymbol{u}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\nabla\mathcal{R}_{\boldsymbol{w}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2\right)\\
&\quad \times\left(\|\nabla\mathcal{R}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\nabla\mathcal{R}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2\right)\\
&\quad + C\tau\left(\|\nabla\mathcal{R}_{\boldsymbol{u}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\nabla\mathcal{R}_{\boldsymbol{w}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\nabla\mathcal{R}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\nabla\mathcal{R}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2\right)
\end{aligned}\tag{4.29}$$

for all $1 \le n \le N$. Multiplying the inequality (4.29) by $\nu(t_n)$ and by applying $\nu(t_n) \le \nu(t_{n-1}) + \tau$, we obtain

$$\begin{aligned}
&\nu(t_n)\left(\|d_t\mathcal{R}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + \|d_t\mathcal{R}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2\right)\tau + \nu(t_n)\left((\mu+\mu_r)\|\nabla\mathcal{R}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + c_1\|\nabla\mathcal{R}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2\right)\\
&\quad - \nu(t_{n-1})\left((\mu+\mu_r)\|\nabla\mathcal{R}_{\boldsymbol{u}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2 + c_1\|\nabla\mathcal{R}_{\boldsymbol{w}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2\right)\\
&\quad + \nu(t_n)\left(2c_2\|\operatorname{div}\mathcal{R}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + 8\mu_r\|\mathcal{R}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2\right)\\
&\quad - \nu(t_{n-1})\left(2c_2\|\operatorname{div}\mathcal{R}_{\boldsymbol{w}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2 + 8\mu_r\|\mathcal{R}_{\boldsymbol{w}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2\right)\\
&\le C\nu(t_{n-1})\left(\|\nabla\mathcal{R}_{\boldsymbol{u}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\nabla\mathcal{R}_{\boldsymbol{w}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2\right)\left(\|\nabla\mathcal{R}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\nabla\mathcal{R}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2\right)\\
&\quad + C\tau\left(\|\nabla\mathcal{R}_{\boldsymbol{u}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\nabla\mathcal{R}_{\boldsymbol{w}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\nabla\mathcal{R}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\nabla\mathcal{R}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2\right)\\
&\quad + C\nu(t_n)\left(\|\mathcal{E}_1^n\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\mathcal{E}_2^n\|_{\boldsymbol{L}^2(\Omega)}^2\right)\tau
\end{aligned}\tag{4.30}$$

for all $1 \le n \le N$.

Summing (4.30) with respect to n from $n = 1$ to $n = m$ and by applying Lemma 4.2, Theorem 4.2 and $(\mathcal{R}_{\boldsymbol{u}}^0, \mathcal{R}_{\boldsymbol{w}}^0) = (\mathbf{0}, \mathbf{0})$, we have

$$\begin{aligned}
&\tau\sum_{n=0}^{m}\nu(t_n)\left(\|d_t\mathcal{R}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + \|d_t\mathcal{R}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2\right)\\
&\quad + \nu(t_m)\left((\mu+\mu_r)\|\nabla\mathcal{R}_{\boldsymbol{u}}^m\|_{\boldsymbol{L}^2(\Omega)}^2 + c_1\|\nabla\mathcal{R}_{\boldsymbol{w}}^m\|_{\boldsymbol{L}^2(\Omega)}^2\right)\\
&\le \tau\sum_{n=0}^{m-1} d_n\nu(t_n)\left((\mu+\mu_r)\|\nabla\mathcal{R}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + c_1\|\nabla\mathcal{R}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2\right) + C\tau^2,
\end{aligned}\tag{4.31}$$

where $d_n = C\tau^{-1}(\|\nabla\mathcal{R}_{\boldsymbol{u}}^{n+1}\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\nabla\mathcal{R}_{\boldsymbol{w}}^{n+1}\|_{\boldsymbol{L}^2(\Omega)}^2)$.

By applying Lemma 4.1 for the inequality (4.31), we obtain

$$\begin{aligned}
&\tau \sum_{n=0}^{m} \nu(t_n) \left(\|d_t \mathcal{R}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + \|d_t \mathcal{R}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 \right) \\
&\quad + \nu(t_m) \left((\mu + \mu_r) \|\nabla \mathcal{R}_{\boldsymbol{u}}^m\|_{\boldsymbol{L}^2(\Omega)}^2 + c_1 \|\nabla \mathcal{R}_{\boldsymbol{w}}^m\|_{\boldsymbol{L}^2(\Omega)}^2 \right) \\
&\le C\tau^2 \exp\left(\tau \sum_{n=0}^{m-1} d_n \right).
\end{aligned} \tag{4.32}$$

By applying Theorem 4.2 for the inequality (4.32), we have (4.27). The proof is complete. $\square$

Theorem 4.4. *Suppose that Hypothesis* 2.1-2.3 *hold. For all* $1 \le m \le N$*, the solution* $(\boldsymbol{u}^n, \boldsymbol{w}^n, p^n)$ *of the problem* (4.1)-(4.2) *satisfies the error estimate*

$$\tau \sum_{n=1}^{m} \nu(t_n) \|p(t_n) - p^n\|_{\boldsymbol{L}^2(\Omega)}^2 \le C\tau^2. \tag{4.33}$$

Proof. By applying the inf-sup condition, the Hölder inequality and (2.2), the Eq. (4.11), we have

$$\begin{aligned}
\beta \|\mathcal{R}_p^n\|_{\boldsymbol{L}^2(\Omega)} \le & \sup_{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)} \frac{d(\boldsymbol{v}, \mathcal{R}_p^n)}{\|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}} \\
&\le C \left(\|d_t \mathcal{R}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla \mathcal{R}_{\boldsymbol{u}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)} \|\nabla \boldsymbol{u}(t_n)\|_{\boldsymbol{L}^2(\Omega)} \right) \\
&\quad + C \|\nabla \boldsymbol{u}^{n-1}\|_{\boldsymbol{L}^2(\Omega)} \|\nabla \mathcal{R}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)} \\
&\quad + C \left(\|\nabla \mathcal{R}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla \mathcal{R}_{\boldsymbol{w}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathcal{E}_1^n\|_{\boldsymbol{L}^2(\Omega)} \right).
\end{aligned} \tag{4.34}$$

Squaring this identities (4.34) and multiplying by $\nu(t_n)\tau$, we can show

$$\begin{aligned}
\nu(t_n) \|\mathcal{R}_p^n\|_{\boldsymbol{L}^2(\Omega)}^2 \tau \le & C \left(\nu(t_n) \|d_t \mathcal{R}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 \tau + \|\nabla \mathcal{R}_{\boldsymbol{u}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2 \|\nabla \boldsymbol{u}(t_n)\|_{\boldsymbol{L}^2(\Omega)}^2 \tau \right) \\
&+ C \left(\|\nabla \boldsymbol{u}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2 \nu(t_n) \|\nabla \mathcal{R}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 \tau + \|\nabla \mathcal{R}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 \tau \right) \\
&+ C \left(\|\nabla \mathcal{R}_{\boldsymbol{w}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2 \tau + \nu(t_n) \|\mathcal{E}_1^n\|_{\boldsymbol{L}^2(\Omega)}^2 \tau \right).
\end{aligned} \tag{4.35}$$

Summing (4.35) with respect to n from $n = 1$ to $n = m$ and by applying Theorems 3.2, 4.3 and $(\mathcal{R}_{\boldsymbol{u}}^0, \mathcal{R}_{\boldsymbol{w}}^0) = (\mathbf{0}, \mathbf{0})$, we obtain

$$\begin{aligned}
\tau \sum_{n=1}^{m} \nu(t_n) \|\mathcal{R}_p^n\|_{\boldsymbol{L}^2(\Omega)}^2 \le & C \left(\tau \sum_{n=1}^{m} \nu(t_n) \|d_t \mathcal{R}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + \tau \sum_{n=1}^{m} \nu(t_n) \|\mathcal{E}_1^n\|_{\boldsymbol{L}^2(\Omega)}^2 \right) \\
&+ C \left(\tau^3 \sum_{n=0}^{m-1} \|\nabla \boldsymbol{u}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + \tau \sum_{n=1}^{m} \left(\|\nabla \mathcal{R}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\nabla \mathcal{R}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 \right) \right).
\end{aligned} \tag{4.36}$$

By applying Theorems 4.1-4.3, Lemma 4.2 and Hypothesis 2.1 for the inequality (4.36), we have (4.33). The proof is complete. $\square$

5. Regularity Results for the Time-Discrete Solutions

In this section, we establish some regularity results for the time-discrete solution $(\boldsymbol{u}^n, \boldsymbol{w}^n, p^n)$. In addition to their own theoretical significance, these regularity results are necessary for the error estimate results of the fully discrete Euler semi-implicit scheme.

We first give a priori estimates for the maximal $\boldsymbol{H}^1$ norm of $(\boldsymbol{u}^n, \boldsymbol{w}^n)$ and the maximal $\boldsymbol{L}^2$ norm of $(d_t\boldsymbol{u}^n, d_t\boldsymbol{w}^n)$.

Theorem 5.1. *Suppose that Hypothesis* 2.1-2.3 *hold. The solution* $(\boldsymbol{u}^n, \boldsymbol{w}^n, p^n)$ *of the problem* (4.1)-(4.2) *satisfies the following estimate:*

$$\sup_{1\le n\le N}\Big(\|\nabla\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}\Big) + \sup_{1\le n\le N}\Big(\|d_t\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \|d_t\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}\Big) \le C. \tag{5.1}$$

Proof. By applying the triangle inequality and $(a+b)^2 \le 2(a^2+b^2)$, Theorem 4.3, we have

$$\begin{aligned}
\|\nabla\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} &\le 2\|\nabla\boldsymbol{u}(t_n)\|^2_{\boldsymbol{L}^2(\Omega)} + 2\|\nabla(\boldsymbol{u}(t_n)-\boldsymbol{u}^n)\|^2_{\boldsymbol{L}^2(\Omega)}\\
&\le 2\|\nabla\boldsymbol{u}(t_n)\|^2_{\boldsymbol{L}^2(\Omega)} + C\nu(t_n)^{-1}\tau^2, \qquad (5.2)\\
\|\nabla\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)} &\le 2\|\nabla\boldsymbol{w}(t_n)\|^2_{\boldsymbol{L}^2(\Omega)} + 2\|\nabla(\boldsymbol{w}(t_n)-\boldsymbol{w}^n)\|^2_{\boldsymbol{L}^2(\Omega)}\\
&\le 2\|\nabla\boldsymbol{w}(t_n)\|^2_{\boldsymbol{L}^2(\Omega)} + C\nu(t_n)^{-1}\tau^2 \qquad (5.3)
\end{aligned}$$

for all $1 \le n \le N$.

Adding (5.2) and (5.3), and by applying Theorem 3.2 and the inequality $\nu(t_n)^{-1}\tau^2 \le C$, we obtain

$$\|\nabla\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)} \le C \tag{5.4}$$

for any $1 \le n \le N$. Similarly, by applying the triangle inequality and $(a+b)^2 \le 2(a^2+b^2)$, Theorem 4.3, we can show

$$\begin{aligned}
\|d_t\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} &\le 2\|d_t\boldsymbol{u}(t_n)\|^2_{\boldsymbol{L}^2(\Omega)} + 2\|d_t(\boldsymbol{u}(t_n)-\boldsymbol{u}^n)\|^2_{\boldsymbol{L}^2(\Omega)}\\
&\le 2\|d_t\boldsymbol{u}(t_n)\|^2_{\boldsymbol{L}^2(\Omega)} + C\nu(t_n)^{-1}\tau, \qquad (5.5)\\
\|d_t\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)} &\le 2\|d_t\boldsymbol{w}(t_n)\|^2_{\boldsymbol{L}^2(\Omega)} + 2\|d_t(\boldsymbol{w}(t_n)-\boldsymbol{w}^n)\|^2_{\boldsymbol{L}^2(\Omega)}\\
&\le 2\|d_t\boldsymbol{w}(t_n)\|^2_{\boldsymbol{L}^2(\Omega)} + C\nu(t_n)^{-1}\tau \qquad (5.6)
\end{aligned}$$

for all $1 \le n \le N$. Adding (5.5) and (5.6) and by applying Theorem 3.3 and $\nu(t_n)^{-1}\tau \le C$, we conclude that

$$\begin{aligned}
&\|d_t\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \|d_t\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}\\
&\le \frac{2}{\tau}\int_{t_{n-1}}^{t_n}\Big(\|\partial_t\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\partial_t\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)}\Big)ds + C\nu(t_n)^{-1}\tau\\
&\le \|\partial_t\boldsymbol{u}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|\partial_t\boldsymbol{w}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + C \le C \qquad (5.7)
\end{aligned}$$

for all $1 \le n \le N$. Combining the inequality (5.4) and (5.7), we have (5.1). The proof is complete. □

In next step, we give a priori estimates for the maximal $\boldsymbol{H}^2$ norm of $(\boldsymbol{u}^n, \boldsymbol{w}^n)$ and the maximal $\boldsymbol{H}^1$ norm of p^n.

Theorem 5.2. *Suppose that Hypothesis* 2.1-2.3 *hold. The solution* $(\boldsymbol{u}^n, \boldsymbol{w}^n, p^n)$ *of the problem* (4.1)-(4.2) *satisfies the following estimate:*

$$\sup_{1\le n\le N}\Big(\|\boldsymbol{u}^n\|^2_{\boldsymbol{H}^2(\Omega)} + \|\boldsymbol{w}^n\|^2_{\boldsymbol{H}^2(\Omega)} + \|p^n\|^2_{H^1(\Omega)}\Big) \le C. \tag{5.8}$$

Proof. According to (4.1) and (4.2), we can show $(\boldsymbol{u}^n, p, \boldsymbol{w}^n)$ satisfy the following system:

$$\begin{cases} d_t\boldsymbol{u}^n + (\boldsymbol{u}^{n-1}\cdot\nabla)\boldsymbol{u}^n - (\mu+\mu_r)\Delta\boldsymbol{u}^n + \nabla p^n = \boldsymbol{f}^n + 2\mu_r\mathbf{curl}\,\boldsymbol{w}^{n-1}, \\ \operatorname{div}\boldsymbol{u}^n = 0, \\ d_t\boldsymbol{w}^n + (\boldsymbol{u}^{n-1}\cdot\nabla)\boldsymbol{w}^n - c_1\Delta\boldsymbol{w}^n - c_2\nabla\operatorname{div}\boldsymbol{w}^n + 4\mu_r\boldsymbol{w}^n = \boldsymbol{g}^n + 2\mu_r\mathbf{curl}\,\boldsymbol{u}^n \end{cases}$$

in $\mathcal{D}'((0,T)\times\Omega)$.

