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Compact Finite Difference Scheme for the Fourth-Order Fractional Subdiffusion System

Seakweng Vong and Zhibo Wang*

Department of Mathematics, University of Macau, Av. Padre Tomás Pereira Taipa, Macau

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Abstract. In this paper, we study a high-order compact difference scheme for the fourth-order fractional subdiffusion system. We consider the situation in which the unknown function and its first-order derivative are given at the boundary. The scheme is shown to have high order convergence. Numerical examples are given to verify the theoretical results.

AMS subject classifications: 26A33, 35R11, 65M06, 65M12, 65M15

Key words: Fourth-order fractional subdiffusion equation, compact difference scheme, energy method, stability, convergence.

1 Introduction

Recently, it has been found that many physical phenomena and processes in engineering can be modeled by fractional differential equations (FDEs), see [1–6]. Therefore, there have been growing interests on the study of these equations. The difference between classical differential equations and FDEs are that the derivatives involved in the equation are of fractional order. Since the definitions of fractional derivatives are completely different from that of the classic derivatives, it becomes necessary to develop the study of FDEs in both theoretical and numerical aspects. This paper concentrates on numerical methods for solving FDEs.

In the past decade, many works have been done on the study of efficient methods for the numerical approximation of FDEs. Our interest lies in the finite difference method. Here, we give a brief review of some recent progress in this direction. In [7], Tadjern and Meesrschaert studied a Crank-Nicolson method with second-oder accuracy in time

*Corresponding author.
Email: swvong@umac.mo (S. Vong), zhibowangok@gmail.com (Z. B. Wang)

by a special extrapolation. The method was then generalized to two-dimensional problems by the alternating directions implicit method [8]. By the use of the energy method and the method of order reduction, Sun and Wu [9] studied the stability and convergence of a fully discrete scheme for a fractional diffusion-wave equation. Based on [9], a compact finite difference scheme was studied in [10]. Compact scheme with high order accuracy is our main concern. In [11], Cui gave a compact finite difference scheme with spatial accuracy of fourth order for fractional diffusion equations. A compact finite difference scheme for fractional subdiffusion equations was shown to converge with order $\mathcal{O}(\tau^{2-\alpha}+h^4)$ in [12], where τ is the temproal grid size and h is the spatial grid size. The technique used in [12] has also been applied to a generalized Cattaneo equation in [13]. FDEs with different types of boundary conditions are also of interest. In a very recent result, Ren et al. [14] established a fourth order compact scheme for the fractional subdiffusion equation with Neumann boundary conditions.

All the results mentioned above concern with equations having second-order space derivatives. However, in some practical applications, the problem must be modeled by fourth-order space derivatives. For example, when modeling the formation of grooves on a flat surface because of grain, fourth-order terms are required [15, 16]. FDEs with fourth-order space derivatives were studied in [17–19] on both bounded and unbounded domain. Homotopy perturbation method was employed in [20] to obtain the approximate solution of a generalized fourth-order fractional diffusion-wave equation. Very recently, a finite difference scheme for fourth-order FDE was studied in [21]. This result was further improved in [22], where a compact finite difference scheme for a fractional diffusion-wave equation was shown to converge with order $\mathcal{O}(\tau^{3-\alpha}+h^4)$, when the values of the unknown function u and u_{xx} are given at the boundary.

Inspired by the results of [21,22], in this paper, we consider, for $\alpha\in(0,1)$, the following fractional subdiffusion equation

$$\frac{\partial^\alpha u}{\partial t^\alpha}+\kappa^2\frac{\partial^4 u}{\partial x^4}=f(x,t),\quad 0\le x\le 1,\quad 0<t\le T, \tag{1.1}$$

subject to the initial condition

$$u(x,0)=\phi(x),\quad 0\le x\le 1, \tag{1.2}$$

and the boundary conditions

$$u(0,t)=g_L(t),\quad u(1,t)=g_R(t),\quad \frac{\partial u(0,t)}{\partial x}=\tilde{g}_L(t),\quad \frac{\partial u(1,t)}{\partial x}=\tilde{g}_R(t),\quad 0<t\le T, \tag{1.3}$$

where $\partial^\alpha u/\partial t^\alpha$ is the Caputo fractional derivative of u which is defined as

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\partial u(x,s)}{\partial s}\frac{ds}{(t-s)^\alpha}$$

with $\Gamma(\cdot)$ being the gamma function.

The difference between the problem we consider and those in [21, 22] is that u_x is now given on the boundary instead of u_{xx}. We remark that the analysis in [21,22] cannot be applied directly in this situation because the structure of the scheme is different and, therefore, further study must be given.

This paper is organized as follows. In Section 2, the high-order compact difference scheme for the problem (1.1)-(1.3) is presented. In Section 3, by the energy method, the stability and convergence of the proposed scheme are analyzed. Numerical experiments are carried out in Section 4 to demonstrate our theoretical results and concluding remarks are given in Section 5.

2 The proposed compact difference scheme

For given integers M and N, let $h=1/M$ and $\tau=T/N$ be the spatial and temporal step sizes, respectively. For $j=0,1,\cdots,M$ and $k=0,1,\cdots,N$, denote $x_j=jh$, $t_k=k\tau$, $U_j^k=u(x_j,t_k)$. To approximate fourth-order derivatives, we first note that the following lemmas hold:

Lemma 2.1. *Suppose $g(x)\in\mathcal{C}^7[x_0,x_1]$. Then*

$$\begin{aligned}&-\frac{2}{3}g^{(4)}(x_0)+\frac{5}{3}g^{(4)}(x_1)-\frac{1}{h^3}\Big\{\frac{1}{h}\Big[-\frac{22}{3}g(x_0)+12g(x_1)\\&-6g(x_2)+\frac{4}{3}g(x_3)\Big]-4g'(x_0)\Big\}\\=&\frac{7}{15}hg^{(5)}(x_0)+\mathcal{O}(h^3).\end{aligned}\tag{2.1}$$

