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More on Fixed Point Theorem of $\{a,b,c\}$-Generalized Nonexpansive Mappings in Normed Spaces

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Abstract. Let X be a weakly Cauchy normed space in which the parallelogram law holds, C be a bounded closed convex subset of X with one contracting point and T be an $\{a,b,c\}$-generalized-nonexpansive mapping from C into C. We prove that the infimum of the set $\{\|x-T(x)\|\}$ on C is zero, study some facts concerning the $\{a,b,c\}$-generalized-nonexpansive mapping and prove that the asymptotic center of any bounded sequence with respect to C is singleton. Depending on the fact that the $\{a,b,0\}$-generalized-nonexpansive mapping from C into C has fixed points, accordingly, another version of the Browder's strong convergence theorem for mappings is given.

Key Words: Fixed point theorem, $\{a,b,c\}$-generalized-nonexpansive mapping, asymptotic center, Browder's strong convergence Theorem.

AMS Subject Classifications: 42B25, 42B20

1 Introduction

The following Banach contraction principle is a basic theorem of fixed point theory.

Theorem 1.1. *If X is a complete metric space and T is a r-contraction mapping from X into itself, then T has a unique fixed point $y\in X$. Moreover, the sequence of iterates $\{T^n(x)\}_{n\in\mathcal{N}}$ is strongly convergent to y for every $x\in X$.*

Mathematicians in the field of fixed point theory try to improve the result of this theorem in which changing the r-contractivity assumption imposed on the given mapping or reduce the completeness condition on the given topological space.

F. Edelstein proved that if X is a compact metric space and T is a contractive mapping from X into itself, then T has a unique fixed point $y\in X$. Moreover, the sequence of iterates $\{T^n(x)\}_{n\in\mathcal{N}}$ is strongly convergent to y for every $x\in X$, see, e.g., [5]. More explicitly,

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changing the r-contractivity condition imposed on the mapping to the contractivity condition on the mapping needs to strengthen the completeness condition to compactness condition on the metric space.

One can see that nonexpansive mappings require compact Hilbert sapce [4]. Also proper convex semi continuous mappings are used in Caristi's Fixed Point Theorem [5].

The nonexpansive mapping may not have fixed point even on uniformly convex (not bounded) Banach spaces in general. In [6], it is proved that the existence of a unique fixed point of the contraction mapping defined on a closed convex subset of a weakly Cauchy normed space. In [2], it is proved the existence of fixed point of a nonexpansive mapping defined on a bounded closed convex subset C of a weakly Cauchy normed space X in which parallelogram law holds. In [7], it gave a generalization of Banach contraction principle in two directions, and proved the existence of a unique fixed point of $\{a,b,c\}$-generalized-contraction mappings defined on the closed subset of a weakly Cauchy normed space.

The problem whether the parallelogram law holds in $\{a,b,c\}$-generalized-nonexpansive mappings defined on a bounded closed convex subset C of a weakly Cauchy normed space X in which the parallelogram law holds or the problem even in uniformly convex Banach space is open and has no affirmative solution.

In this paper, we will extend my study in this field, introduce the concept of $\{a,b,c\}$-generalized-nonexpansive mapping defined on a bounded closed convex subset of a weakly Cauchy normed space and show that $\inf\{\|x-T(x)\| : x\in C\}=0$ provided the contracting point is in C.

We will also show that the asymptotic center of any bounded sequence with respect to a closed convex subset of a weakly normed space in which the parallelogram law holds is singleton. Depending on this fact the $\{a,b,0\}$-generalized-nonexpansive mapping from C into C has fixed points, accordingly, another version of the Browder's strong convergence theorem for mappings is given.

2 Notations and basic definitions

Let X be a linear space and $f: X\to(-\infty,\infty]$ be a function from X into $(-\infty,\infty]$. Then [8]:

(1) f is said to be proper if and only if there is $x\in X$ such that $f(x)<\infty$;

(2) f is said to be lower semicontinuous if and only if the set $\{x\in X: f(x)\le\alpha\}$ is a closed convex subset of X for any real number α;

(3) f is said to be convex if and only if $f(tx+(1-t)y\le tf(x)+91-t)f(y)$ for any $x,y\in X$ and $t\in[0,1]$.