By applying the Hölder inequality and the Young inequality, (2.2) and (2.8), Hypothesis 2.3, we conclude that

$$\begin{aligned}
&\|\boldsymbol{u}^n\|_{\boldsymbol{H}^2(\Omega)} + \|p^n\|_{H^1(\Omega)} \\
\le{}& C\|d_t\boldsymbol{u}^n\|_{\boldsymbol{L}^2(\Omega)} + C\|\nabla\boldsymbol{u}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}\|\nabla\boldsymbol{u}^n\|^{\frac12}_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{u}^n\|^{\frac12}_{\boldsymbol{H}^2(\Omega)} \\
&+ C\|\boldsymbol{f}^n\|_{\boldsymbol{L}^2(\Omega)} + C\|\nabla\boldsymbol{w}^{n-1}\|_{\boldsymbol{L}^2(\Omega)} \\
\le{}& C\|d_t\boldsymbol{u}^n\|_{\boldsymbol{L}^2(\Omega)} + C\|\boldsymbol{f}^n\|_{\boldsymbol{L}^2(\Omega)} + C\|\nabla\boldsymbol{w}^{n-1}\|_{\boldsymbol{L}^2(\Omega)} \\
&+ C\|\nabla\boldsymbol{u}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)}\|\nabla\boldsymbol{u}^n\|_{\boldsymbol{L}^2(\Omega)} + \frac12\|\boldsymbol{u}^n\|_{\boldsymbol{H}^2(\Omega)},
\end{aligned} \tag{5.9}$$

$$\begin{aligned}
&\|\boldsymbol{w}^n\|_{\boldsymbol{H}^2(\Omega)} + \|\operatorname{div}\boldsymbol{w}^n\|_{H^1(\Omega)} \\
\le{}& C\|d_t\boldsymbol{w}^n\|_{\boldsymbol{L}^2(\Omega)} + C\|\nabla\boldsymbol{u}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}\|\nabla\boldsymbol{w}^n\|^{\frac12}_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{w}^n\|^{\frac12}_{\boldsymbol{H}^2(\Omega)} \\
&+ C\|\boldsymbol{g}^n\|_{\boldsymbol{L}^2(\Omega)} + C\|\nabla\boldsymbol{u}^n\|_{\boldsymbol{L}^2(\Omega)} + C\|\boldsymbol{w}^n\|_{\boldsymbol{L}^2(\Omega)} \\
\le{}& C\|d_t\boldsymbol{w}^n\|_{\boldsymbol{L}^2(\Omega)} + C\|\boldsymbol{g}^n\|_{\boldsymbol{L}^2(\Omega)} + C\|\nabla\boldsymbol{u}^n\|_{\boldsymbol{L}^2(\Omega)} + C\|\boldsymbol{w}^n\|_{\boldsymbol{L}^2(\Omega)} \\
&+ C\|\nabla\boldsymbol{u}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)}\|\nabla\boldsymbol{w}^n\|_{\boldsymbol{L}^2(\Omega)} + \frac12\|\boldsymbol{w}^n\|_{\boldsymbol{H}^2(\Omega)}
\end{aligned} \tag{5.10}$$

for all $1\le n\le N$. By applying Hypothesis 2.1, we obtain

$$\|\boldsymbol{f}^n\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{g}^n\|_{\boldsymbol{L}^2(\Omega)} \le \|\boldsymbol{f}\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|\boldsymbol{g}\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} \le C \tag{5.11}$$

for any $1\le n\le N$.

Combining (5.9)-(5.10) and applying Theorem 5.1 and the inequality (5.11), we have (5.8). The proof is complete. □

In the following theorem, we will give a priori estimate for the discrete $L^2(\boldsymbol{H}^1)$ norm of $(d_t\boldsymbol{u}^n, d_t\boldsymbol{w}^n)$.

Theorem 5.3. *Suppose that Hypothesis 2.1-2.3 hold. For all $2\le m\le N$, the solution $(\boldsymbol{u}^n, \boldsymbol{w}^n, p^n)$ of the problem (4.1)-(4.2) satisfies the following estimate:*

$$\tau\sum_{n=2}^{m}\Big(\|\nabla d_t\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla d_t\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}\Big) \le C. \tag{5.12}$$

Proof. Taking increments on (4.1)-(4.2) for $n\ge 2$, we can show

$$\begin{aligned}
&(d_t^2\boldsymbol{u}^n, \boldsymbol{v}) + b(d_t\boldsymbol{u}^{n-1}, \boldsymbol{u}^n, \boldsymbol{v}) + b(\boldsymbol{u}^{n-2}, d_t\boldsymbol{u}^n, \boldsymbol{v}) \\
&\quad + (\mu+\mu_r)a(d_t\boldsymbol{u}^n, \boldsymbol{v}) - d(\boldsymbol{v}, d_tp^n) + d(d_t\boldsymbol{u}^n, q) \\
={}& (d_t\boldsymbol{f}^n, \boldsymbol{v}) + 2\mu_r(\mathbf{curl}d_t\boldsymbol{w}^{n-1}, \boldsymbol{v}),
\end{aligned} \tag{5.13}$$

$$\begin{aligned}
&(d_t^2\boldsymbol{w}^n, \boldsymbol{\psi}) + b(d_t\boldsymbol{u}^{n-1}, \boldsymbol{w}^n, \boldsymbol{\psi}) + b(\boldsymbol{u}^{n-2}, d_t\boldsymbol{w}^n, \boldsymbol{\psi}) \\
&\quad + c_1a(d_t\boldsymbol{w}^n, \boldsymbol{\psi}) + c_2(\operatorname{div}d_t\boldsymbol{w}^n, \operatorname{div}\boldsymbol{\psi}) + 4\mu_r(d_t\boldsymbol{w}^n, \boldsymbol{\psi}) \\
={}& (d_t\boldsymbol{g}^n, \boldsymbol{\psi}) + 2\mu_r(\mathbf{curl}d_t\boldsymbol{u}^n, \boldsymbol{\psi})
\end{aligned} \tag{5.14}$$

for any $(\boldsymbol{v}, q, \boldsymbol{\psi}) \in \boldsymbol{H}_0^1(\Omega)\times L_0^2(\Omega)\times\boldsymbol{H}_0^1(\Omega)$.

Let $(\boldsymbol{v},q)=2(d_t\boldsymbol{u}^n,d_tp^n)\tau$ in (5.13) and $\boldsymbol{\psi}=2d_t\boldsymbol{w}^n\tau$ in (5.14) and adding together, we get

$$\begin{aligned}
&\|d_t\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)}+\|d_t\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}+\|d_t\boldsymbol{u}^n-d_t\boldsymbol{u}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)}+\|d_t\boldsymbol{w}^n-d_t\boldsymbol{w}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)}\\
&\quad+2\tau(\mu+\mu_r)\|\nabla d_t\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)}+2\tau c_1\|\nabla d_t\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}+2\tau c_2\|\operatorname{div} d_t\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}\\
&\quad+8\mu_r\tau\|d_t\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}+2\tau b(d_t\boldsymbol{u}^{n-1},\boldsymbol{u}^n,d_t\boldsymbol{u}^n)+2\tau b(d_t\boldsymbol{u}^{n-1},\boldsymbol{w}^n,d_t\boldsymbol{w}^n)\\
=&\|d_t\boldsymbol{u}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)}+\|d_t\boldsymbol{w}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)}+2\tau(d_t\boldsymbol{f}^n,d_t\boldsymbol{u}^n)+2\tau(d_t\boldsymbol{g}^n,d_t\boldsymbol{w}^n)\\
&\quad+4\mu_r\tau(\mathbf{curl}d_t\boldsymbol{w}^{n-1},d_t\boldsymbol{u}^n)+4\mu_r\tau(\mathbf{curl}d_t\boldsymbol{u}^n,d_t\boldsymbol{w}^n).
\end{aligned}\tag{5.15}$$

By applying the Hölder inequality, the Young inequality and (2.1)-(2.2), we have

$$\begin{aligned}
&2\left|b(d_t\boldsymbol{u}^{n-1},\boldsymbol{u}^n,d_t\boldsymbol{u}^n)\right|\le\frac{\mu}{2}\|\nabla d_t\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)}+C\|\boldsymbol{u}^n\|^2_{\boldsymbol{H}^2(\Omega)}\|d_t\boldsymbol{u}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)},\\
&2\left|b(d_t\boldsymbol{u}^{n-1},\boldsymbol{w}^n,d_t\boldsymbol{w}^n)\right|\le\frac{c_1}{2}\|\nabla d_t\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}+C\|\boldsymbol{w}^n\|^2_{\boldsymbol{H}^2(\Omega)}\|d_t\boldsymbol{u}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)},\\
&2\left|(d_t\boldsymbol{f}^n,d_t\boldsymbol{u}^n)\right|\le\frac{\mu}{2}\|\nabla d_t\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)}+C\|d_t\boldsymbol{f}^n\|^2_{\boldsymbol{L}^2(\Omega)},\\
&2\left|(d_t\boldsymbol{g}^n,d_t\boldsymbol{w}^n)\right|\le\frac{c_1}{2}\|\nabla d_t\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}+C\|d_t\boldsymbol{g}^n\|^2_{\boldsymbol{L}^2(\Omega)},\\
&4\mu_r\left|(\mathbf{curl}d_t\boldsymbol{w}^{n-1},d_t\boldsymbol{u}^n)\right|\le4\mu_r\|d_t\boldsymbol{w}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)}+\mu_r\|\nabla d_t\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)},\\
&4\mu_r\left|(\mathbf{curl}d_t\boldsymbol{u}^n,d_t\boldsymbol{w}^n)\right|\le4\mu_r\|d_t\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}+\mu_r\|\nabla d_t\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)}.
\end{aligned}$$

Combining the above inequalities and (5.15), we conclude that

$$\begin{aligned}
&\|d_t\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)}+(1+4\mu_r\tau)\|d_t\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}+\mu\tau\|\nabla d_t\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)}+c_1\tau\|\nabla d_t\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}\\
\le&\|d_t\boldsymbol{u}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)}+(1+4\mu_r\tau)\|d_t\boldsymbol{w}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)}+C\tau\left(\|d_t\boldsymbol{f}^n\|^2_{\boldsymbol{L}^2(\Omega)}+\|d_t\boldsymbol{g}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right)\\
&\quad+C\tau\left(\|\boldsymbol{u}^n\|^2_{\boldsymbol{H}^2(\Omega)}+\|\boldsymbol{w}^n\|^2_{\boldsymbol{H}^2(\Omega)}\right)\left(\|d_t\boldsymbol{u}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)}+\|d_t\boldsymbol{w}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)}\right)
\end{aligned}\tag{5.16}$$

for any $2\le n\le N$. Summing (5.16) with respect to n from $n=2$ to $n=m$, we obtain

$$\begin{aligned}
&\|d_t\boldsymbol{u}^m\|^2_{\boldsymbol{L}^2(\Omega)}+\|d_t\boldsymbol{w}^m\|^2_{\boldsymbol{L}^2(\Omega)}+\tau\sum_{n=2}^{m}\left(\mu\|\nabla d_t\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)}+c_1\|\nabla d_t\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right)\\
\le&C\tau\sum_{n=1}^{m-1}\left(\|\boldsymbol{u}^{n+1}\|^2_{\boldsymbol{H}^2(\Omega)}+\|\boldsymbol{w}^{n+1}\|^2_{\boldsymbol{H}^2(\Omega)}\right)\left(\|d_t\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)}+\|d_t\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right)\\
&\quad+C\tau\sum_{n=1}^{m}\left(\|d_t\boldsymbol{f}^n\|^2_{\boldsymbol{L}^2(\Omega)}+\|d_t\boldsymbol{g}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right)+\left(\|d_t\boldsymbol{u}^1\|^2_{\boldsymbol{L}^2(\Omega)}+\|d_t\boldsymbol{w}^1\|^2_{\boldsymbol{L}^2(\Omega)}\right)
\end{aligned}\tag{5.17}$$

for any $2\le m\le N$. By using of Hypothesis 2.1, we can show

$$\begin{aligned}
&\tau\sum_{n=1}^{m}\left(\|d_t\boldsymbol{f}^n\|^2_{\boldsymbol{L}^2(\Omega)}+\|d_t\boldsymbol{g}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right)\\
\le&C\Big(\|\partial_t\boldsymbol{f}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}+\|\partial_t\boldsymbol{g}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}\Big)\le C.
\end{aligned}\tag{5.18}$$

By applying Lemma 4.1, Theorem 5.1 and (5.18) for the inequality (5.17), we have (5.12). The proof is complete. □

In next theorem, we will give a priori estimate for the maximal $\boldsymbol{H}^1$ norm of $(d_t\boldsymbol{u}^n,d_t\boldsymbol{w}^n)$ and the discrete $L^2(\boldsymbol{L}^2)$ norm of $(d_t^2\boldsymbol{u}^n,d_t^2\boldsymbol{w}^n)$.