Similarly, if $g(x)\in\mathcal{C}^7[x_{M-1},x_M]$, we have

$$\begin{aligned}&\frac{5}{3}g^{(4)}(x_{M-1})-\frac{2}{3}g^{(4)}(x_M)-\frac{1}{h^3}\Big\{\frac{1}{h}\Big[\frac{4}{3}g(x_{M-3})-6g(x_{M-2})\\&+12g(x_{M-1})-\frac{22}{3}g(x_M)\Big]+4g'(x_M)\Big\}\\=&-\frac{7}{15}hg^{(5)}(x_M)+\mathcal{O}(h^3).\end{aligned}\tag{2.2}$$

Proof. By Taylor expansion, we easily get

$$\begin{aligned}&\frac{1}{h^4}\Big[-\frac{22}{3}g(x_0)+12g(x_1)-6g(x_2)+\frac{4}{3}g(x_3)\Big]\\=&\frac{4}{h^3}g'(x_0)+g^{(4)}(x_0)+\frac{6}{5}g^{(5)}(x_0)h+\frac{5}{6}g^{(6)}(x_0)h^2+\mathcal{O}(h^3),\end{aligned}$$

which gives

$$\begin{aligned}&\frac{1}{h^3}\left\{\frac{1}{h}\Big[-\frac{22}{3}g(x_0)+12g(x_1)-6g(x_2)+\frac{4}{3}g(x_3)\Big]-4g'(x_0)\right\}\\=&g^{(4)}(x_0)+\frac{6}{5}g^{(5)}(x_0)h+\frac{5}{6}g^{(6)}(x_0)h^2+\mathcal{O}(h^3).\end{aligned}\tag{2.3}$$

Moreover, by noting that

$$g^{(4)}(x_1)=g^{(4)}(x_0)+g^{(5)}(x_0)h+\frac{g^{(6)}(x_0)}{2}h^2+\mathcal{O}(h^3),$$

we have

$$-\frac{2}{3}g^{(4)}(x_0)+\frac{5}{3}g^{(4)}(x_1)=g^{(4)}(x_0)+\frac{5}{3}g^{(5)}(x_0)h+\frac{5}{6}g^{(6)}(x_0)h^2+\mathcal{O}(h^3).\tag{2.4}$$

Subtraction of (2.3) from (2.4) yields (2.1). We can similarly get (2.2). $\square$

Lemma 2.2 (see [22]). *Suppose $g(x)\in\mathcal{C}^8[x_{i-1},x_{i+1}]$, for $2\le i\le M-2$. Then*

$$\begin{aligned}&\frac{1}{6}[g^{(4)}(x_{i-1})+4g^{(4)}(x_i)+g^{(4)}(x_{i+1})]\\=&\frac{g(x_{i-2})-4g(x_{i-1})+6g(x_i)-4g(x_{i+1})+g(x_{i+2})}{h^4}+\mathcal{O}(h^4).\end{aligned}$$

These lemmas inspire us, when imposing the condition that $u_0=u_M=0$, to define

$$\delta_x^4u_j^k=\begin{cases}\frac{1}{h^4}(12u_1^k-6u_2^k+\frac{4}{3}u_3^k), & j=1,\ 0\le k\le N,\\ \frac{1}{h^4}(-4u_1^k+6u_2^k-4u_3^k+u_4^k), & j=2,\ 0\le k\le N,\\ \frac{1}{h^4}(u_{j-2}^k-4u_{j-1}^k+6u_j^k-4u_{j+1}^k+u_{j+2}^k), & 3\le j\le M-3,\ 0\le k\le N,\\ \frac{1}{h^4}(u_{M-4}^k-4u_{M-3}^k+6u_{M-2}^k-4u_{M-1}^k), & j=M-2,\ 0\le k\le N,\\ \frac{1}{h^4}(\frac{4}{3}u_{M-3}^k-6u_{M-2}^k+12u_{M-1}^k), & j=M-1,\ 0\le k\le N,\end{cases}$$

and an average operator $\mathcal{A}$, which is used to improve the spatial convergence order,

$$\mathcal{A}u_i=\begin{cases}\frac{5}{3}u_1, & i=1,\\ \frac{1}{6}(u_{i-1}+4u_i+u_{i+1}), & 2\le i\le M-2,\\ \frac{5}{3}u_{M-1}, & i=M-1.\end{cases}$$

For approximating the fractional derivative, the following lemma comes into play.

Lemma 2.3 (see [9]). *Suppose $\alpha\in(0,1)$, $g\in\mathcal{C}^2[0,t_n]$, let*

$$R(g(t_n))\equiv\frac{1}{\Gamma(1-\alpha)}\int_0^{t_n}\frac{g'(s)}{(t_n-s)^\alpha}ds-\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\Big[a_0g(t_n)-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})g(t_k)-a_{n-1}g(0)\Big],$$

then

$$|R(g(t_n))|\le\frac{1}{\Gamma(2-\alpha)}\Big[\frac{1-\alpha}{12}+\frac{2^{2-\alpha}}{2-\alpha}-(1+2^{-\alpha})\Big]\max_{0\le t\le t_n}|g''(t)|\tau^{2-\alpha},$$

where $a_k=(k+1)^{1-\alpha}-k^{1-\alpha}$, satisfying

$$1=a_0>a_1>a_2>\cdots>a_n>\cdots\to 0,$$
$$(1-\alpha)(k+1)^{-\alpha}<a_k<(1-\alpha)k^{-\alpha}.$$

This leads us to define the discrete fractional derivative operator $\mathcal{D}$ as

$$\mathcal{D}u_j^k=\frac{1}{\mu}\Big[u_j^k-\sum_{l=1}^{k-1}(a_{k-l-1}-a_{k-l})u_j^l-a_{k-1}u_j^0\Big],\quad 1\le j\le M-1,\quad 1\le k\le N,$$

where $\mu=\tau^\alpha\Gamma(2-\alpha)$.