Let X be a normed space and A be a mapping from X into itself. Then

(i) A is said to be an r-contraction, a real number r with, $0\le r<1$, if and only if

$$\|A(x)-A(y)\|\le r\|x-y\| \text{ for every } x \text{ and } y, \quad x,y\in X.$$

(ii) A is said to be nonexpansive if and only if

$$\|A(x)-A(y)\| \leq \|x-y\| \text{ for every } x \text{ and } y, \quad x,y \in X.$$

(iii) A is said to be an $\{a,b,c\}$-generalized-contraction, where a,b and c are positive real numbers, $0 \leq a,b,c$ and $a+b+c<1$, if and only if

$$\|A(x)-A(y)\| \leq a\|x-y\|+b\|A(x)-x\|+c\|A(y)-y\| \text{ for every } x \text{ and } y, \quad x,y \in X.$$

(iv) A is said to be $\{a,b,c\}$-generalized-nonexpansive, where $a,b,$ and c are positive real numbers, $0 \leq a,b,c$ and $a+b+c=1$, if and only if

$$\|A(x)-A(y)\| \leq a\|x-y\|+b\|A(x)-x\|+c\|A(y)-y\| \text{ for every } x \text{ and } y, \quad x,y \in X.$$

Remark 2.1. Every r-contraction mapping is $\{r,0,0\}$-generalized-contraction; Every non-expansive mapping is $\{1,0,0\}$-generalized-nonexpansive.

Definition 2.1. Let X be a normed space, C be bounded closed convex subset of X, $0 \leq t$, $a,b,c<1$, $b,c \neq 1$ and $a+b+c=1$. Then

(1) a point $x_0 \in C$ is said to be *abct*-contracting with respect to C if and only if there is a real number λ, $\lambda < (1-\{ta+b+c\})/(1-t)$, satisfying

$$b\|x_0-x\|+c\|x_0-y\| \leq \|x-y\|, \quad x \neq y, \quad x,y \in C.$$

(2) X is said to be weakly Cauchy if and only if every Cauchy sequence in X is weakly convergent to an element $x \in X$. For some properties of such a space see [1] and [6].

(3) If $\{x_n\}_{n \in \mathcal{N}}$ is a bounded sequence in X and f is the real valued function on C defined by:

$$f(x) := \limsup_{n\to\infty} \|x_n - x\|,$$

then

(a) the asymptotic radius of $\{x_n\}_{n\in\mathcal{N}}$ is denoted by $r(C,\{x_n\}_{n\in\mathcal{N}})$ and is defined as:

$$r(C,\{x_n\}_{n\in\mathcal{N}}) := \inf\{f(x) : x \in C\};$$

(b) the asymptotic center of $\{x_n\}_{n\in\mathcal{N}}$ is denoted by $A(C,\{x_n\}_{n\in\mathcal{N}})$ and is defined as:

$$A(C,\{x_n\}_{n\in\mathcal{N}}) := \{y \in C : f(y) = r(C,\{x_n\}_{n\in\mathcal{N}})\}.$$

Some properties of the asymptotic radius and asymptotic center of a bounded sequence in uniformly convex Banach space are given in [4] as follows:

Theorem 2.1. (see [4]) *Every bounded sequence in a uniformly convex Banach space has a unique asymptotic center with respect to any closed convex subset of X.*

We will need the next theorem to prove the parallel result in the case of weakly Cauchy normed space in which the parallelogram law holds.

Theorem 2.2. (see [2]) *Let X be a weakly Cauchy normed space in which the parallelogram law holds. If $\{C_n\}_{n\in\mathcal{N}}$ is a descending sequence of closed bounded convex subsets of X, then their intersection is not empty, $\bigcap_{n\in\mathcal{N}} C_n \neq \Phi$.*

Recall the following lemma:

Lemma 2.1. (see [8]) *Let X be a normed space in which the parallelogram law holds, and C be a closed convex subset of X. If T is a nonexpansive mapping from C into C and the set of fixed points $F(T)$ is nonempty, then $F(T)$ is a closed convex subset of X.*

We will also need the following interesting theorems:

Theorem 2.3. (see [1]) *Let X be weakly Cauchy normed space in which the parallelogram law holds. Then every nonempty closed convex subset of X is Chebyshev. Or equivalently the metric projection P_C exists on every nonempty closed convex subset C of X. That is, for every $x\in X$ there exists a unique element $P_C(x)=y\in C$ called the best approximation element of x in C such that*

$$\|x-y\|=\mathrm{dist}(x,C)=:\inf\big\{\|x-z\|:z\in C\big\}.$$

Theorem 2.4. *Let X be a weakly Cauchy normed space in which the parallelogram law holds. If $\{C_n\}_{n\in\mathcal{N}}$ is a descending sequence of closed bounded convex subsets of X, then their intersection is not empty, $\bigcap_{n\in\mathcal{N}} C_n \neq \Phi$.*

3 Main results

We have the following lemma:

Lemma 3.1. *Let X be a weakly Cauchy normed space in which the parallelogram law holds, and C be a closed convex subset of X. If T is a nonexpansive mapping from C into C and the set of fixed points $F(T)$ is nonempty, then $F(T)$ is a Chebyshev subset of X.*

Proof. It is obviously clear if we use Theorem 2.3 and Lemma 2.1. □

The next theorem is a core theorem for the upcoming theorems.

Theorem 3.1. *In a weakly Cauchy normed space X in which the parallelogram law holds every bounded sequence has a unique asymptotic center with respect to any closed convex subset of X.*

Proof. Let C be a bounded closed convex subset of X. We will show that a function f can attain its minimum at some $y\in C$. Let $\{t_n\}_{n\in\mathcal{N}}$ be a sequence of positive real numbers that converges to zero. According to the Theorem 2.2, the following sequence of descending bounded closed convex subsets of X has a nonempty intersection:

$$C_n:=\big\{x:f(x)\le \inf_{x\in C} f(x)+t_n\big\}.$$

Thus there is $y \in X$ such that $f(y) \leq \inf_{x \in C} f(x) + t_n$, $\forall n \in \mathcal{N}$. Consequently, f attains its minimum at y.

Using the parallelogram law, for every $x, y \in X$, we have the following strict inequality

$$\Big\| x_n - \frac{x+y}{2} \Big\| = \Big\| \frac{x_n - x}{2} + \frac{x_n - y}{2} \Big\| < \max\big\{ \|x_n - x\|, \|x_n - y\| \big\}.$$

This strict inequality implies the following strict inequality

$$f\Big(\frac{y+z}{2}\Big) < \max\{f(x), f(y)\} \text{ for every } y, z \in X \text{ and } y \neq z. \tag{3.1}$$

To show that such y is unique, consider $z \in X$ such that $y \neq z$ and that $f(z) = \inf\{f(x) : x \in C\}$, using the inequality (3.1) gives the following obvious contradiction:

$$f\Big(\frac{y+z}{2}\Big) < \max\{f(x), f(y)\} = f(y) = f(z) = \inf\big\{f(x) : x \in C\big\}.$$

So, the proof is complete. $\square$

Remark 3.1. Let X be a metric space with a metric d, T be a nonexpansive mapping on X into itself, and $\{x_n\}_{n \in \mathcal{N}}$ be a sequence in X such that the sequence $\{d(x_n, T(x_n))\}_{n \in \mathcal{N}}$ converges to zero. If $\{x_n\}_{n \in \mathcal{N}}$ is strongly convergent to a point $y \in X$, then y is a fixed point of T.

In fact, if y is the limit point of $\{x_n\}_{n \in \mathcal{N}}$, we have the following:

$$\begin{aligned} d(y, f(y)) &\leq d(x_n, y) + d(x_n, T(x_n)) + d(T(x_n), T(y)) \\ &\leq 2d(x_n, y) + d(x_n, T(x_n)) \to 0. \end{aligned}$$

This proves that $T(y) = y$.

Corollary 3.1. Let X be a weakly Cauchy normed space in which the parallelogram law holds, C a bounded closed convex subset of X, T is a mapping from C into C such that

$$\|T(x) - T(y)\| \leq a\|x - y\| + b\|T(x) - x\|$$

for some real numbers a, b, $a + b = 1$, and $\{x_n\}_{n \in \mathcal{N}}$ a bounded sequence in X such that

$$\lim_{n \to \infty} \|T(x_n) - x_n\| = 0.$$

Then T has fixed points.