Theorem 5.4. *Suppose that Hypothesis* 2.1-2.3 *hold. For all* $2 \le m \le N$*, the solution* $(\boldsymbol{u}^n, \boldsymbol{w}^n, p^n)$ *of the problem* (4.1)-(4.2) *satisfies the following estimate:*

$$\begin{aligned}&\sup_{2\le n\le m} \nu(t_n)\left(\|\nabla d_t \boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla d_t \boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right)\\ &\quad + \tau\sum_{n=2}^{m} \nu(t_n)\left(\|d_t^2 \boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \|d_t^2 \boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right) \le C. \end{aligned} \tag{5.19}$$

Proof. Taking $(\boldsymbol{v}, q) = 2\tau(d_t^2\boldsymbol{u}^n, 0)$ in (5.13) and $\boldsymbol{\psi} = 2\tau d_t^2 \boldsymbol{w}^n$ in (5.14) and adding together the two equations, we obtain

$$\begin{aligned}&2\tau\|d_t^2\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + 2\tau\|d_t^2\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)} + 2b(d_t\boldsymbol{u}^{n-1}, \boldsymbol{u}^n, d_t^2\boldsymbol{u}^n)\tau + 2b(\boldsymbol{u}^{n-2}, d_t\boldsymbol{u}^n, d_t^2\boldsymbol{u}^n)\tau\\ &\quad + d_t((\mu+\mu_r)\|\nabla d_t\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + c_1\|\nabla d_t \boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)})\tau + 2b(d_t\boldsymbol{u}^{n-1}, \boldsymbol{w}^n, d_t^2\boldsymbol{w}^n)\tau\\ &\quad + d_t(c_2\|\operatorname{div} d_t\boldsymbol{w}^n\|^2_{L^2(\Omega)} + 8\mu_r\|d_t\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)})\tau + 2b(\boldsymbol{u}^{n-2}, d_t\boldsymbol{w}^n, d_t^2\boldsymbol{w}^n)\\ &\le 2\tau(d_t\boldsymbol{f}^n, d_t^2\boldsymbol{u}^n) + 2\tau(d_t\boldsymbol{g}^n, d_t^2\boldsymbol{w}^n) + 4\mu_r\tau(\mathbf{curl} d_t\boldsymbol{w}^{n-1}, d_t^2\boldsymbol{u}^n) + 4\mu_r\tau(\mathbf{curl} d_t\boldsymbol{u}^n, d_t^2\boldsymbol{w}^n)\end{aligned} \tag{5.20}$$

for any $n \ge 2$.

By applying the Hölder inequality, the Young inequality and (2.2), we derive

$$\begin{aligned}2\left|b(d_t\boldsymbol{u}^{n-1}, \boldsymbol{u}^n, d_t^2\boldsymbol{u}^n)\right| &\le 2\|d_t\boldsymbol{u}^{n-1}\|_{\boldsymbol{L}^6(\Omega)}\|\nabla\boldsymbol{u}^n\|_{\boldsymbol{L}^3(\Omega)}\|d_t^2\boldsymbol{u}^n\|_{\boldsymbol{L}^2(\Omega)}\\ &\le \frac{1}{4}\|d_t^2\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + C\|\boldsymbol{u}^n\|^2_{\boldsymbol{H}^2(\Omega)}\|\nabla d_t\boldsymbol{u}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)},\\ 2\left|b(\boldsymbol{u}^{n-2}, d_t\boldsymbol{u}^n, d_t^2\boldsymbol{u}^n)\right| &\le 2\|\boldsymbol{u}^{n-2}\|_{\boldsymbol{L}^\infty(\Omega)}\|\nabla d_t\boldsymbol{u}^n\|_{\boldsymbol{L}^2(\Omega)}\|d_t^2\boldsymbol{u}^n\|_{\boldsymbol{L}^2(\Omega)}\\ &\le \frac{1}{4}\|d_t^2\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + C\|\boldsymbol{u}^{n-2}\|^2_{\boldsymbol{H}^2(\Omega)}\|\nabla d_t\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)},\\ 2\left|b(d_t\boldsymbol{u}^{n-1}, \boldsymbol{w}^n, d_t^2\boldsymbol{w}^n)\right| &\le 2\|d_t\boldsymbol{u}^{n-1}\|_{\boldsymbol{L}^6(\Omega)}\|\nabla\boldsymbol{w}^n\|_{\boldsymbol{L}^3(\Omega)}\|d_t^2\boldsymbol{w}^n\|_{\boldsymbol{L}^2(\Omega)}\\ &\le \frac{1}{4}\|d_t^2\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)} + C\|\boldsymbol{w}^n\|^2_{\boldsymbol{H}^2(\Omega)}\|\nabla d_t\boldsymbol{u}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)},\\ 2\left|b(\boldsymbol{u}^{n-2}, d_t\boldsymbol{w}^n, d_t^2\boldsymbol{w}^n)\right| &\le 2\|\boldsymbol{u}^{n-2}\|_{\boldsymbol{L}^\infty(\Omega)}\|\nabla d_t\boldsymbol{w}^n\|_{\boldsymbol{L}^2(\Omega)}\|d_t^2\boldsymbol{w}^n\|_{\boldsymbol{L}^2(\Omega)}\\ &\le \frac{1}{4}\|d_t^2\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)} + C\|\boldsymbol{u}^{n-2}\|^2_{\boldsymbol{H}^2(\Omega)}\|\nabla d_t\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)},\\ 2\left|(d_t\boldsymbol{f}^n, d_t^2\boldsymbol{u}^n)\right| &\le \frac{1}{4}\|d_t^2\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + C\|d_t\boldsymbol{f}^n\|^2_{\boldsymbol{L}^2(\Omega)},\\ 2\left|(d_t\boldsymbol{g}^n, d_t^2\boldsymbol{w}^n)\right| &\le \frac{1}{4}\|d_t^2\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)} + C\|d_t\boldsymbol{g}^n\|^2_{\boldsymbol{L}^2(\Omega)},\\ 4\mu_r\left|(\mathbf{curl} d_t\boldsymbol{w}^{n-1}, d_t^2\boldsymbol{u}^n)\right| &\le \frac{1}{4}\|d_t^2\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + C\|\nabla d_t\boldsymbol{w}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)},\\ 4\mu_r\left|(\mathbf{curl} d_t\boldsymbol{u}^n, d_t^2\boldsymbol{w}^n)\right| &\le \frac{1}{4}\|d_t^2\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)} + C\|\nabla d_t\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)}.\end{aligned}$$

Combining the above inequalities and (5.20), we conclude that

$$\begin{aligned}&\tau\|d_t^2\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \tau\|d_t^2\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}\\ &\quad + d_t\left((\mu+\mu_r)\|\nabla d_t\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + c_1\|\nabla d_t\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right)\tau\\ &\quad + d_t\left(c_2\|\operatorname{div} d_t\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)} + 8\mu_r\|d_t\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right)\tau\\ &\le C\left(\|\nabla d_t\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla d_t\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right)\tau\end{aligned}$$

$$
\begin{aligned}
&+ C\|\boldsymbol{u}^{n-2}\|^2_{\boldsymbol{H}^2(\Omega)} \left(\|\nabla d_t \boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla d_t \boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right)\tau \\
&+ C\left(\|\nabla d_t \boldsymbol{u}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla d_t \boldsymbol{w}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)}\right)\tau \\
&+ C\|\boldsymbol{u}^{n}\|^2_{\boldsymbol{H}^2(\Omega)} \left(\|\nabla d_t \boldsymbol{u}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla d_t \boldsymbol{w}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)}\right)\tau \\
&+ C\|\boldsymbol{w}^{n}\|^2_{\boldsymbol{H}^2(\Omega)} \left(\|\nabla d_t \boldsymbol{u}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla d_t \boldsymbol{w}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)}\right)\tau \\
&+ C\left(\|d_t \boldsymbol{f}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \|d_t \boldsymbol{g}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right)\tau
\end{aligned}
\tag{5.21}
$$

for any $2 \leq n \leq N$. Multiplying the inequality (5.21) by $\nu(t_n)$, and by applying $\nu(t_n) \leq \nu(t_{n-1}) + \tau$, we can show

$$
\begin{aligned}
&\nu(t_n)\left(\|d_t^2 \boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \|d_t^2 \boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right)\tau \\
&\quad + \nu(t_n)\left((\mu+\mu_r)\|\nabla d_t \boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + c_1\|\nabla d_t \boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right) \\
&\quad - \nu(t_{n-1})\left((\mu+\mu_r)\|\nabla d_t \boldsymbol{u}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)} + c_1\|\nabla d_t \boldsymbol{w}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)}\right) \\
&\quad + \nu(t_n)\left(c_2\|\operatorname{div} d_t \boldsymbol{w}^n\|^2_{L^2(\Omega)} + 8\mu_r\|d_t \boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right) \\
&\quad - \nu(t_{n-1})\left(c_2\|\operatorname{div} d_t \boldsymbol{w}^{n-1}\|^2_{L^2(\Omega)} + 8\mu_r\|d_t \boldsymbol{w}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)}\right) \\
&\leq C\left(\|d_t \boldsymbol{f}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \|d_t \boldsymbol{g}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right)\tau + C\left(1 + \|\boldsymbol{u}^{n-2}\|^2_{\boldsymbol{H}^2(\Omega)} + \|\boldsymbol{u}^{n}\|^2_{\boldsymbol{H}^2(\Omega)} + \|\boldsymbol{w}^{n}\|^2_{\boldsymbol{H}^2(\Omega)}\right) \\
&\quad \times \left(\|\nabla d_t \boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla d_t \boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla d_t \boldsymbol{u}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla d_t \boldsymbol{w}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)}\right)\tau
\end{aligned}
\tag{5.22}
$$

for any $2 \leq n \leq N$. Summing (5.22) with respect to n from $n = 2$ to $n = m$, and by applying Theorem 5.2 and $\nu(t_1) = \tau$, we can show

$$
\begin{aligned}
&\nu(t_m)\left((\mu+\mu_r)\|\nabla d_t \boldsymbol{u}^m\|^2_{\boldsymbol{L}^2(\Omega)} + c_1\|\nabla d_t \boldsymbol{w}^m\|^2_{\boldsymbol{L}^2(\Omega)}\right) \\
&\quad + \tau\sum_{n=2}^{m}\nu(t_n)\left(\|d_t^2 \boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \|d_t^2 \boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right) \\
&\leq C\tau\sum_{n=2}^{m}\left(\|\nabla d_t \boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla d_t \boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right) \\
&\quad + \nu(t_1)\left((\mu+\mu_r)\|\nabla d_t \boldsymbol{u}^1\|^2_{\boldsymbol{L}^2(\Omega)} + c_1\|\nabla d_t \boldsymbol{w}^1\|^2_{\boldsymbol{L}^2(\Omega)}\right) \\
&\quad + C\tau\sum_{n=2}^{m}\left(\|d_t \boldsymbol{f}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \|d_t \boldsymbol{g}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right) \\
&\quad + \nu(t_1)\left(c_2\|\operatorname{div} d_t \boldsymbol{w}^1\|^2_{L^2(\Omega)} + 8\mu_r\|d_t \boldsymbol{w}^1\|^2_{\boldsymbol{L}^2(\Omega)}\right) \\
&\leq C\tau\sum_{n=1}^{m}\left(\|\nabla d_t \boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla d_t \boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right) \\
&\quad + C\tau\sum_{n=1}^{m}\left(\|d_t \boldsymbol{f}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \|d_t \boldsymbol{g}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right)
\end{aligned}
$$

for any $2 \leq m \leq N$. Using the triangle inequality and Theorems 4.2 and 3.3, we get that

$$
\tau\left(\|\nabla d_t \boldsymbol{u}^1\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla d_t \boldsymbol{w}^1\|^2_{\boldsymbol{L}^2(\Omega)}\right)
$$

$$\leq C\tau^{-1}\left(\|\nabla(\boldsymbol{u}^1-\boldsymbol{u}(t_1))\|^2_{\boldsymbol{L}^2(\Omega)}+\|\nabla(\boldsymbol{w}^1-\boldsymbol{w}(t_1))\|^2_{\boldsymbol{L}^2(\Omega)}\right)$$
$$+C\int_0^T\left(\|\nabla\boldsymbol{u}_t\|^2_{\boldsymbol{L}^2(\Omega)}+\|\nabla\boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)}\right)dt\leq C. \tag{5.23}$$

By applying Theorem 5.3 and (5.18), (5.23), we have (5.19). The proof is complete. □

The following theorem gives a priori estimate for the discrete $L^2(\boldsymbol{H}^2)$ norm of $(d_t\boldsymbol{u}^n, d_t\boldsymbol{w}^n)$ and the discrete $L^2(H^1)$ norm of d_tp^n.