Now we can discretize equation (1.1) as

$$\mathcal{D}u_j^k+\kappa^2\delta_x^4u_j^k=f_j^k,\quad 1\le j\le M-1,\quad 1\le k\le N. \tag{2.5}$$

The truncation error for the scheme (2.5) is of order $\mathcal{O}(\tau^{3-\alpha}+h^2)$. Notice that, in order to use Lemmas 2.1 and 2.2 to raise the accuracy in the spatial direction, we need to know the values of fifth-order derivatives at the boundaries. To this end, we differentiate equation (1.1) with respect to x to get

$$\frac{\partial^\alpha}{\partial t^\alpha}\Big[\frac{\partial u(x,t)}{\partial x}\Big]+\kappa^2\frac{\partial^5u(x,t)}{\partial x^5}=f_x(x,t).$$

Following the idea in [14], with x tending to 0^+ and the boundary conditions, we get

$$\kappa^2\frac{\partial^5u(0,t)}{\partial x^5}=f_x(0,t)-\frac{\partial^\alpha\tilde{g}_L}{\partial t^\alpha}.$$

Similarly, for the boundary on the right, we have

$$\kappa^2\frac{\partial^5u(1,t)}{\partial x^5}=f_x(1,t)-\frac{\partial^\alpha\tilde{g}_R}{\partial t^\alpha}.$$

We further recall that, when defining $\mathcal{A}$ and δ_x^4, we have imposed the condition that u is equal to zero on the boundary. Therefore, we must put their values back to the corresponding equations when discretizing the problem (1.1)-(1.3).

With all these lemmas and discussions, we can have the average operator $\mathcal{A}$ acting on both sides of (2.5) to get the following compact difference scheme for our problem (1.1)-(1.3):

Set $u_j^0=\phi(x_j)$. For $k>0$, we solve

$$\mathcal{A}\mathcal{D}u_j^k+\kappa^2\delta_x^4u_j^k=r_j^k, \tag{2.6}$$

where

$$r_j^k=\begin{cases}\frac{2}{3}\mathcal{D}g_L^k+\kappa^2\Big(\frac{22}{3h^4}g_L^k+\frac{4}{h^3}\tilde{g}_L^k\Big)+\frac{7}{15}h\Big[(f_x)_0^k-\Big(\frac{\partial^\alpha\tilde{g}_L}{\partial t^\alpha}\Big)^k\Big]-\frac{2}{3}f_0^k+\mathcal{A}f_1^k, & j=1,\\ -\kappa^2\frac{1}{h^4}g_L^k+\mathcal{A}f_2^k, & j=2,\\ \mathcal{A}f_j^k, & 3\le j\le M-3,\\ -\kappa^2\frac{1}{h^4}g_R^k+\mathcal{A}f_{M-2}^k, & j=M-2,\\ \frac{2}{3}\mathcal{D}g_R^k+\kappa^2\Big(\frac{22}{3h^4}g_R^k+\frac{4}{h^3}\tilde{g}_R^k\Big)-\frac{7}{15}h\Big[(f_x)_M^k-\Big(\frac{\partial^\alpha\tilde{g}_R}{\partial t^\alpha}\Big)^k\Big]-\frac{2}{3}f_M^k+\mathcal{A}f_{M-1}^k, & j=M-1.\end{cases}$$

One can see that the truncation error for the first and the last equation of (2.6) is $\mathcal{O}(\tau^{2-\alpha}+h^3)$, while the truncation error for the remaining equations is $\mathcal{O}(\tau^{2-\alpha}+h^4)$. By the technique similar to that in [14], in the next section, we show that the approximating solution tends to the true solution with order $\mathcal{O}(\tau^{2-\alpha}+h^4)$.

3 Stability and convergence of the compact scheme

In this section, the stability and convergence of the finite difference scheme are investigated.

For mesh functions $u,v\in\mathcal{V}_h$, where $\mathcal{V}_h=\{w|w=(w_0,w_1,\cdots,w_{M-1},w_M)^T,\ w_0=w_M=0\}$, we first introduce some notations:

$$(u,v)=h\Big(\frac{9}{20}u_1v_1+\sum_{i=2}^{M-2}u_iv_i+\frac{9}{20}u_{M-1}v_{M-1}\Big),\qquad \|u\|=\sqrt{(u,u)},$$

$$\delta_xu_{i+\frac{1}{2}}=\begin{cases}\frac{u_1}{h}, & i=0,\\ \frac{u_{i+1}-u_i}{h}, & 1\le i\le M-2,\\ \frac{-u_{M-1}}{h}, & i=M-1,\end{cases}\qquad \langle\delta_xu,\delta_xu\rangle=h\sum_{i=0}^{M-1}\big(\delta_xu_{i+\frac{1}{2}}\big)^2,$$

$$\|u\|_1=\sqrt{\langle\delta_xu,\delta_xu\rangle},\qquad \|u\|_\infty=\max_{0\le i\le M}|u_i|,$$

$$\delta_x^2u_i=\begin{cases}\frac{-2u_1+u_2}{h^2}, & i=1,\\ \frac{u_{i-1}-2u_i+u_{i+1}}{h^2}, & 2\le i\le M-2,\\ \frac{u_{M-2}-2u_{M-1}}{h^2}, & i=M-1.\end{cases}$$

We remark that the norm $\|u\|$ is equivalent to the one defined in the usual sense. The coefficient $9/20$ is introduced for the sake of some delicate estimates (see Lemma 3.4 below).

We need the following lemmas for our analysis.

Lemma 3.1. *Let $w \in \mathcal{V}_h$, then*

$$\frac{3\sin\frac{\pi h}{2}}{\sqrt{5}h}\|w\|_1 \leq \|\delta_x^2 w\|.$$

Proof. The counterpart of this lemma for the norm defined in the usual sense can be found in [21,22]. With those results, the lemma can be proved easily. □

Lemma 3.2. *Let $w \in \mathcal{V}_h$, for any positive constant ε, it holds that*

$$\|w\|_\infty^2 \leq \varepsilon\|w\|_1^2 + \frac{5}{9\varepsilon}\|w\|^2.$$

Proof. For $1 \leq l \leq M-1$, we have

$$w_l^2 = 2h\sum_{i=0}^{l-1}\frac{w_{i+1}+w_i}{2}\delta_x w_{i+\frac{1}{2}}, \quad w_l^2 = -2h\sum_{i=l}^{M-1}\frac{w_{i+1}+w_i}{2}\delta_x w_{i+\frac{1}{2}},$$

which yield

$$\begin{aligned} w_l^2 \leq& h\sum_{i=0}^{M-1}\left|\frac{w_{i+1}+w_i}{2}\right|\cdot|\delta_x w_{i+\frac{1}{2}}| \\ \leq& \varepsilon h\sum_{i=0}^{M-1}(\delta_x w_{i+\frac{1}{2}})^2 + \frac{1}{4\varepsilon}h\sum_{i=0}^{M-1}\Big(\frac{w_{i+1}+w_i}{2}\Big)^2 \\ \leq& \varepsilon\|w\|_1^2 + \frac{5}{9\varepsilon}h\Big(\frac{9}{20}w_1^2 + \sum_{i=2}^{M-2}w_i^2 + \frac{9}{20}w_{M-1}^2\Big). \end{aligned}$$