Theorem 5.5. *Suppose that Hypothesis* 2.1-2.3 *hold. For all* $1\leq m\leq N$*, the solution* $(\boldsymbol{u}^n, \boldsymbol{w}^n, p^n)$ *of the problem* (4.1)-(4.2) *satisfies the following estimate:*

$$\tau\sum_{n=1}^m\nu(t_n)\left\{\|d_t\boldsymbol{u}^n\|^2_{\boldsymbol{H}^2(\Omega)}+\|d_t\boldsymbol{w}^n\|^2_{\boldsymbol{H}^2(\Omega)}+\|d_tp^n\|^2_{H^1(\Omega)}\right\}\leq C. \tag{5.24}$$

Proof. According to (5.13) and (5.14), we can show $(\boldsymbol{u}^n, \boldsymbol{w}^n, p^n)$ satisfy the following system:

$$\begin{cases} d_t^2\boldsymbol{u}^n+(d_t\boldsymbol{u}^{n-1}\cdot\nabla)\boldsymbol{u}^n+(\boldsymbol{u}^{n-2}\cdot\nabla)d_t\boldsymbol{u}^n-(\mu+\mu_r)\Delta d_t\boldsymbol{u}^n\\ \quad +\nabla d_tp^n=d_t\boldsymbol{f}^n+2\mu_r\mathbf{curl}d_t\boldsymbol{w}^{n-1},\quad \operatorname{div} d_t\boldsymbol{u}^n=0,\\ d_t^2\boldsymbol{w}^n+(d_t\boldsymbol{u}^{n-1}\cdot\nabla)\boldsymbol{w}^n+(\boldsymbol{u}^{n-2}\cdot\nabla)d_t\boldsymbol{w}^n-c_1\Delta d_t\boldsymbol{w}^n\\ \quad -c_2\nabla\operatorname{div} d_t\boldsymbol{w}^n+4\mu_rd_t\boldsymbol{w}^n=d_t\boldsymbol{g}^n+2\mu_r\mathbf{curl}d_t\boldsymbol{u}^n\end{cases}$$

in $\mathcal{D}'((0,T)\times\Omega)$. By applying the Hölder inequality and the Young inequality, (2.2), Hypothesis 2.3, we conclude that

$$\begin{aligned}&\|d_t\boldsymbol{u}^n\|_{\boldsymbol{H}^2(\Omega)}+\|d_tp^n\|_{H^1(\Omega)}\\ \leq\ & C\|d_t^2\boldsymbol{u}^n\|_{\boldsymbol{L}^2(\Omega)}+C\|\boldsymbol{u}^n\|_{\boldsymbol{H}^2(\Omega)}\|\nabla d_t\boldsymbol{u}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}\\ &+C\|\boldsymbol{u}^{n-2}\|_{\boldsymbol{H}^2(\Omega)}\|\nabla d_t\boldsymbol{u}^n\|_{\boldsymbol{L}^2(\Omega)}+C\|d_t\boldsymbol{f}^n\|_{\boldsymbol{L}^2(\Omega)}+C\|\nabla d_t\boldsymbol{w}^{n-1}\|_{\boldsymbol{L}^2(\Omega)},\end{aligned} \tag{5.25}$$
$$\begin{aligned}&\|d_t\boldsymbol{w}^n\|_{\boldsymbol{H}^2(\Omega)}+\|\operatorname{div} d_t\boldsymbol{w}^n\|_{H^1(\Omega)}\\ \leq\ & C\|d_t^2\boldsymbol{w}^n\|_{\boldsymbol{L}^2(\Omega)}+C\|\boldsymbol{w}^n\|_{\boldsymbol{H}^2(\Omega)}\|\nabla d_t\boldsymbol{u}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}\\ &+C\|\boldsymbol{u}^{n-2}\|_{\boldsymbol{H}^2(\Omega)}\|\nabla d_t\boldsymbol{w}^n\|_{\boldsymbol{L}^2(\Omega)}+C\|d_t\boldsymbol{g}^n\|_{\boldsymbol{L}^2(\Omega)}+C\|\nabla d_t\boldsymbol{u}^n\|_{\boldsymbol{L}^2(\Omega)}\end{aligned} \tag{5.26}$$

for any $2\leq n\leq N$.

Squaring this identities (5.25) and (5.26), and multiplying by $\nu(t_n)$, together with Theorem 5.2, we get

$$\begin{aligned}&\nu(t_n)\|d_t\boldsymbol{u}^n\|^2_{\boldsymbol{H}^2(\Omega)}+\nu(t_n)\|d_tp^n\|^2_{H^1(\Omega)}\\ \leq\ & C\nu(t_n)\|d_t^2\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)}+C\|\nabla d_t\boldsymbol{u}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)}+C\|\nabla d_t\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)}\\ &+C\|d_t\boldsymbol{f}^n\|^2_{\boldsymbol{L}^2(\Omega)}+C\|\nabla d_t\boldsymbol{w}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)},\end{aligned} \tag{5.27}$$
$$\begin{aligned}&\nu(t_n)\|d_t\boldsymbol{w}^n\|^2_{\boldsymbol{H}^2(\Omega)}+\nu(t_n)\|\operatorname{div} d_t\boldsymbol{w}^n\|^2_{H^1(\Omega)}\\ \leq\ & C\nu(t_n)\|d_t^2\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}+C\|\nabla d_t\boldsymbol{u}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)}+C\|\nabla d_t\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}\\ &+C\|d_t\boldsymbol{g}^n\|^2_{\boldsymbol{L}^2(\Omega)}+C\|\nabla d_t\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)}\end{aligned} \tag{5.28}$$

for any $2\leq n\leq N$. Summing (5.27)-(5.28) with respect to n from $n=2$ to $n=m$, we have

$$\tau\sum_{n=2}^m\nu(t_n)\left(\|d_t\boldsymbol{u}^n\|^2_{\boldsymbol{H}^2(\Omega)}+\|d_t\boldsymbol{w}^n\|^2_{\boldsymbol{H}^2(\Omega)}+\|d_tp^n\|^2_{H^1(\Omega)}\right)$$

$$
\begin{aligned}
&\le C\tau\sum_{n=1}^{m}\left(\|d_t\boldsymbol{f}^n\|^2_{\boldsymbol{L}^2(\Omega)}+\|d_t\boldsymbol{g}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right)\\
&\quad+C\tau\sum_{n=2}^{m}\nu(t_n)\left(\|d_t^2\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)}+\|d_t^2\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right)\\
&\quad+C\tau\sum_{n=1}^{m}\left(\|\nabla d_t\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)}+\|\nabla d_t\boldsymbol{w}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right)
\end{aligned}
$$

for any $2\le m\le N$. By applying Theorems 5.3-5.4 and (5.18), (5.23), we have

$$
\tau\sum_{n=2}^{m}\nu(t_n)\left(\|d_t\boldsymbol{u}^n\|^2_{\boldsymbol{H}^2(\Omega)}+\|d_t\boldsymbol{w}^n\|^2_{\boldsymbol{H}^2(\Omega)}+\|d_tp^n\|^2_{H^1(\Omega)}\right)\le C.
$$

When $n=1$, using $\nu(t_1)=\tau$ and Theorem 5.2, the estimate (3.17), we obtain

$$
\begin{aligned}
&\tau\nu(t_1)\left(\|d_t\boldsymbol{u}^1\|^2_{\boldsymbol{H}^2(\Omega)}+\|d_t\boldsymbol{w}^1\|^2_{\boldsymbol{H}^2(\Omega)}+\|d_tp^1\|^2_{H^1(\Omega)}\right)\\
&\le 2\left(\|\boldsymbol{u}^1\|^2_{\boldsymbol{H}^2(\Omega)}+\|\boldsymbol{w}^1\|^2_{\boldsymbol{H}^2(\Omega)}+\|p^1\|^2_{H^1(\Omega)}\right)\\
&\quad+2\left(\|\boldsymbol{u}(0)\|^2_{\boldsymbol{H}^2(\Omega)}+\|\boldsymbol{w}(0)\|^2_{\boldsymbol{H}^2(\Omega)}+\|p(0)\|^2_{H^1(\Omega)}\right)\le C.
\end{aligned}
$$

Combining the above two inequalities, we can show the estimate (5.24). The proof is thus complete. $\square$

6. The Fully Discrete Euler Semi-Implicit Scheme

In this section, we study the fully discrete Euler semi-implicit mixed finite element scheme for the MNSE, which has been proposed and studied in [29]. We establish the $\boldsymbol{L}^2$-$\boldsymbol{H}^1$ error estimates for the finite element solutions of the MNSE unconditionally.

6.1. Spatial discretization

Now we introduce some notations for the fully discrete scheme. Let $\mathcal{T}_h$ be a quasi-uniform tetrahedral partition of Ω with $\overline{\Omega}=\cup_{K\in\mathcal{T}_h}\overline{K}$. The mesh size is denoted by $h=\max_{K\in\mathcal{T}_h}h_K$, where h_K is the mesh size of the tetrahedron K. Moreover, each tetrahedron K is supposed to be the image of a reference tetrahedron $\hat{K}$ under an affine map $\mathcal{F}_K$. We also defined $\mathbb{P}_n(K)$ is the space of polynomials of degree n on K, while $\mathbb{P}_n^3(K)=[\mathbb{P}_n(K)]^3$. We introduce the fnite element spaces $(\mathbb{U}_h,\mathbb{Q}_h)\subset(\boldsymbol{H}_0^1(\Omega)\times L_0^2(\Omega))$ with for the linear velocity with pressure, which satisfies the discrete inf-sup condition: there exists a constant β such that

$$
\inf_{0\ne q_h\in\mathbb{Q}_h}\sup_{0\ne\boldsymbol{v}_h\in\mathbb{U}_h}\frac{(\operatorname{div}\boldsymbol{v}_h,q_h)}{\|q_h\|_{L^2(\Omega)}\|\nabla\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)}}\ge\beta, \tag{6.1}
$$

where β is positive constants depending only on Ω. The space $\mathbb{W}_h\subset\boldsymbol{H}_0^1(\Omega)$ will be used to approximate the angular velocity. There are many finite element pairs satisfy the discrete inf-sup condition, such as $\boldsymbol{P}_2$-P_0 element, Mini-element and Taylor-Hood element (see, e.g., [13]). In this paper, we choose the following finite element space:

$$
\mathbb{U}_h=\left\{\boldsymbol{v}\in\boldsymbol{C}^0(\overline{\Omega}):\boldsymbol{v}\circ\mathcal{F}_K|_K\in[\mathbb{P}_1(\hat{K})\oplus\operatorname{span}(\hat{\mathcal{B}})]^3,\ \forall K\in\mathcal{T}_h\right\}\cap\boldsymbol{H}_0^1(\Omega),
$$

$$\mathbb{Q}_h = \left\{ q \in C^0(\overline{\Omega}) : q \circ \mathcal{F}_K|_K \in \mathbb{P}_1(\hat{K}),\ \forall K \in \mathcal{T}_h \right\} \cap L_0^2(\Omega),$$
$$\mathbb{W}_h = \left\{ \boldsymbol{v} \in \boldsymbol{C}^0(\overline{\Omega}) : \boldsymbol{v}|_K \in \mathbb{P}_1^3(K),\ \forall K \in \mathcal{T}_h \right\} \cap \boldsymbol{H}_0^1(\Omega),$$

where $\hat{\mathcal{B}} \in H_0^1(\hat{K})$ is standard bubble function and $0 \leq \hat{\mathcal{B}} \leq 1$ and $\hat{\mathcal{B}}(\hat{\lambda}) = 1$, which $\hat{\lambda}$ is the barycenter of $\hat{K}$. The subspace $\mathbb{U}_{0,h}$ of $\mathbb{U}_h$, which given by

$$\mathbb{U}_{0,h} = \left\{ \boldsymbol{v} \in \mathbb{U}_h : d(\boldsymbol{v}, q_h) = 0,\ q_h \in \mathbb{Q}_h \right\}.$$

We assume that the initial data is smooth and initialize the scheme

$$\boldsymbol{u}_h^0 = \mathcal{P}_{\mathbb{U}_{0,h}} \boldsymbol{u}_0, \quad \boldsymbol{w}_h^0 = \mathcal{P}_{\mathbb{W}_h} \boldsymbol{w}_0, \quad p_h^0 = \mathcal{P}_{\mathbb{Q}_h} p_0.$$

Denote $\mathcal{P}_{\mathbb{A}_h}$ as $\boldsymbol{L}^2$-orthogonal projection operator from $\boldsymbol{L}^2(\Omega)$ into $\mathbb{A}_h$, where $\mathbb{A}_h$ is either $\mathbb{U}_{0,h}$, $\mathbb{W}_h$. $\mathcal{P}_{\mathbb{Q}_h}$ as L^2-orthogonal projection operator from $L^2(\Omega)$ into $\mathbb{Q}_h$. Then, for every $1 \leq n \leq N$ we compute $(\boldsymbol{u}_h^n, \boldsymbol{w}_h^n, p_h^n) \in \mathbb{U}_h \times \mathbb{W}_h \times \mathbb{Q}_h$ that solves

$$\begin{aligned}
&(d_t \boldsymbol{u}_h^n, \boldsymbol{v}_h) + b_h(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \boldsymbol{v}_h) + (\mu + \mu_r) a(\boldsymbol{u}_h^n, \boldsymbol{v}_h) - d(\boldsymbol{v}_h, p_h^n) + d(\boldsymbol{u}_h^n, q_h) \\
&= (\boldsymbol{f}^n, \boldsymbol{v}_h) + 2\mu_r (\mathbf{curl}\, \boldsymbol{w}_h^{n-1}, \boldsymbol{v}_h), \qquad (6.2)\\
&(d_t \boldsymbol{w}_h^n, \boldsymbol{\psi}_h) + b_h(\boldsymbol{u}_h^{n-1}, \boldsymbol{w}_h^n, \boldsymbol{\psi}_h) + c_1 a(\boldsymbol{w}_h^n, \boldsymbol{\psi}_h) + c_2 (\operatorname{div} \boldsymbol{w}_h^n, \operatorname{div} \boldsymbol{\psi}_h) + 4\mu_r (\boldsymbol{w}_h^n, \boldsymbol{\psi}_h^n) \\
&= (\boldsymbol{g}^n, \boldsymbol{\psi}_h) + 2\mu_r (\mathbf{curl}\, \boldsymbol{u}_h^n, \boldsymbol{\psi}_h) \qquad (6.3)
\end{aligned}$$

for all $(\boldsymbol{v}_h, \boldsymbol{\psi}_h, q_h) \in \mathbb{U}_h \times \mathbb{W}_h \times \mathbb{Q}_h$, where $b_h : \left[\boldsymbol{H}_0^1(\Omega)\right]^3 \to \mathbb{R}$ denote that

$$b_h(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) + \frac{1}{2}(\operatorname{div} \boldsymbol{u}, \boldsymbol{v} \cdot \boldsymbol{w}) = \frac{1}{2}\left[b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) - b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v})\right]$$

for any $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega)$.

Remark 6.1. We use a semi-implicit treatment to the nonlinear convection term and implicit to the pressure in the above fully discrete scheme, which in turn leads to a Stokes solver. In fact, there have been quite a few existing works for the standard Navier-Stokes equations, in which the nonlinear convection term is treated fully explicitly, and the Stokes solver is decoupled into two Poisson solvers, see, e.g., [6, 8, 33, 38]. Some ideas in these interesting works may be applicable to the MNSE system, see [32, 35].

The trilinear form $b_h(\cdot, \cdot, \cdot)$ satisfies

$$b_h(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = 0, \quad b_h(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -b_h(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}) \qquad (6.4)$$

for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega)$ and for all $\boldsymbol{u} \in \boldsymbol{H}_{\mathrm{div}}^1(\Omega)$,

$$b_h(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}).$$

In addition, the trilinear form $b_h(\cdot, \cdot, \cdot)$ satisfies the following estimates:

$$\begin{aligned}
|b_h(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| &\leq C \|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \|\nabla \boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)}, &(6.5)\\
|b_h(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| &\leq C \|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{v}\|_{\boldsymbol{H}^2(\Omega)} \|\nabla \boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)}, &(6.6)\\
|b_h(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| &\leq C \|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{v}\|_{\boldsymbol{H}^2(\Omega)} \|\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)} &(6.7)
\end{aligned}$$

for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega)$.

The projection $\mathcal{P}_{\mathbb{A}_h}$ satisfies the following approximate properties (see, e.g., [1,13,16,19]):

$$\|\boldsymbol{v}-\mathcal{P}_{\mathbb{U}_{0,h}}\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}+h\|\nabla(\boldsymbol{v}-\mathcal{P}_{\mathbb{U}_{0,h}}\boldsymbol{v})\|_{\boldsymbol{L}^2(\Omega)}\le Ch^2\|\boldsymbol{v}\|_{\boldsymbol{H}^2(\Omega)}, \tag{6.8}$$

$$\|\boldsymbol{\psi}-\mathcal{P}_{\mathbb{W}_h}\boldsymbol{\psi}\|_{\boldsymbol{L}^2(\Omega)}+h\|\nabla(\boldsymbol{w}-\mathcal{P}_{\mathbb{W}_h}\boldsymbol{w})\|_{\boldsymbol{L}^2(\Omega)}\le Ch^2\|\boldsymbol{\psi}\|_{\boldsymbol{H}^2(\Omega)}, \tag{6.9}$$

$$\|q-\mathcal{P}_{\mathbb{Q}_h}q\|_{L^2(\Omega)}\le Ch\|q\|_{H^1(\Omega)}, \tag{6.10}$$

where $\boldsymbol{v},\boldsymbol{\psi}\in\boldsymbol{H}^1_{\rm div}(\Omega)\cap\boldsymbol{H}^2(\Omega)$ and $q\in L^2_0(\Omega)\cap H^1(\Omega)$.

We define the Stokes projection $(\boldsymbol{u},p)$ as the pair $(\mathcal{R}_h(\boldsymbol{u},p),\mathcal{Q}_h(\boldsymbol{u},p))\in\mathbb{U}_h\times\mathbb{Q}_h$ that solves

$$\overline{\mu}a(\mathcal{R}_h(\boldsymbol{u},p),\boldsymbol{v}_h)-d(\boldsymbol{v}_h,\mathcal{Q}_h(\boldsymbol{u},p))=\overline{\mu}a(\boldsymbol{u},\boldsymbol{v}_h)-d(\boldsymbol{v}_h,p), \tag{6.11}$$

$$d(\mathcal{R}_h(\boldsymbol{u},p),q_h)=d(\boldsymbol{u},q_h) \tag{6.12}$$

for any $(\boldsymbol{v}_h,q_h)\in\mathbb{U}_h\times\mathbb{Q}_h$ and $\overline{\mu}=\mu+\mu_r$, which has the following well-known approximation properties (see, e.g., [13,18]):

$$\begin{aligned}&\|\boldsymbol{u}-\mathcal{R}_h(\boldsymbol{u},p)\|_{\boldsymbol{L}^2(\Omega)}+h\|\nabla(\boldsymbol{u}-\mathcal{R}_h(\boldsymbol{u},p))\|_{\boldsymbol{L}^2(\Omega)}+\|p-\mathcal{Q}_h(\boldsymbol{u},p)\|_{L^2(\Omega)}\\&\le Ch^{l+1}(\|\boldsymbol{u}\|_{\boldsymbol{H}^{l+1}(\Omega)}+\|p\|_{H^l(\Omega)}),\quad l=0,1\end{aligned} \tag{6.13}$$

for $(\boldsymbol{u},p)\in[\boldsymbol{H}^{l+1}(\Omega)\cap\boldsymbol{H}^1_0(\Omega)]\times H^l(\Omega)$ with C independent of h, $\boldsymbol{u}$ and p.

The Ritz projection $\mathscr{R}_h:\mathbb{W}\to\mathbb{W}_h$ defined by

$$c_1a(\mathscr{R}_h\boldsymbol{w}-\boldsymbol{w},\boldsymbol{\psi}_h)+c_2(\operatorname{div}(\mathscr{R}_h\boldsymbol{w}-\boldsymbol{w}),\operatorname{div}\boldsymbol{\psi}_h)=0,\quad\forall\boldsymbol{\psi}_h\in\mathbb{W}_h, \tag{6.14}$$

which has the following well-known approximation properties (see, e.g., [13,37]):

$$\|\boldsymbol{w}-\mathscr{R}_h\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)}+h\|\nabla(\boldsymbol{w}-\mathscr{R}_h\boldsymbol{w})\|_{\boldsymbol{L}^2(\Omega)}\le Ch^2\|\boldsymbol{w}\|_{\boldsymbol{H}^2(\Omega)} \tag{6.15}$$

for $\boldsymbol{w}\in\boldsymbol{H}^2(\Omega)\cap\boldsymbol{H}^1_0(\Omega)$.

It can be shown easily that the above fully discrete finite element solution satisfies a discrete energy estimate.

Theorem 6.1. *Suppose that Hypothesis* 2.1 *holds. For any* $1\le m\le N$, *the solution* $(\boldsymbol{u}_h^n,\boldsymbol{w}_h^n,p_h^n)$ *of the problem* (6.2)-(6.3) *satisfies the discrete energy estimate*

$$\begin{aligned}&\|\boldsymbol{u}_h^m\|^2_{\boldsymbol{L}^2(\Omega)}+\|\boldsymbol{w}_h^m\|^2_{\boldsymbol{L}^2(\Omega)}+\tau\sum_{n=1}^m\Big(\tau\|d_t\boldsymbol{u}_h^n\|^2_{\boldsymbol{L}^2(\Omega)}+\tau\|d_t\boldsymbol{w}_h^n\|^2_{\boldsymbol{L}^2(\Omega)}\Big)\\&\quad+\tau\sum_{n=1}^m\Big(\mu\|\nabla\boldsymbol{u}_h^n\|^2_{\boldsymbol{L}^2(\Omega)}+c_1\|\nabla\boldsymbol{w}_h^n\|^2_{\boldsymbol{L}^2(\Omega)}\Big)\le C.\end{aligned} \tag{6.16}$$

Proof. Taking $(\boldsymbol{v}_h,q_h)=2\tau(\boldsymbol{u}_h^n,p_h^n)$ in (6.2) and $\boldsymbol{\psi}=2\tau\boldsymbol{w}_h^n$ in (6.3), respectively. Then the proof is almost the same as Theorem 4.1. $\square$

6.2. Error estimates

In this subsection, we will prove the $\boldsymbol{L}^2$-$\boldsymbol{H}^1$ error estimates of $(\boldsymbol{u}_h^n,\boldsymbol{w}_h^n)$ and the L^2-error estimates of p_h^n. For convenience, setting $\mathcal{E}_{\boldsymbol{u}}^n=\boldsymbol{u}^n-\boldsymbol{u}_h^n$, $\mathcal{E}_{\boldsymbol{w}}^n=\boldsymbol{w}^n-\boldsymbol{w}_h^n$ and $\mathcal{E}_p^n=p^n-p_h^n$. Our analysis relies on the error splitting argument which can be described as

$$\begin{aligned}&\bar{\boldsymbol{e}}_{\boldsymbol{u}}^n=\boldsymbol{u}^n-\mathcal{P}_{\mathbb{U}_{0,h}}\boldsymbol{u}^n, && \bar{\boldsymbol{e}}_{\boldsymbol{w}}^n=\boldsymbol{w}^n-\mathcal{P}_{\mathbb{W}_h}\boldsymbol{w}^n, && \bar{e}_p^n=p^n-\mathcal{P}_{\mathbb{Q}_h}p^n,\\&\boldsymbol{e}_{\boldsymbol{u}}^n=\mathcal{P}_{\mathbb{U}_{0,h}}\boldsymbol{u}^n-\boldsymbol{u}_h^n, && \boldsymbol{e}_{\boldsymbol{w}}^n=\mathcal{P}_{\mathbb{W}_h}\boldsymbol{w}^n-\boldsymbol{w}_h^n, && e_p^n=\mathcal{P}_{\mathbb{Q}_h}p^n-p_h^n,\\&\bar{\boldsymbol{E}}_{\boldsymbol{u}}^n=\boldsymbol{u}^n-\mathcal{R}_h(\boldsymbol{u}^n,p^n), && \bar{\boldsymbol{E}}_{\boldsymbol{w}}^n=\boldsymbol{w}^n-\mathscr{R}_h\boldsymbol{w}^n, && \bar{E}_p^n=p^n-\mathcal{Q}_h(\boldsymbol{u}^n,p^n),\\&\boldsymbol{E}_{\boldsymbol{u}}^n=\mathcal{R}_h(\boldsymbol{u}^n,p^n)-\boldsymbol{u}_h^n, && \boldsymbol{E}_{\boldsymbol{w}}^n=\mathscr{R}_h\boldsymbol{w}^n-\boldsymbol{w}_h^n, && E_p^n=\mathcal{Q}_h(\boldsymbol{u}^n,p^n)-p_h^n.\end{aligned}$$

Subtracting (6.2) and (6.3) from (4.1) and (4.2), respectively, we can get the error equations

$$
\begin{aligned}
&(d_t\mathcal{E}_{\boldsymbol{u}}^n,\boldsymbol{v}_h)+b_h(\mathcal{E}_{\boldsymbol{u}}^{n-1},\boldsymbol{u}^n,\boldsymbol{v}_h)+b_h(\boldsymbol{u}_h^{n-1},\mathcal{E}_{\boldsymbol{u}}^n,\boldsymbol{v}_h)+(\mu+\mu_r)a(\mathcal{E}_{\boldsymbol{u}}^n,\boldsymbol{v}_h)\\
&\quad -d(\boldsymbol{v}_h,\mathcal{E}_p^n)+d(\mathcal{E}_{\boldsymbol{u}}^n,q_h)-2\mu_r(\mathbf{curl}\mathcal{E}_{\boldsymbol{w}}^{n-1},\boldsymbol{v}_h)=0, \qquad (6.17)\\
&(d_t\mathcal{E}_{\boldsymbol{w}}^n,\boldsymbol{\psi}_h)+b_h(\mathcal{E}_{\boldsymbol{u}}^{n-1},\boldsymbol{w}^n,\boldsymbol{\psi}_h)+b_h(\boldsymbol{u}_h^{n-1},\mathcal{E}_{\boldsymbol{w}}^n,\boldsymbol{\psi}_h)+c_1a(\mathcal{E}_{\boldsymbol{w}}^n,\boldsymbol{\psi}_h)\\
&\quad +c_2(\operatorname{div}\mathcal{E}_{\boldsymbol{w}}^n,\operatorname{div}\boldsymbol{\psi}_h)+4\mu_r(\mathcal{E}_{\boldsymbol{w}}^n,\boldsymbol{\psi}_h)-2\mu_r(\mathbf{curl}\mathcal{E}_{\boldsymbol{u}}^n,\boldsymbol{\psi}_h)=0 \qquad (6.18)
\end{aligned}
$$

for all $(\boldsymbol{v}_h,\boldsymbol{\psi}_h,q_h)\in\mathbb{U}_h\times\mathbb{W}_h\times\mathbb{Q}_h$.

We are now in a position to state and prove the error estimates for the fully discrete finite element method.

Theorem 6.2. *Suppose that Hypothesis* 2.1-2.3 *hold. For all* $1\le m\le N$*, the solution* $(\boldsymbol{u}_h^n,\boldsymbol{w}_h^n,p_h^n)$ *of the problem* (6.2)-(6.3) *satisfies the error estimates*

$$
\begin{aligned}
&\sup_{1\le n\le m}\Big(\|\boldsymbol{u}^n-\boldsymbol{u}_h^n\|^2_{\boldsymbol{L}^2(\Omega)}+\|\boldsymbol{w}^n-\boldsymbol{w}_h^n\|^2_{\boldsymbol{L}^2(\Omega)}\Big) \qquad (6.19)\\
&\quad +\tau\sum_{n=1}^m\Big(\|\nabla(\boldsymbol{u}^n-\boldsymbol{u}_h^n)\|^2_{\boldsymbol{L}^2(\Omega)}+\|\nabla(\boldsymbol{w}^n-\boldsymbol{w}_h^n)\|^2_{\boldsymbol{L}^2(\Omega)}\Big)\le Ch^2.
\end{aligned}
$$