The lemma therefore follows. □

Lemma 3.3. *Let $w \in \mathcal{V}_h$, then*

$$\frac{1}{12}\|w\|^2 \leq \|\mathcal{A}w\|^2 \leq 4\|w\|^2.$$

Proof. Note that

$$\begin{aligned} \|\mathcal{A}w\|^2 =& h\Big\{\frac{5}{4}w_1^2 + \sum_{i=2}^{M-2}\Big[w_i^2 + \frac{1}{3}w_i(w_{i-1}-2w_i+w_{i+1}) + \frac{1}{36}(w_{i-1}-2w_i+w_{i+1})^2\Big] + \frac{5}{4}w_{M-1}^2\Big\} \\ \geq& h\Big\{\frac{5}{4}w_1^2 + \sum_{i=2}^{M-2}\Big[w_i^2 + \frac{1}{6}w_i(w_{i-1}-2w_i+w_{i+1}) - \frac{1}{4}w_i^2\Big] + \frac{5}{4}w_{M-1}^2\Big\} \\ \geq& h\Big\{\frac{5}{4}w_1^2 + \sum_{i=2}^{M-2}\Big[\frac{3}{12}w_i^2 - \frac{1}{12}(w_{i-1}^2 + w_{i+1}^2)\Big] + \frac{5}{4}w_{M-1}^2\Big\} \geq \frac{1}{12}\|w\|^2. \end{aligned}$$

The proof for the other end of the inequality is straight forward and is thus omitted. □

Our last lemma is the main ingredients for studying the stability and convergence of the proposed scheme.

Lemma 3.4. *Let $w \in \mathcal{V}_h$, then*

$$(\delta_x^4 w, \mathcal{A}w) \geq \frac{2}{15}\|w\|_1^2.$$

Proof. Note that

$$\sum_{j=2}^{M-2} \delta_x^4 w_j \mathcal{A}w_j = \frac{1}{6}\bigg(\sum_{j=2}^{M-2} w_{j-1}\delta_x^4 w_j + 4\sum_{j=2}^{M-2} w_j \delta_x^4 w_j + \sum_{j=2}^{M-2} w_{j+1}\delta_x^4 w_j\bigg). \tag{3.1}$$

The terms on the right of (3.1) can be written as

$$\begin{aligned}
\sum_{j=2}^{M-2} w_{j-1}\delta_x^4 w_j =& \frac{1}{h^2}\bigg(\sum_{j=2}^{M-2} w_{j-1}\delta_x^2 w_{j-1} - 2\sum_{j=2}^{M-2} w_{j-1}\delta_x^2 w_j + \sum_{j=2}^{M-2} w_{j-1}\delta_x^2 w_{j+1}\bigg)\\
=& \frac{1}{h^2}[w_1\delta_x^2 w_1 + (w_2 - 2w_1)\delta_x^2 w_2 + w_{M-3}\delta_x^2 w_{M-1}\\
&+ (w_{M-4} - 2w_{M-3})\delta_x^2 w_{M-2}] + \sum_{j=2}^{M-4} \delta_x^2 w_j \delta_x^2 w_{j+1},\\
4\sum_{j=2}^{M-2} w_j \delta_x^4 w_j =& \frac{4}{h^2}[w_2\delta_x^2 w_1 + (w_3 - 2w_2)\delta_x^2 w_2 + w_{M-2}\delta_x^2 w_{M-1}\\
&+ (w_{M-3} - 2w_{M-2})\delta_x^2 w_{M-2}] + 4\sum_{j=2}^{M-4} (\delta_x^2 w_{j+1})^2,
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=2}^{M-2} w_{j+1}\delta_x^4 w_j =& \frac{1}{h^2}[w_3\delta_x^2 w_1 + (w_4 - 2w_3)\delta_x^2 w_2 + w_{M-1}\delta_x^2 w_{M-1}\\
&+ (w_{M-2} - 2w_{M-1})\delta_x^2 w_{M-2}] + \sum_{j=2}^{M-4} \delta_x^2 w_{j+1}\delta_x^2 w_{j+2}.
\end{aligned}$$

Putting these back to (3.1) and moving some terms inside the summation, we have

$$\begin{aligned}
\sum_{j=2}^{M-2} \delta_x^4 w_j \mathcal{A}w_j =& \frac{1}{h^2}(w_2\delta_x^2 w_1 - w_1\delta_x^2 w_2 - w_{M-1}\delta_x^2 w_{M-2} + w_{M-2}\delta_x^2 w_{M-1})\\
&+ \frac{1}{6}\bigg[\delta_x^2 w_1 \delta_x^2 w_2 + \delta_x^2 w_{M-2}\delta_x^2 w_{M-1} + 2\sum_{j=2}^{M-3} \delta_x^2 w_j \delta_x^2 w_{j+1} + 4\sum_{j=2}^{M-2} (\delta_x^2 w_j)^2\bigg].
\end{aligned}$$

Notice that

$$\begin{aligned}
&\frac{1}{6}\left[2\sum_{j=2}^{M-3}\delta_x^2 w_j\delta_x^2 w_{j+1}+4\sum_{j=2}^{M-2}(\delta_x^2 w_j)^2\right]\geq\frac{1}{6}[(\delta_x^2 w_2)^2+(\delta_x^2 w_{M-2})^2]+\frac{1}{3}\sum_{j=2}^{M-2}(\delta_x^2 w_j)^2,\\
&\frac{1}{h^2}(w_2\delta_x^2 w_1-w_1\delta_x^2 w_2)+\frac{1}{6}\delta_x^2 w_1\delta_x^2 w_2+\frac{9}{20}\mathcal{A}w_1\delta_x^4 w_1\\
&=\frac{1}{6h^2}(2w_1-w_2)\delta_x^2 w_2+\frac{1}{h^4}\Big(8w_1^2-\frac{9}{2}w_1w_2+w_2^2\Big)\\
&\geq\frac{1}{6h^4}\Big(-\frac{4}{3}w_1^2-w_2^2\Big)-\frac{1}{6}(\delta_x^2 w_2)^2+\frac{1}{h^4}\Big(8w_1^2-\frac{243}{32}w_1^2-\frac{2}{3}w_2^2+w_2^2\Big)\\
&\geq-\frac{1}{6}(\delta_x^2 w_2)^2+\frac{1}{6h^4}(w_1^2+w_2^2),
\end{aligned}$$

where one can see that the coefficient 9/20 has been introduced so that the product w_1w_3 can be canceled. Similarly,

$$\begin{aligned}
&\frac{1}{h^2}(w_{M-2}\delta_x^2 w_{M-1}-w_{M-1}\delta_x^2 w_{M-2})+\frac{1}{6}\delta_x^2 w_{M-2}\delta_x^2 w_{M-1}+\frac{9}{20}\mathcal{A}w_{M-1}\delta_x^4 w_{M-1}\\
\geq&-\frac{1}{6}(\delta_x^2 w_{M-2})^2+\frac{1}{6h^4}(w_{M-1}^2+w_{M-2}^2).
\end{aligned}$$

By Lemma 3.1, we can thus conclude that

$$\begin{aligned}
(\delta_x^4 w,\mathcal{A}w)\geq&\frac{h}{3}\sum_{j=2}^{M-2}(\delta_x^2 w_j)^2+\frac{1}{6h^3}(w_1^2+w_2^2+w_{M-1}^2+w_{M-2}^2)\\
\geq&\frac{h}{3}\sum_{j=2}^{M-2}(\delta_x^2 w_j)^2+\frac{1}{30h^3}[(w_2-2w_1)^2+(w_{M-2}-2w_{M-1})^2]\\
\geq&\frac{2}{27}\|\delta_x^2 w\|^2\geq\frac{2}{27}\cdot\frac{9}{5}\frac{\sin^2\frac{\pi}{2}h}{h^2}\|w\|_1^2\geq\frac{2}{15}\|w\|_1^2.
\end{aligned}$$

So, we complete the proof. □

We are now ready to prove the main result of this paper.

Theorem 3.1. *For a fixed k, suppose that $w^k\in\mathcal{V}_h$ and $\{w_j^k\}$ is a solution of the following difference system*

$$\mathcal{A}\mathcal{D}w_j^k+\kappa^2\delta_x^4 w_j^k=s_j^k,\quad w_j^0=\phi(x_j). \tag{3.2}$$

Then

$$\|\mathcal{A}w^k\|^2\leq B,$$

where

$$
\begin{aligned}
B=&2\|\mathcal{A}w^0\|^2+\frac{2T^\alpha\Gamma(1-\alpha)}{19\kappa^2\varepsilon_1}h^2\Big(\max_{1\le k\le N}|s_1^k|^2+\max_{1\le k\le N}|s_{M-1}^k|^2\Big)\\
&+4T^{2\alpha}\Gamma(1-\alpha)^2\max_{1\le k\le N}\|s^k\|_0^2,\\
\epsilon_1=&\min\Big\{\frac{1}{45},\frac{1}{320\kappa^2T^\alpha\Gamma(1-\alpha)}\Big\},\qquad \|s^k\|_0^2=h\sum_{j=2}^{M-2}(s_j^k)^2.
\end{aligned}
$$

Proof. Taking the inner product of (3.2) with $\mu\mathcal{A}w^k$, we get

$$
\mu(\mathcal{A}\mathcal{D}w^k,\mathcal{A}w^k)+\mu\kappa^2(\delta_x^4w^k,\mathcal{A}w^k)=\mu(s^k,\mathcal{A}w^k),\quad 1\le k\le N. \tag{3.3}
$$

Note that

$$
\begin{aligned}
&\mu(\mathcal{A}\mathcal{D}w^k,\mathcal{A}w^k)=\|\mathcal{A}w^k\|^2-\sum_{l=1}^{k-1}(a_{k-l-1}-a_{k-l})(\mathcal{A}w^l,\mathcal{A}w^k)-a_{k-1}(\mathcal{A}w^0,\mathcal{A}w^k),\\
&\mu\kappa^2(\delta_x^4w^k,\mathcal{A}w^k)\ge\frac{2\mu\kappa^2}{15}\|w^k\|_1^2,
\end{aligned}
$$

and, by letting $\varepsilon=1$ in Lemma 3.2 and Lemma 3.3,

$$
\begin{aligned}
\mu(s^k,\mathcal{A}w^k)=&\frac{9\mu h}{20}s_1^k\mathcal{A}w_1^k+\mu h\sum_{i=2}^{M-2}s_i^k\mathcal{A}w_i^k+\frac{9\mu h}{20}s_{M-1}^k\mathcal{A}w_{M-1}^k\\
\le&\mu\kappa^2\varepsilon_1(\mathcal{A}w_1^k)^2+\frac{81\mu}{1600\kappa^2\varepsilon_1}(hs_1^k)^2+\frac{a_{k-1}}{8}\|\mathcal{A}w^k\|_0^2+\frac{2\mu^2}{a_{k-1}}\|s^k\|_0^2+\mu\kappa^2\varepsilon_1(\mathcal{A}w_{M-1}^k)^2\\
&+\frac{81\mu}{1600\kappa^2\varepsilon_1}(hs_{M-1}^k)^2\\
\le&6\mu\kappa^2\varepsilon_1\|w^k\|_\infty^2+\frac{a_{k-1}}{8}\|\mathcal{A}w^k\|^2+\frac{\mu}{19\kappa^2\varepsilon_1}[(hs_1^k)^2+(hs_{M-1}^k)^2]+\frac{2\mu^2}{a_{k-1}}\|s^k\|_0^2\\
\le&6\mu\kappa^2\varepsilon_1\|w^k\|_1^2+40\mu\kappa^2\varepsilon_1\|\mathcal{A}w^k\|^2+\frac{a_{k-1}}{8}\|\mathcal{A}w^k\|^2+\frac{\mu}{19\kappa^2\varepsilon_1}[(hs_1^k)^2+(hs_{M-1}^k)^2]\\
&+\frac{2\mu^2}{a_{k-1}}\|s^k\|_0^2.
\end{aligned}
$$