Proof. Taking $(\boldsymbol{v}_h,q_h)=2\tau(\boldsymbol{e}_{\boldsymbol{u}}^n,e_p^n)$ in (6.17) and $\boldsymbol{\psi}_h=2\tau\boldsymbol{e}_{\boldsymbol{w}}^n$ in (6.18) and adding together, noticing that

$$
\begin{aligned}
-d(\boldsymbol{e}_{\boldsymbol{u}}^n,\mathcal{E}_p^n)+d(\mathcal{E}_{\boldsymbol{u}}^n,e_p^n)&=-d(\boldsymbol{e}_{\boldsymbol{u}}^n,\bar{e}_p^n)-d(\boldsymbol{e}_{\boldsymbol{u}}^n,e_p^n)+d(\boldsymbol{e}_{\boldsymbol{u}}^n,e_p^n)+d(\bar{\boldsymbol{e}}_{\boldsymbol{u}}^n,e_p^n)\\
&=-d(\boldsymbol{e}_{\boldsymbol{u}}^n,\bar{e}_p^n)+d(\bar{\boldsymbol{e}}_{\boldsymbol{u}}^n,e_p^n)=-d(\boldsymbol{e}_{\boldsymbol{u}}^n,\bar{e}_p^n),
\end{aligned}
$$

applying the identities

$$2a(a-b)=|a|^2-|b|^2+|a-b|^2$$

and (6.4), we have

$$
\begin{aligned}
&\|\boldsymbol{e}_{\boldsymbol{u}}^n\|^2_{\boldsymbol{L}^2(\Omega)}+\|\boldsymbol{e}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)}-\big(\|\boldsymbol{e}_{\boldsymbol{u}}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)}+\|\boldsymbol{e}_{\boldsymbol{w}}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)}\big)+2\tau b_h\left(\mathcal{E}_{\boldsymbol{u}}^{n-1},\boldsymbol{u}^n,\boldsymbol{e}_{\boldsymbol{u}}^n\right)\\
&\quad+2\tau b_h\left(\boldsymbol{u}_h^{n-1},\bar{\boldsymbol{e}}_{\boldsymbol{u}}^n,\boldsymbol{e}_{\boldsymbol{u}}^n\right)+2\tau b_h\left(\mathcal{E}_{\boldsymbol{u}}^{n-1},\boldsymbol{w}^n,\boldsymbol{e}_{\boldsymbol{w}}^n\right)+2\tau b_h\left(\boldsymbol{u}_h^{n-1},\bar{\boldsymbol{e}}_{\boldsymbol{w}}^n,\boldsymbol{e}_{\boldsymbol{w}}^n\right)\\
&\quad+\bar{\mu}\Big(\|\nabla\mathcal{E}_{\boldsymbol{u}}^n\|^2_{\boldsymbol{L}^2(\Omega)}+\|\nabla\boldsymbol{e}_{\boldsymbol{u}}^n\|^2_{\boldsymbol{L}^2(\Omega)}\Big)\tau+c_1\Big(\|\nabla\mathcal{E}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)}+\|\nabla\boldsymbol{e}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)}\Big)\tau\\
&\quad+c_2\Big(\|\operatorname{div}\mathcal{E}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)}+\|\operatorname{div}\boldsymbol{e}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)}\Big)\tau+4\mu_r\Big(\|\mathcal{E}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)}+\|\boldsymbol{e}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)}\Big)\tau\\
&\le\bar{\mu}\|\nabla\bar{\boldsymbol{e}}_{\boldsymbol{u}}^n\|^2_{\boldsymbol{L}^2(\Omega)}\tau+c_1\|\nabla\bar{\boldsymbol{e}}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)}\tau+c_2\|\operatorname{div}\bar{\boldsymbol{e}}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)}+4\mu_r\|\bar{\boldsymbol{e}}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)}\\
&\quad+4\mu_r(\mathbf{curl}\mathcal{E}_{\boldsymbol{w}}^{n-1},\boldsymbol{e}_{\boldsymbol{u}}^n)\tau+4\mu_r(\mathbf{curl}\mathcal{E}_{\boldsymbol{u}}^n,\boldsymbol{e}_{\boldsymbol{w}}^n)\tau+2d(\boldsymbol{e}_{\boldsymbol{u}}^n,\bar{e}_p^n). \qquad (6.20)
\end{aligned}
$$

By virtue of the Hölder inequality and the Young inequality, (2.2) and (6.5)-(6.6), we can show

$$
\begin{aligned}
&2\left|b_h(\mathcal{E}_{\boldsymbol{u}}^{n-1},\boldsymbol{u}^n,\boldsymbol{e}_{\boldsymbol{u}}^n)\right|\le\frac{\mu}{3}\|\nabla\boldsymbol{e}_{\boldsymbol{u}}^n\|^2_{\boldsymbol{L}^2(\Omega)}+C\|\boldsymbol{u}^n\|^2_{\boldsymbol{H}^2(\Omega)}\|\mathcal{E}_{\boldsymbol{u}}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)},\\
&2\left|b_h(\boldsymbol{u}_h^{n-1},\bar{\boldsymbol{e}}_{\boldsymbol{u}}^n,\boldsymbol{e}_{\boldsymbol{u}}^n)\right|\le\frac{\mu}{3}\|\nabla\boldsymbol{e}_{\boldsymbol{u}}^n\|^2_{\boldsymbol{L}^2(\Omega)}+C\|\nabla\boldsymbol{u}_h^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)}\|\nabla\bar{\boldsymbol{e}}_{\boldsymbol{u}}^n\|^2_{\boldsymbol{L}^2(\Omega)},\\
&2\left|b_h(\mathcal{E}_{\boldsymbol{u}}^{n-1},\boldsymbol{w}^n,\boldsymbol{e}_{\boldsymbol{w}}^n)\right|\le\frac{c_1}{2}\|\nabla\boldsymbol{e}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)}+C\|\boldsymbol{w}^n\|^2_{\boldsymbol{H}^2(\Omega)}\|\mathcal{E}_{\boldsymbol{u}}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)},\\
&2\left|b_h(\boldsymbol{u}_h^{n-1},\bar{\boldsymbol{e}}_{\boldsymbol{w}}^n,\boldsymbol{e}_{\boldsymbol{w}}^n)\right|\le\frac{c_1}{2}\|\nabla\boldsymbol{e}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)}+C\|\nabla\boldsymbol{u}_h^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)}\|\nabla\bar{\boldsymbol{e}}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)},
\end{aligned}
$$

$$
\begin{aligned}
&4\mu_r \left|(\mathbf{curl}\mathcal{E}_{\boldsymbol{w}}^{n-1}, \boldsymbol{e}_{\boldsymbol{u}}^n)\right| \le \mu_r \|\nabla \boldsymbol{e}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + 4\mu_r \|\mathcal{E}_{\boldsymbol{w}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2,\\
&4\mu_r \left|(\mathbf{curl}\mathcal{E}_{\boldsymbol{u}}^{n}, \boldsymbol{e}_{\boldsymbol{w}}^n)\right| \le 4\mu_r \|\boldsymbol{e}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + \mu_r \|\nabla\mathcal{E}_{\boldsymbol{u}}^{n}\|_{\boldsymbol{L}^2(\Omega)}^2,\\
&2\left|d(\boldsymbol{e}_{\boldsymbol{u}}^n, \bar{e}_p^n)\right| \le \frac{\mu}{3}\|\nabla \boldsymbol{e}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + C\|\bar{e}_p^n\|_{\boldsymbol{L}^2(\Omega)}^2.
\end{aligned}
$$

Combining the above inequalities and (6.20), we derive

$$
\begin{aligned}
&\|\boldsymbol{e}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\boldsymbol{e}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 - \left(\|\boldsymbol{e}_{\boldsymbol{u}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\boldsymbol{e}_{\boldsymbol{w}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2\right)\\
&\quad + \mu\tau\|\nabla\mathcal{E}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + c_1\tau\|\nabla\mathcal{E}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2\\
&\le C\left(1 + \|\boldsymbol{u}^n\|_{\boldsymbol{H}^2(\Omega)}^2 + \|\boldsymbol{w}^n\|_{\boldsymbol{H}^2(\Omega)}^2\right)\left(\|\mathcal{E}_{\boldsymbol{u}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\mathcal{E}_{\boldsymbol{w}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2\right)\tau\\
&\quad + C\left(1 + \|\nabla \boldsymbol{u}_h^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2\right)\left(\|\nabla\bar{\boldsymbol{e}}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\nabla\bar{\boldsymbol{e}}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\bar{e}_p^n\|_{\boldsymbol{L}^2(\Omega)}^2\right)\tau
\end{aligned}
\tag{6.21}
$$

for all $1 \le n \le N$.

Summing (6.21) with respect to n from $n = 1$ to $n = m$, by applying (6.8)-(6.10), Theorems 6.1, 5.2 and $(\boldsymbol{e}_{\boldsymbol{u}}^0, \boldsymbol{e}_{\boldsymbol{w}}^0) = (\mathbf{0}, \mathbf{0})$, we have

$$
\begin{aligned}
&\|\mathcal{E}_{\boldsymbol{u}}^m\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\mathcal{E}_{\boldsymbol{w}}^m\|_{\boldsymbol{L}^2(\Omega)}^2 + \tau\sum_{n=1}^{m}\left(\mu\|\nabla\mathcal{E}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + c_1\|\nabla\mathcal{E}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2\right)\\
&\le \tau\sum_{n=0}^{m-1} d_n\left(\|\mathcal{E}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\mathcal{E}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2\right) + Ch^2,
\end{aligned}
\tag{6.22}
$$

where $d_n = C(1 + \|\boldsymbol{u}^{n+1}\|_{\boldsymbol{H}^2(\Omega)}^2 + \|\boldsymbol{w}^{n+1}\|_{\boldsymbol{H}^2(\Omega)}^2)$ and $1 \le m \le N$. By applying Lemma 4.1 to (6.22), we obtain

$$
\begin{aligned}
&\|\mathcal{E}_{\boldsymbol{u}}^m\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\mathcal{E}_{\boldsymbol{w}}^m\|_{\boldsymbol{L}^2(\Omega)}^2 + \tau\sum_{n=1}^{m}\left(\mu\|\nabla\mathcal{E}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + c_1\|\nabla\mathcal{E}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2\right)\\
&\le Ch^2\exp\left(\tau\sum_{n=0}^{m-1} d_n\right).
\end{aligned}
\tag{6.23}
$$

Combining (6.23) and Theorem 5.2, we have (6.19). The proof is complete. □

Remark 6.2. Here we only give a proof of a first order convergence in space in the L^2 norm. We have not got the optimal error estimate of the L^2 norm in space because of our technical reasons, where we establish the error estimate of L^2 norm and H^1 norm in a unified way. It is possible to get a second order convergence in the L^2 norm in space to use other techniques, which is not the scope of this paper and we will address in a future work.

Theorem 6.3. *Suppose that Hypothesis* 2.1-2.3 *hold. For all* $1 \le m \le N$, *the solution* $(\boldsymbol{u}_h^n, \boldsymbol{w}_h^n, p_h^n)$ *of the problem* (6.2)-(6.3) *satisfies the error estimates*

$$
\begin{aligned}
&\sup_{1\le n\le m}\left(\nu(t_n)\|\nabla(\boldsymbol{u}^n - \boldsymbol{u}_h^n)\|_{\boldsymbol{L}^2(\Omega)}^2 + \nu(t_n)\|\nabla(\boldsymbol{w}^n - \boldsymbol{w}_h^n)\|_{\boldsymbol{L}^2(\Omega)}^2\right)\\
&\quad + \tau\sum_{n=1}^{m}\nu(t_n)\left(\|d_t(\boldsymbol{u}^n - \boldsymbol{u}_h^n)\|_{\boldsymbol{L}^2(\Omega)}^2 + \|d_t(\boldsymbol{w}^n - \boldsymbol{w}_h^n)\|_{\boldsymbol{L}^2(\Omega)}^2\right) \le Ch^2.
\end{aligned}
\tag{6.24}
$$

Proof. By applying (6.11)-(6.12) and (6.14), we can rewrite the system (6.17)-(6.18) that

$$\begin{aligned}
&(d_t\mathcal{E}_{\boldsymbol{u}}^n, \boldsymbol{v}_h) + b_h(\mathcal{E}_{\boldsymbol{u}}^{n-1}, \boldsymbol{u}^n, \boldsymbol{v}_h) + b_h(\boldsymbol{u}_h^{n-1}, \mathcal{E}_{\boldsymbol{u}}^n, \boldsymbol{v}_h) + \bar{\mu} a(\boldsymbol{E}_{\boldsymbol{u}}^n, \boldsymbol{v}_h) \\
&\quad - d(\boldsymbol{v}_h, E_p^n) + d(\boldsymbol{E}_{\boldsymbol{u}}^n, q_h) - 2\mu_r(\mathbf{curl}\mathcal{E}_{\boldsymbol{w}}^{n-1}, \boldsymbol{v}_h) = 0, \qquad (6.25)\\
&(d_t\mathcal{E}_{\boldsymbol{w}}^n, \boldsymbol{\psi}_h) + b_h(\mathcal{E}_{\boldsymbol{u}}^{n-1}, \boldsymbol{w}^n, \boldsymbol{\psi}_h) + b_h(\boldsymbol{u}_h^{n-1}, \mathcal{E}_{\boldsymbol{w}}^n, \boldsymbol{\psi}_h) + c_1 a(\boldsymbol{E}_{\boldsymbol{w}}^n, \boldsymbol{\psi}_h) \\
&\quad + c_2(\operatorname{div} \boldsymbol{E}_{\boldsymbol{w}}^n, \operatorname{div} \boldsymbol{\psi}_h) + 4\mu_r(\mathcal{E}_{\boldsymbol{w}}^n, \boldsymbol{\psi}_h) - 2\mu_r(\mathbf{curl}\mathcal{E}_{\boldsymbol{u}}^n \boldsymbol{\psi}_h) = 0 \qquad (6.26)
\end{aligned}$$

for any $(\boldsymbol{v}_h, \boldsymbol{\psi}_h, q_h) \in \mathbb{U}_h \times \mathbb{W}_h \times \mathbb{Q}_h$.