Therefore, we have, from (3.3), that

$$
\begin{aligned}
&\|\mathcal{A}w^k\|^2+\frac{2\mu\kappa^2}{15}\|w^k\|_1^2\\
\le&\sum_{l=1}^{k-1}(a_{k-l-1}-a_{k-l})(\mathcal{A}w^l,\mathcal{A}w^k)+a_{k-1}(\mathcal{A}w^0,\mathcal{A}w^k)+6\mu\kappa^2\varepsilon_1\|w^k\|_1^2\\
&+40\mu\kappa^2\varepsilon_1\|\mathcal{A}w^k\|^2+\frac{a_{k-1}}{8}\|\mathcal{A}w^k\|^2+\frac{\mu}{19\kappa^2\varepsilon_1}[(hs_1^k)^2+(hs_{M-1}^k)^2]+\frac{2\mu^2}{a_{k-1}}\|s^k\|_0^2.
\end{aligned}
$$

Since $\varepsilon_1 \le 1/45$, we get that $6\mu\kappa^2\varepsilon_1 \le 2\mu\kappa^2/15$. Notice

$$\frac{\mu}{a_{k-1}} < \frac{\tau^\alpha k^\alpha \Gamma(2-\alpha)}{1-\alpha} \le T^\alpha\Gamma(1-\alpha) \quad \text{and} \quad \varepsilon_1 \le \frac{1}{320\kappa^2 T^\alpha \Gamma(1-\alpha)},$$

we have the followings,

$$40\mu\kappa^2\varepsilon_1 \le 40 a_{k-1} T^\alpha \Gamma(1-\alpha)\kappa^2\varepsilon_1 \le \frac{a_{k-1}}{8}.$$

We thus get

$$\begin{aligned}\|\mathcal{A}w^k\|^2 + \frac{2\mu\kappa^2}{15}\|w^k\|_1^2 \le& \sum_{l=1}^{k-1}(a_{k-l-1}-a_{k-l})\frac{\|\mathcal{A}w^l\|^2+\|\mathcal{A}w^k\|^2}{2} + a_{k-1}\|\mathcal{A}w^0\|^2 \\ &+ \frac{a_{k-1}}{4}\|\mathcal{A}w^k\|^2 + \frac{2\mu\kappa^2}{15}\|w^k\|_1^2 + \frac{a_{k-1}}{4}\|\mathcal{A}w^k\|^2 \\ &+ \frac{T^\alpha\Gamma(1-\alpha)}{19\kappa^2\varepsilon_1}a_{k-1}[(hs_1^k)^2 + (hs_{M-1}^k)^2] + 2T^{2\alpha}\Gamma(1-\alpha)^2 a_{k-1}\|s^k\|_0^2,\end{aligned}$$

which is,

$$\begin{aligned}\|\mathcal{A}w^k\|^2 \le& \sum_{l=1}^{k-1}(a_{k-l-1}-a_{k-l})\|\mathcal{A}w^l\|^2 + 2a_{k-1}\|\mathcal{A}w^0\|^2 \\ &+ \frac{2T^\alpha\Gamma(1-\alpha)}{19\kappa^2\varepsilon_1}a_{k-1}[(hs_1^k)^2 + (hs_{M-1}^k)^2] + 4T^{2\alpha}\Gamma(1-\alpha)^2 a_{k-1}\|s^k\|_0^2 \\ \le& \sum_{l=1}^{k-1}(a_{k-l-1}-a_{k-l})\|\mathcal{A}w^l\|^2 + a_{k-1}B.\end{aligned}$$

By induction, we finally get the desired result. □

From Theorem 3.1, we know that if $\{s_j^k\}$ is identically zero, then so is $\{w_j^k\}$. We can thus conclude that the coefficient matrix of the difference scheme is invertible.

We now consider the convergence of the finite difference scheme (3.2). Let

$$e_j^k = U_j^k - u_j^k, \quad 0 \le j \le M, \quad 0 \le k \le N.$$

We then have

Theorem 3.2. *Assume that* $u(x,t) \in \mathcal{C}_{x,t}^{8,2}([0,1]\times[0,T])$ *is the solution of* (1.1)-(1.3) *and* $\{u_j^k | 0 \le j \le M,\ 0 \le k \le N\}$ *is a solution of the finite difference scheme* (2.6), *respectively. Then there exists a positive constant* $\tilde{c}$ *such that*

$$\|e^k\| \le \tilde{c}(\tau^{2-\alpha} + h^4), \quad 0 \le k \le N.$$

Proof. We can easily get that, for each k, $e^k \in \mathcal{V}_h$ and satisfies

$$\begin{aligned}
&\mathcal{A}\mathcal{D}e_j^k+\kappa^2\delta_x^4 e_j^k=R_j^k, && 1\le j\le M-1, \quad 1\le k\le N,\\
&e_j^0=0, && 0\le j\le M.
\end{aligned}$$

The analysis in the end of Section 2 shows that there exists a positive number c independent of h and τ such that

$$\begin{aligned}
&|R_1^k|\le c(\tau^{2-\alpha}+h^3), && 1\le k\le N,\\
&|R_j^k|\le c(\tau^{2-\alpha}+h^4), && 2\le j\le M-2, \quad 1\le k\le N,\\
&|R_{M-1}^k|\le c(\tau^{2-\alpha}+h^3), && 1\le k\le N.
\end{aligned}$$

By Theorem 3.1 and Lemma 3.3, we can get that the following holds

$$\begin{aligned}
\|e^k\|^2 &\le 12\left[\frac{2T^\alpha\Gamma(1-\alpha)}{19\kappa^2\varepsilon_1}h^2\big(\max_{1\le k\le N}|R_1^k|^2+\max_{1\le k\le N}|R_{M-1}^k|^2\big)+4T^{2\alpha}\Gamma(1-\alpha)^2\max_{1\le k\le N}\|R^k\|_0^2\right]\\
&\le\frac{48T^\alpha\Gamma(1-\alpha)c^2}{19\kappa^2\varepsilon_1}h^2(\tau^{2-\alpha}+h^3)^2+48T^{2\alpha}\Gamma(1-\alpha)^2c^2(\tau^{2-\alpha}+h^4)^2\\
&\le\tilde{c}^2(\tau^{2-\alpha}+h^4)^2,
\end{aligned}$$

where

$$\tilde{c}=4c\sqrt{\frac{3T^\alpha\Gamma(1-\alpha)}{19\kappa^2\varepsilon_1}+3T^{2\alpha}\Gamma(1-\alpha)^2}.$$

This completes the proof. ☐

4 Numerical experiments

In this section, we carry out numerical experiments for the finite difference scheme (2.6) to illustrate our theoretical statements. Two examples are considered. All our tests were done in MATLAB. The maximum norm errors between the exact and the numerical solutions

$$E_\infty(h,\tau)=\max_{0\le k\le N}\|U^k-u^k\|_\infty$$

are shown. Furthermore, the temporal convergence order, denoted by

$$Rate1=\log_2\left(\frac{E_\infty(h,2\tau)}{E_\infty(h,\tau)}\right)$$

for sufficiently small h, and the spatial convergence order, denoted by

$$Rate2=\log_2\left(\frac{E_\infty(2h,\tau)}{E_\infty(h,\tau)}\right),$$

when τ is sufficiently small, are reported. The numerical results given by these examples justify our theoretical analysis.

Example 4.1. Consider the following problem

$$\begin{aligned}
&\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}+\frac{\partial^4 u(x,t)}{\partial x^4}=f(x,t), \quad 0\le x\le 1, \quad 0<t\le 1,\\
&u(x,0)=0, \quad 0\le x\le 1,\\
&u(0,t)=0, \quad u(1,t)=0, \quad \frac{\partial u(0,t)}{\partial x}=0, \quad \frac{\partial u(1,t)}{\partial x}=0, \quad 0<t\le 1,
\end{aligned}$$

where the source term $f(x,t)$ is given by

$$f(x,t)=\frac{\Gamma(\alpha+3)}{2}t^2e^x x^2(1-x)^2+e^x t^{\alpha+2}(x^4+14x^3+49x^2+32x-12).$$

It is easy to verify that the exact solution of the above system is $u(x,t)=e^x x^2(1-x)^2t^{\alpha+2}$.

Firstly, Fig. 1 plots the curves of the exact solutions and numerical solutions for the problem at $t=1$ with $\alpha=0.5$. Fig. 2 shows the exact solutions and numerical solutions for $t\in[0,1]$, when $\alpha=0.5$.

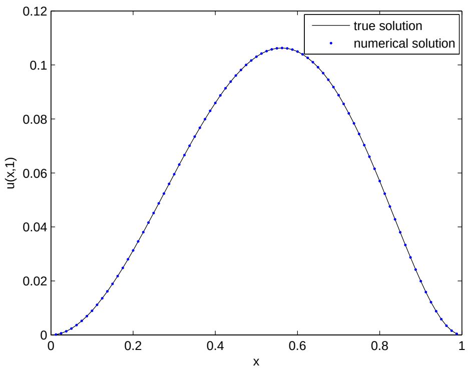

Figure 1: Exact solutions and numerical solutions for Example 4.1 at $t=1$, when $\alpha=0.5$, $h=\tau=1/80$.

Convergence order in temporal direction with $h=1/500$ is reported in Table 1, while in Table 2, the convergence order in spatial direction with $\tau=1/50000$, $\alpha=0.5$ is listed.

In the next example, we consider the equation subject to non-zero boundary conditions.

Example 4.2. Consider the equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}+\frac{\partial^4 u(x,t)}{\partial x^4}=f(x,t), \quad 0\le x\le 1, \quad 0<t\le 1,$$

subject to the following conditions:

$$\begin{aligned}
&u(x,0)=\beta\cos(\pi x), \quad 0\le x\le 1,\\
&u(0,t)=\beta, \quad u(1,t)=-\beta, \quad \frac{\partial u(0,t)}{\partial x}=0, \quad \frac{\partial u(1,t)}{\partial x}=0, \quad 0<t\le 1,
\end{aligned}$$

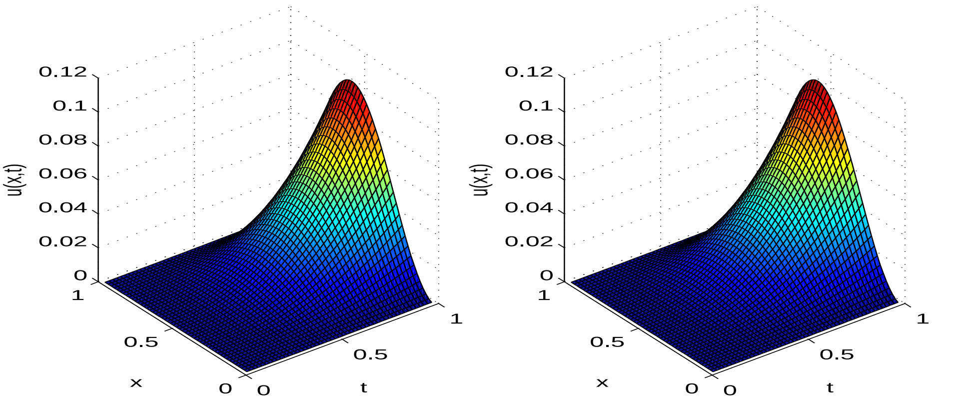

Figure 2: The exact solution (left) and numerical solution (right) for Example 4.1 for $t\in[0,1]$, when $\alpha=0.5$, $h=\tau=1/50$.

Table 1: Numerical convergence orders in temporal direction with $h=1/500$ for Example 4.1.