Taking $(\boldsymbol{v}_h, q_h) = (2d_t\boldsymbol{E}_{\boldsymbol{u}}^n\tau, 0)$ in (6.25) and $\boldsymbol{\psi}_h = 2d_t\boldsymbol{E}_{\boldsymbol{w}}^n\tau$ in (6.26) and adding together, noticing $d(d_t\boldsymbol{E}_{\boldsymbol{u}}^n, E_p^n) = 0$, and applying the identity

$$2a(a-b) = |a|^2 - |b|^2 + |a-b|^2,$$

we can show

$$\begin{aligned}
&2\tau\|d_t\boldsymbol{E}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + 2\tau\|d_t\boldsymbol{E}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + d_t(\bar{\mu}\|\nabla\boldsymbol{E}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 \\
&\quad + c_1\|\nabla\boldsymbol{E}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + c_2\|\operatorname{div} \boldsymbol{E}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2)\tau \\
&\quad + 2b_h(\mathcal{E}_{\boldsymbol{u}}^{n-1}, \boldsymbol{u}^n, d_t\boldsymbol{E}_{\boldsymbol{u}}^n)\tau + 2b_h(\boldsymbol{u}_h^{n-1}, \mathcal{E}_{\boldsymbol{u}}^n, d_t\boldsymbol{E}_{\boldsymbol{u}}^n)\tau \\
&\quad - 4\mu_r(\mathbf{curl}\mathcal{E}_{\boldsymbol{w}}^{n-1}, d_t\boldsymbol{E}_{\boldsymbol{u}}^n)\tau \\
&\quad + 2b_h(\mathcal{E}_{\boldsymbol{u}}^{n-1}, \boldsymbol{w}^n, d_t\boldsymbol{E}_{\boldsymbol{w}}^n)\tau + 2b_h(\boldsymbol{u}_h^{n-1}, \mathcal{E}_{\boldsymbol{w}}^n, d_t\boldsymbol{E}_{\boldsymbol{w}}^n)\tau \\
&\quad - 4\mu_r(\mathbf{curl}\mathcal{E}_{\boldsymbol{u}}^n, d_t\boldsymbol{E}_{\boldsymbol{w}}^n)\tau + 8\mu_r(\mathcal{E}_{\boldsymbol{w}}^n, d_t\boldsymbol{E}_{\boldsymbol{w}}^n)\tau \\
&\le -2(d_t\bar{\boldsymbol{E}}_{\boldsymbol{u}}^n, d_t\boldsymbol{E}_{\boldsymbol{u}}^n)\tau - 2(d_t\bar{\boldsymbol{E}}_{\boldsymbol{w}}^n, d_t\boldsymbol{E}_{\boldsymbol{w}}^n)\tau. \qquad (6.27)
\end{aligned}$$

By applying the inverse inequality, the Young inequality and (6.5), (6.7), we obtain

$$\begin{aligned}
&2\left|b_h(\mathcal{E}_{\boldsymbol{u}}^{n-1}, \boldsymbol{u}^n, d_t\boldsymbol{E}_{\boldsymbol{u}}^n) + b_h(\boldsymbol{u}_h^{n-1}, \mathcal{E}_{\boldsymbol{u}}^n, d_t\boldsymbol{E}_{\boldsymbol{u}}^n)\right| \\
\le\ & \frac{1}{3}\|d_t\boldsymbol{E}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + C\|\boldsymbol{u}^n\|_{\boldsymbol{H}^2(\Omega)}^2\|\nabla\mathcal{E}_{\boldsymbol{u}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2 \\
&\quad + C\|\boldsymbol{u}^{n-1}\|_{\boldsymbol{H}^2(\Omega)}^2\|\nabla\mathcal{E}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + Ch^{-1}\|\nabla\mathcal{E}_{\boldsymbol{u}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2\|\nabla\mathcal{E}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2, \\
&2\left|b_h(\mathcal{E}_{\boldsymbol{u}}^{n-1}, \boldsymbol{w}^n, d_t\boldsymbol{E}_{\boldsymbol{w}}^n) + b_h(\boldsymbol{u}_h^{n-1}, \mathcal{E}_{\boldsymbol{w}}^n, d_t\boldsymbol{E}_{\boldsymbol{w}}^n)\right| \\
\le\ & \frac{1}{4}\|d_t\boldsymbol{E}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + C\|\boldsymbol{w}^n\|_{\boldsymbol{H}^2(\Omega)}^2\|\nabla\mathcal{E}_{\boldsymbol{u}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2 \\
&\quad + C\|\boldsymbol{u}^{n-1}\|_{\boldsymbol{H}^2(\Omega)}^2\|\nabla\mathcal{E}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + Ch^{-1}\|\nabla\mathcal{E}_{\boldsymbol{u}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2\|\nabla\mathcal{E}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2.
\end{aligned}$$

Similarly, by applying the Young ineuqality and (2.2), we have

$$\begin{aligned}
&4\mu_r\left|(\mathbf{curl}\mathcal{E}_{\boldsymbol{w}}^{n-1}, d_t\boldsymbol{E}_{\boldsymbol{u}}^n)\right| \le \frac{1}{3}\|d_t\boldsymbol{E}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + C\|\nabla\mathcal{E}_{\boldsymbol{w}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2, \\
&4\mu_r\left|(\mathbf{curl}\mathcal{E}_{\boldsymbol{u}}^n, d_t\boldsymbol{E}_{\boldsymbol{w}}^n)\right| \le \frac{1}{4}\|d_t\boldsymbol{E}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + C\|\nabla\mathcal{E}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2, \\
&8\mu_r\left|(\mathcal{E}_{\boldsymbol{w}}^n, d_t\boldsymbol{E}_{\boldsymbol{w}}^n)\right| \le \frac{1}{4}\|d_t\boldsymbol{E}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + C\|\nabla\mathcal{E}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2, \\
&2\left|(d_t\bar{\boldsymbol{E}}_{\boldsymbol{u}}^n, d_t\boldsymbol{E}_{\boldsymbol{u}}^n)\right| \le \frac{1}{3}\|d_t\boldsymbol{E}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + C\|d_t\bar{\boldsymbol{E}}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2, \\
&2\left|(d_t\bar{\boldsymbol{E}}_{\boldsymbol{w}}^n, d_t\boldsymbol{E}_{\boldsymbol{w}}^n)\right| \le \frac{1}{4}\|d_t\boldsymbol{E}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + C\|d_t\bar{\boldsymbol{E}}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2.
\end{aligned}$$

Combining the above inequalities and (6.27), multiplying the result by $\nu(t_n)$, by applying $\nu(t_n) \le \nu(t_{n-1}) + \tau$ and Theorem 5.2, we have

$$\begin{aligned}
&\nu(t_n)\left(\|d_t\boldsymbol{E}_{\boldsymbol{u}}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \|d_t\boldsymbol{E}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right)\tau \\
&\quad + d_t(\nu(t_n)\left(\bar{\mu}\|\nabla\boldsymbol{E}_{\boldsymbol{u}}^n\|^2_{\boldsymbol{L}^2(\Omega)} + c_1\|\nabla\boldsymbol{E}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)} + c_2\|\operatorname{div}\boldsymbol{E}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)})\right)\tau \\
&\le C\left(\|\nabla\bar{\boldsymbol{E}}_{\boldsymbol{u}}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla\bar{\boldsymbol{E}}_{\boldsymbol{w}}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)}\right)\tau + C\left(\|\nabla\mathcal{E}_{\boldsymbol{u}}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla\mathcal{E}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right)\tau \\
&\quad + C\left(\|\nabla\mathcal{E}_{\boldsymbol{u}}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla\mathcal{E}_{\boldsymbol{w}}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)}\right)\tau + C\nu(t_n)\left(\|d_t\bar{\boldsymbol{E}}_{\boldsymbol{u}}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \|d_t\bar{\boldsymbol{E}}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right)\tau \\
&\quad + Ch^{-1}\nu(t_{n-1})\left(\|\nabla\mathcal{E}_{\boldsymbol{u}}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla\mathcal{E}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right)\left(\bar{\mu}\|\nabla\boldsymbol{E}_{\boldsymbol{u}}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)} + c_1\|\nabla\boldsymbol{E}_{\boldsymbol{w}}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)}\right)\tau \\
&\quad + Ch^{-1}\left(\|\nabla\mathcal{E}_{\boldsymbol{u}}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla\mathcal{E}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)}\right)\left(\bar{\mu}\|\nabla\bar{\boldsymbol{E}}_{\boldsymbol{u}}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)} + c_1\|\nabla\bar{\boldsymbol{E}}_{\boldsymbol{w}}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)}\right)\tau. \quad (6.28)
\end{aligned}$$

By applying (6.13) and (6.15), we can show

$$\begin{aligned}
&\|\nabla\bar{\boldsymbol{E}}_{\boldsymbol{u}}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla\bar{\boldsymbol{E}}_{\boldsymbol{w}}^{n-1}\|^2_{\boldsymbol{L}^2(\Omega)} \\
&\le Ch^2\left(\|\boldsymbol{u}^{n-1}\|^2_{\boldsymbol{H}^2(\Omega)} + \|\boldsymbol{w}^{n-1}\|^2_{\boldsymbol{H}^2(\Omega)} + \|p^{n-1}\|^2_{\boldsymbol{H}^1(\Omega)}\right), \quad (6.29) \\
&\|d_t\bar{\boldsymbol{E}}_{\boldsymbol{u}}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \|d_t\bar{\boldsymbol{E}}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)} \\
&\le Ch^4\left(\|d_t\boldsymbol{u}^n\|^2_{\boldsymbol{H}^2(\Omega)} + \|d_t\boldsymbol{w}^n\|^2_{\boldsymbol{H}^2(\Omega)} + \|d_tp^n\|^2_{\boldsymbol{H}^1(\Omega)}\right) \quad (6.30)
\end{aligned}$$

for any $1 \le n \le N$.

Summing (6.28) with respect to n from $n = 1$ to $n = m$, and by applying (6.29)-(6.30), Theorems 5.2, 5.5, 6.2 and $\nu(t_0) = 0$, we have

$$\begin{aligned}
&\nu(t_m)\left(\bar{\mu}\|\nabla\boldsymbol{E}_{\boldsymbol{u}}^m\|^2_{\boldsymbol{L}^2(\Omega)} + c_1\|\nabla\boldsymbol{E}_{\boldsymbol{w}}^m\|^2_{\boldsymbol{L}^2(\Omega)}\right) \\
&\quad + \tau\sum_{n=1}^{m}\nu(t_n)\Big(\|d_t\boldsymbol{E}_{\boldsymbol{u}}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \|d_t\boldsymbol{E}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)}\Big) \\
&\le \tau\sum_{n=0}^{m-1}d_n\nu(t_n)\Big(\bar{\mu}\|\nabla\boldsymbol{E}_{\boldsymbol{u}}^n\|^2_{\boldsymbol{L}^2(\Omega)} + c_1\|\nabla\boldsymbol{E}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)}\Big) + Ch^2, \quad (6.31)
\end{aligned}$$

where $d_n = Ch^{-1}(\|\nabla\mathcal{E}_{\boldsymbol{u}}^{n+1}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla\mathcal{E}_{\boldsymbol{w}}^{n+1}\|^2_{\boldsymbol{L}^2(\Omega)})$ and $1 \le m \le N$.

By applying Lemma 4.1 and Theorem 6.2 to (6.31), we conclude that

$$\begin{aligned}
&\nu(t_m)\Big(\bar{\mu}\|\nabla\boldsymbol{E}_{\boldsymbol{u}}^m\|^2_{\boldsymbol{L}^2(\Omega)} + c_1\|\nabla\boldsymbol{E}_{\boldsymbol{w}}^m\|^2_{\boldsymbol{L}^2(\Omega)}\Big) \\
&\quad + \tau\sum_{n=0}^{m}\nu(t_n)\Big(\|d_t\boldsymbol{E}_{\boldsymbol{u}}^n\|^2_{\boldsymbol{L}^2(\Omega)} + \|d_t\boldsymbol{E}_{\boldsymbol{w}}^n\|^2_{\boldsymbol{L}^2(\Omega)}\Big) \\
&\le Ch^2\exp\left(\tau\sum_{n=0}^{m-1}d_n\right) \le Ch^2. \quad (6.32)
\end{aligned}$$

Combining (6.29)-(6.30) and (6.32), we can show (6.24). The proof is complete. □

Theorem 6.4. *Suppose that Hypothesis 2.1-2.3 hold. For all $1 \le m \le N$, the solution ($\boldsymbol{u}_h^n$, $\boldsymbol{w}_h^n, p_h^n$) of the problem (6.2)-(6.3) satisfies the error estimates*

$$\tau\sum_{n=1}^{m}\nu(t_n)\|p^n - p_h^n\|^2_{\boldsymbol{L}^2(\Omega)} \le Ch^2. \quad (6.33)$$

Proof. By applying the discrete inf-sup condition (6.1), the Hölder inequality and (2.2), Eq. (6.17), we have

$$\begin{aligned}\beta\|E_p^n\|_{L^2(\Omega)} \leq\, & C\|d_t\mathcal{E}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)} + C\|\nabla\mathcal{E}_{\boldsymbol{u}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}\|\nabla\boldsymbol{u}^n\|_{\boldsymbol{L}^2(\Omega)}\\ & + C\|\nabla\mathcal{E}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)} + C\|\nabla\mathcal{E}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}\\ & + C\left(\|\nabla\boldsymbol{u}^{n-1}\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla\mathcal{E}_{\boldsymbol{u}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}\right)\|\nabla\mathcal{E}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)} + C\|\bar{E}_p^n\|_{\boldsymbol{L}^2(\Omega)}. \end{aligned} \tag{6.34}$$

Squaring this identities (6.34) and multiplying the result by $\nu(t_n)\tau$, by applying Theorem 5.1, we obtain

$$\begin{aligned}\nu(t_n)\|E_p^n\|_{L^2(\Omega)}^2\tau \leq\, & C\nu(t_n)\|d_t\mathcal{E}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2\tau\\ & + C\left(\|\nabla\mathcal{E}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\nabla\mathcal{E}_{\boldsymbol{w}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\nabla\mathcal{E}_{\boldsymbol{u}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2\right)\tau\\ & + C\nu(t_n)\|\nabla\mathcal{E}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2\|\nabla\mathcal{E}_{\boldsymbol{u}}^{n-1}\|_{\boldsymbol{L}^2(\Omega)}^2\tau + C\|\bar{E}_p^n\|_{\boldsymbol{L}^2(\Omega)}^2\tau. \end{aligned} \tag{6.35}$$

Summing (6.35) with respect to n from $n = 1$ to $n = m$ and by applying Theorems 6.2-6.3 and (6.13), we can show

$$\tau\sum_{n=1}^m \nu(t_n)\|E_p^n\|_{L^2(\Omega)}^2 \leq C\tau\sum_{n=1}^m \nu(t_n)\|d_t\mathcal{E}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + Ch^2\tau\sum_{n=0}^{m-1}\|\nabla\mathcal{E}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}^2 + Ch^2. \tag{6.36}$$

A repeated application of Theorems 6.2-6.3 to (6.36), which implies that

$$\tau\sum_{n=1}^m \nu(t_n)\|E_p^n\|_{L^2(\Omega)}^2 \leq Ch^2 + Ch^2\|\nabla\mathcal{E}_{\boldsymbol{u}}^0\|_{\boldsymbol{L}^2(\Omega)}^2. \tag{6.37}$$

Combining (6.8), (6.13) and (6.37), Hypothesis 2.1, we have (6.32). The proof is complete. □

Finally, combining the Theorems 4.2-4.4 and Theorems 6.2-6.4, we can get the final unconditional error estimates.