α	τ	$E_\infty(h,\tau)$	Rate1
	1/10	5.2354e-6	*
	1/20	1.9190e-6	1.4479
$\alpha=0.5$	1/40	6.8823e-7	1.4794
	1/80	2.5251e-7	1.4466
	1/160	8.8027e-8	1.5203
	1/10	1.4947e-5	*
	1/20	6.2160e-6	1.2658
$\alpha=0.7$	1/40	2.5627e-6	1.2783
	1/80	1.0473e-6	1.2910
	1/160	4.2522e-7	1.3003
	1/10	3.8832e-5	*
	1/20	1.8456e-5	1.0731
$\alpha=0.9$	1/40	8.6927e-6	1.0862
	1/80	4.0794e-6	1.0915
	1/160	1.9105e-6	1.0944

Table 2: Numerical convergence orders in spatial direction with $\tau=1/50000$ when $\alpha=0.5$ for Example 4.1.

h	$E_\infty(h,\tau)$	Rate2
1/5	3.6349e-2	*
1/10	2.3726e-3	3.9374
1/20	1.4936e-4	3.9895
1/40	9.3776e-6	3.9935
1/80	5.8628e-7	3.9995

where $\beta=0.05$,

$$
\begin{aligned}
f(x,t)=&\frac{\Gamma(\alpha+3)}{2}t^2x(1-x)\sin(\pi x)+\beta\pi^4\cos(\pi x)\\
&+t^{\alpha+2}[12\pi^2\sin(\pi x)-4\pi^3(1-2x)\cos(\pi x)+\pi^4x(1-x)\sin(\pi x)].
\end{aligned}
$$

Table 3: Numerical convergence orders in temporal direction with $h=1/500$ for Example 4.2.

α	τ	$E_\infty(h,\tau)$	Rate1
	1/10	1.2313e-5	$*$
	1/20	4.5141e-6	1.4476
$\alpha=0.5$	1/40	1.6199e-6	1.4785
	1/80	5.9508e-7	1.4447
	1/160	2.0823e-7	1.5147
	1/10	3.5150e-5	$*$
	1/20	1.4619e-5	1.2657
$\alpha=0.7$	1/40	6.0278e-6	1.2781
	1/80	2.4641e-6	1.2906
	1/160	1.0015e-6	1.2990
	1/10	9.1319e-5	$*$
	1/20	4.3403e-5	1.0731
$\alpha=0.9$	1/40	2.0443e-5	1.0862
	1/80	9.5944e-6	1.0914
	1/160	4.4939e-6	1.0942

Table 4: Numerical convergence orders in spatial direction with $\tau=1/20000$ when $\alpha=0.5$ for Example 4.2.

h	$E_\infty(h,\tau)$	Rate2
1/5	6.3310e-2	$*$
1/10	3.7500e-3	4.0774
1/20	2.2821e-4	4.0385
1/40	1.4169e-5	4.0096
1/80	8.8424e-7	4.0021

As in the previous example, we consider the errors between the exact solution

$$u(x,t)=x(1-x)\sin(\pi x)t^{\alpha+2}+\beta\cos(\pi x)$$

and the approximate one.

Fig. 3 plots the curves of the exact solutions and numerical solutions for this problem at $t=1$ with $\alpha=0.5$. The exact solutions and numerical solutions for $t\in[0,1]$ are shown in Fig. 4.

The convergence order in temporal direction with $h=1/500$, and that in spatial direction with $\tau=1/20000$, $\alpha=0.5$ are given in Table 3 and Table 4 respectively.

5 Conclusions

In this paper, we propose a high-order compact difference scheme for the fourth-order fractional subdiffusion system. The unknown function and its first-order derivative are considered to be given at the boundary. This boundary condition is different from those considered in previous works and further analysis must be given. By the energy method, we show that the scheme is convergent with a global order $\mathcal{O}(\tau^{2-\alpha}+h^4)$, where τ is the

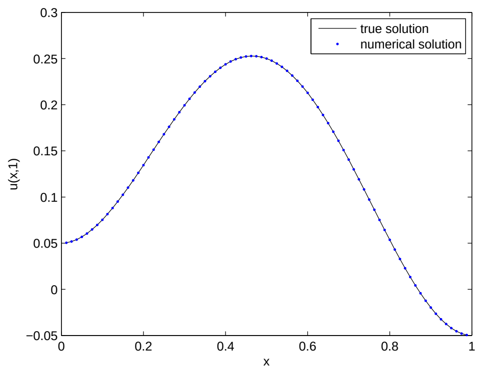

Figure 3: Exact solutions and numerical solutions for Example 4.2 at $t=1$, when $\alpha=0.5$, $h=\tau=1/80$.

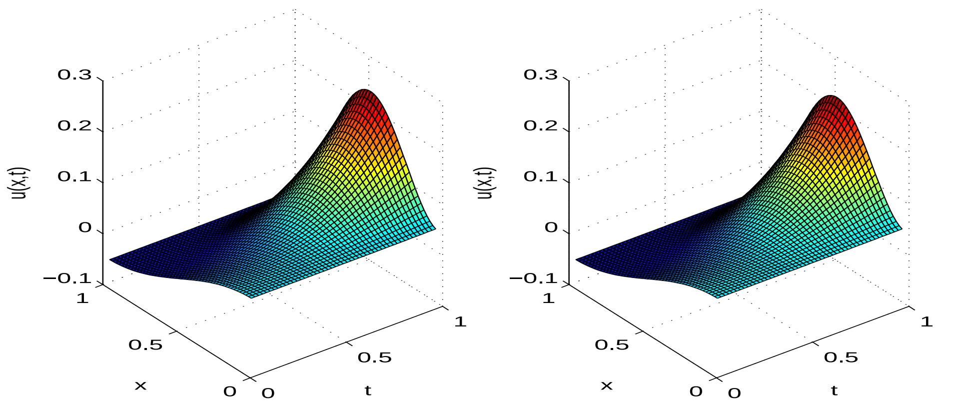

Figure 4: Exact solutions and numerical solutions for Example 4.2 for $t\in[0,1]$, when $\alpha=0.5$, $h=\tau=1/50$.

temporal grid size and h is the spatial grid size. Numerical experiments are included to support the accuracy of the proposed scheme.

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