Theorem 6.5. *Suppose that Hypothesis* 2.1-2.3 *hold. For all* $1 \leq m \leq N$, *the solution* $(\boldsymbol{u}_h^n, \boldsymbol{w}_h^n, p_h^n)$ *of the problem* (6.2)-(6.3) *satisfies the error estimates*

$$\begin{aligned}&\sup_{1\leq n\leq m}\left(\|\boldsymbol{u}(t_n) - \boldsymbol{u}_h^n\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\boldsymbol{w}(t_n) - \boldsymbol{w}_h^n\|_{\boldsymbol{L}^2(\Omega)}^2\right)\\ &\quad + \tau\sum_{n=1}^m\left(\|\nabla(\boldsymbol{u}(t_n) - \boldsymbol{u}_h^n)\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\nabla(\boldsymbol{w}(t_n) - \boldsymbol{w}_h^n)\|_{\boldsymbol{L}^2(\Omega)}^2\right) \leq C(\tau^2 + h^2),\\ &\sup_{1\leq n\leq m}\left(\nu(t_n)\|\nabla(\boldsymbol{u}(t_n) - \boldsymbol{u}_h^n)\|_{\boldsymbol{L}^2(\Omega)}^2 + \nu(t_n)\|\nabla(\boldsymbol{w}(t_n) - \boldsymbol{w}_h^n)\|_{\boldsymbol{L}^2(\Omega)}^2\right)\\ &\quad + \tau\sum_{n=1}^m\nu(t_n)\left(\|d_t(\boldsymbol{u}(t_n) - \boldsymbol{u}_h^n)\|_{\boldsymbol{L}^2(\Omega)}^2 + \|d_t(\boldsymbol{w}(t_n) - \boldsymbol{w}_h^n)\|_{\boldsymbol{L}^2(\Omega)}^2\right) \leq C(\tau^2 + h^2),\\ &\tau\sum_{n=1}^m\nu(t_n)\|p(t_n) - p_h^n\|_{L^2(\Omega)}^2 \leq C(\tau^2 + h^2).\end{aligned}$$

7. Numerical Experiments

In this section, we present a series of numerical experiments to verify the convergence results of the scheme. Our test is based on the adaptive finite element package "parallel hierarchical grid" (PHG) ([41, 42]), and the computations are carried out on the cluster LSSC-IV of the State Key Laboratory on Scientific and Engineering Computing, Chinese Academic of Sciences. In all examples, the domain under consideration is $\Omega = [0,1]^3$ and the mesh is obtained by a uniform tetrahedral partition. The first two examples are used to verify the optimal error estimates of the fully discrete scheme. The last one is to simulate the three dimensional lid-driven cavity flow. Due to the lack of Mini element in PHG package, we employ the lowest Taylor-Hood element, i.e., the continuous P_2 finite element for discretizing the linear velocity $\boldsymbol{u}$ and the continuous P_1 finite element for discretizing the pressure p, and the continuous P_2 finite element for discretizing the angular velocity $\boldsymbol{w}$.

Example 7.1. This example is to test the time discretization error for the Euler semi-implicit scheme. We set $\mu = \mu_r = 1.0$, $c_1 = 2.0$, $c_2 = 1.0$, f and g are chosen so that the exact solution is

$$\begin{aligned}
&\boldsymbol{u}(x,y,z,t) = (z\cos(t), x\exp(-t), yt)\,,\\
&p(x,y,z,t) = 0,\\
&\boldsymbol{w}(x,y,z,t) = (t, \cos(t), \exp(-t))\,.
\end{aligned}$$

It can be seen that the exact solution $\boldsymbol{u}$ is linear in space and $p, \boldsymbol{w}$ are constant in space, so the time discretization error plays a dominant role. In this example, we compute the errors at each time step and write down the errors at the last moment. The terminal time is $T = 1.0$ and the mesh size is $h = \frac{\sqrt{3}}{2}$. Table 7.1 shows that the results for the linear velocity, pressure, and angular velocity, we can see that all the errors are first-order.

Table 7.1: Errors and convergence rates for Example 7.1.

Δt	$\lVert \boldsymbol{u}(T) - \boldsymbol{u}_h^N \rVert_{\boldsymbol{H}^1}$	order	$\lVert \boldsymbol{w}(T) - \boldsymbol{w}_h^N \rVert_{\boldsymbol{H}^1}$	order
1/40	1.01e-04	-	4.75e-04	-
1/80	5.05e-05	0.99	2.36e-04	1.01
1/160	2.59e-05	0.96	1.18e-04	1.00
Δt	$\lVert p(T) - p_h^N \rVert_{L^2}$	order		
1/40	1.98e-03	-		
1/80	9.80e-04	1.01		
1/160	4.86e-04	1.01		

Example 7.2. This example is to test the spatial discretization error for the scheme. The parameters are the same to Example 7.1, f and g are chosen so that the exact solution is

$$\begin{aligned}
\boldsymbol{u}(x,y,z,t) &= (\sin(2\pi x + t)\sin(2\pi y + t), \cos(2\pi x + t)\cos(2\pi y + t), 0)\,,\\
p(x,y,z,t) &= \sin(\pi x) - \sin(\pi y),\\
\boldsymbol{w}(x,y,z,t) &= (0, 0, -4\pi\sin(2\pi x + t)\cos(2\pi y + t))\,.
\end{aligned}$$

The initial time step is $\Delta t_0 = \frac{1}{160}$ and the initial mesh size is $h_0 = \frac{\sqrt{3}}{4}$, the terminal time is also $T = 1.0$. Here we set $\Delta t = Ch^2$ to obtain the convergence rate. Table 7.2 shows that the numerical results, we can see that all the errors are second-order.

Table 7.2: Errors and convergence rates for Example 7.2.

$(\Delta t, h)$	$\left\|\boldsymbol{u}(T)-\boldsymbol{u}_h^N\right\|_{\boldsymbol{H}^1}$	order	$\left\|\boldsymbol{w}(T)-\boldsymbol{w}_h^N\right\|_{\boldsymbol{H}^1}$	order
$(\Delta t_0, h_0)$	2.75	-	21.04	-
$(\Delta t_0/4, h_0/2)$	0.74	1.89	6.51	1.69
$(\Delta t_0/16, h_0/4)$	0.19	1.96	1.68	1.96
$(\Delta t, h)$	$\left\|p(T)-p_h^N\right\|_{L^2}$	order		
$(\Delta t_0, h_0)$	1.09	-		
$(\Delta t_0/4, h_0/2)$	0.17	2.65		
$(\Delta t_0/16, h_0/4)$	0.04	2.04		

Tables 7.1 and 7.2 show that the fully discretization solution converges with the optimal rates, i.e.,

$$\begin{aligned}&\left\|\boldsymbol{u}(T)-\boldsymbol{u}_h^N\right\|_{\boldsymbol{H}^1} \approx \mathcal{O}(\Delta t+h^2),\\&\left\|\boldsymbol{w}(T)-\boldsymbol{w}_h^N\right\|_{\boldsymbol{H}^1} \approx \mathcal{O}(\Delta t+h^2),\\&\left\|p(T)-p_h^N\right\|_{L^2} \approx \mathcal{O}(\Delta t+h^2).\end{aligned}$$

Example 7.3. This example computes the benchmark problem of lid-driven cavity flow. We set the physics parameters $c_1 = 2.0$, $c_2 = 1.0$ and the source terms $f = (0,0,0)$ and $g = (50\cos(\pi z), 50\sin(\pi y), 100\exp(x))$. In order to show the effect of the kinematic viscosity, we take $\overline{\mu}$ as $\frac{1}{100}, \frac{1}{400}, \frac{1}{1000}$ in the computation, and we set $\overline{\mu} = 2\mu_r$. The initial values are given by

$$\begin{aligned}&\boldsymbol{u}_0 = \begin{cases}(0,0,0), & \text{if } 0 \le z < 1,\\ (1,0,0), & \text{if } z = 1,\end{cases}\\&\boldsymbol{w}_0 = (0,0,0).\end{aligned}$$

Together with the boundary conditions are set by: $\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{u}_0$ and $\boldsymbol{w}|_{\partial\Omega} = \boldsymbol{w}_0$. In this example, we fix a tetrahedral mesh with 196608 elements, and the time step is $\Delta t = 0.01$, the terminal time T is chosen such that the discretization solution satisfies

$$\left\|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\right\|_{\boldsymbol{L}^2} + \left\|\boldsymbol{w}_h^n - \boldsymbol{w}_h^{n-1}\right\|_{\boldsymbol{L}^2} \le 10^{-8},$$

that is, the variables reach almost steady state. We find that the kinematic viscosity $\overline{\mu}$ greatly affects the terminal time T. When $\overline{\mu} = \frac{1}{100}$, $T = 7.65$. When $\overline{\mu} = \frac{1}{400}$, $T = 18.34$. When

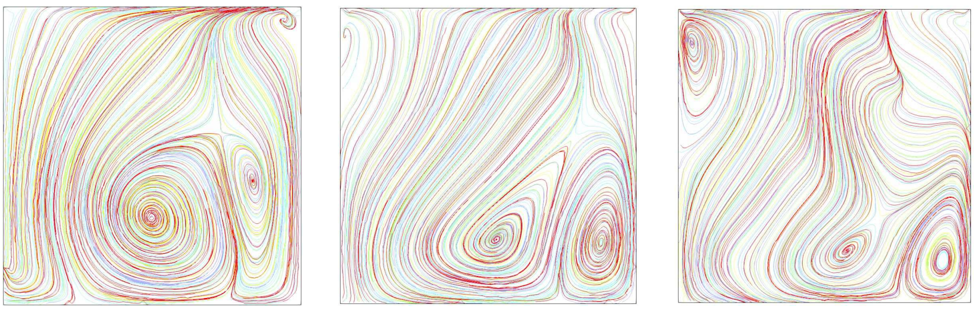

Fig. 7.1. Streamlines of $\boldsymbol{u}_h$ at $x = 0.5$ when $\overline{\mu} = 1/100, 1/400, 1/1000$ (from left to right).

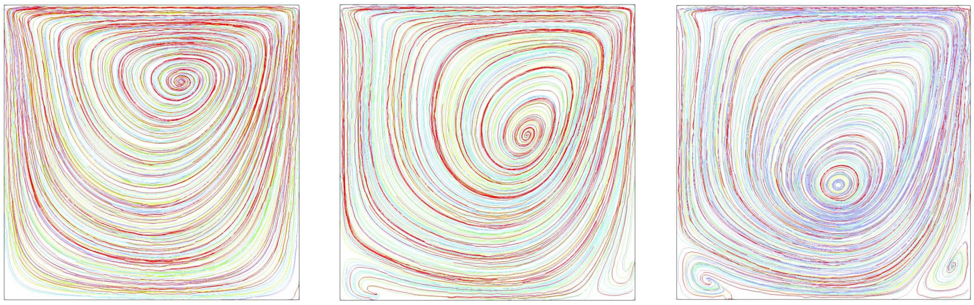

Fig. 7.2. Streamlines of $\boldsymbol{u}_h$ at $y = 0.5$ when $\overline{\mu} = 1/100, 1/400, 1/1000$ (from left to right).

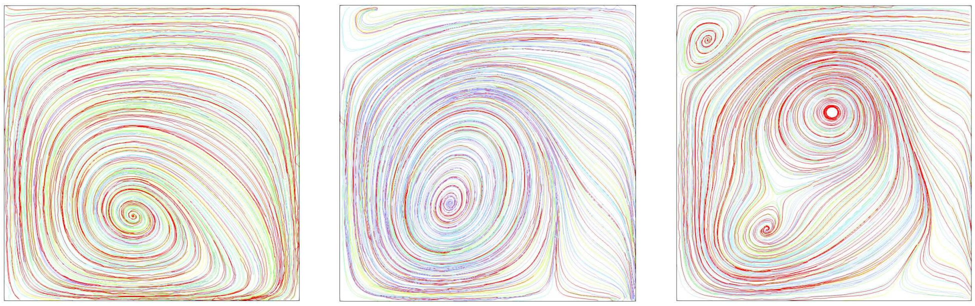

Fig. 7.3. Streamlines of $\boldsymbol{u}_h$ at $z = 0.5$ when $\overline{\mu} = 1/100, 1/400, 1/1000$ (from left to right).

$\overline{\mu} = \frac{1}{1000}$, $T = 31.34$. Figs. 7.1-7.3 show the streamlines of $\boldsymbol{u}_h$ projected onto the cross section for case $\overline{\mu} = \frac{1}{100}, \frac{1}{400}, \frac{1}{1000}$ at time T. We can clearly see that as $\overline{\mu}$ decreases, the number and distribution of vortexes have changed at three cross sections.

8. Conclusions

We introduced and analyzed a first order fully discrete mixed finite element scheme for the micropolar Navier-Stokes equations. The regularity results of the solution of MNSE were established by applying the energy method. We have shown the $\boldsymbol{L}^2$-$\boldsymbol{H}^1$ error estimates for the Euler semi-implicit time discrete solution of MNSE in this paper. In addition, the regularity estimates of time discrete solution were shown rigorously. Finally, we have shown the unconditional $\boldsymbol{L}^2$-$\boldsymbol{H}^1$ error estimates for the finite element solution of MNSE.

In this paper, we only consider a first order fully discrete scheme for MNSE, a second order accurate numerical scheme could be analyzed for the MNSE system by a similar technique developed in [5,33] and the references therein. To improve the convergence order and accuracy in the spatial discretization, a possible way is to employ uniform mesh to derive some superconvergence results as what have been done for standard Navier-Stokes equation in [26]. For the 2D case, another possible way is to use the vorticity-stream function formulation, which have been proven to be a very efficient and accurate calculation method, see [7,27,39].

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