Proof. Using Theorem 2.1 proves that the sequence $\{x_n\}_{n \in \mathcal{N}}$ has a unique asymptotic center. Let y be the unique asymptotic center of $\{x_n\}_{n \in \mathcal{N}}$, we have the following inequalities

$$\begin{aligned} \|x_n - f(y)\| &\leq \|x_n - T(x_n)\| + \|T(x_n) - T(y)\| \\ &\leq \|x_n - T(x_n)\| + a\|x_n - y\| + b\|x_n - T(x_n)\| \\ &= (1+b)\|x_n - T(x_n)\| + a\|x_n - y\|. \end{aligned}$$

Thus $f(T(y)) \leq f(y)$. Using the uniqueness of such a y proves that $T(y) = y$. $\square$

Theorem 3.2. *Let X be a weakly Cauchy normed space, C a closed convex subset of X, and T a $\{a,b,c\}$-generalized-nonexpansive mapping from C into C, $b,c\neq 1$. Assume in addition that x_0 is an abct-contracting point with respect to C and A is the mapping from C into C that defined by*

$$A(x)=(1-t)x_0+tT(x).$$

Then there is a real number r, $0<r<1$, for which A fulfils the following inequalities

(1) For every natural number n, we have

$$\|A^n(x)-A^{n-1}(x)\|\leq r^{n-1}\|A(x)-x\|. \tag{3.2}$$

(2) For given natural numbers n and m, $n\leq m$, we have

$$\|A^m(x)-A^n(x)\|\leq \frac{r^n}{1-r}\|A(x)-x\|. \tag{3.3}$$

(3) For given natural number l and $x,y\in C$, we have

$$\begin{aligned}&\|A^l(x)-A^l(y)\|\\ \leq&[1-(b+c)]^l\|x-y\|+\frac{r}{r-[1-(b+c)]}\cdot[r^l-\{1-(b+c)\}^l]\,[b\|A(x)-x\|\\ &+c\|A(y)-y\|].\end{aligned} \tag{3.4}$$

(4) For given natural numbers l and m, $l\leq m$, and $x,y\in C$, we have

$$\|A^m(x)-A^l(x)\|\leq \sum_{k=l-1}^{m-1} r^k\|A(x)-x\|+\|A^l(x)-A^l(y)\|. \tag{3.5}$$

Proof. (1) Let $x\in C$. If $A(x)=x$, then there is nothing to prove; so let $A(x)\neq x$. As T is $\{a,b,c\}$-generalized nonexpansive, we have

$$\begin{aligned}&\|A^2(x)-A^1(x)\|=t\|T(A(x))-T(x)\|\\ \leq&t\{a\|A(x)-x\|+b\|T(A(x))-A(x)\|+c\|T(x)-x\|\}\\ \leq&(ta)\|A(x)-x\|+b\|tT(A(x))-tA(x)\|+c\|tT(x)-tx\|\\ \leq&(ta)\|A(x)-x\|+b\|[A^2(x)-A^1(x)]+(1-t)[x_0-A(x)]\|\\ &\quad+c\|[A(x)-x]+(1-t)[x_0-x]\|\\ \leq&(ta+c)\|A(x)-x\|+b\|A^2(x)-A^1(x)\|+(1-t)\{b\|x_0-A(x)\|+c\|x_0-x\|\},\end{aligned}$$

which gives

$$\|A^2(x)-A^1(x)\|\leq\Big[\frac{ta+c}{1-b}\Big]\|A(x)-x\|+\Big[\frac{1-t}{1-b}\Big]\{b\|x_0-A(x)\|+c\|x_0-x\|\}.$$

Since x_0 is an *abct*-contracting point with respect to C, there is λ satisfying

$$\lambda < \frac{[1-(ta+b+c)]}{(1-t)}$$

such that

$$b\|x_0 - A(x)\| + c\|x_0 - x\| \le \|A(x) - x\|.$$

We then obtain

$$\begin{aligned}\|A^2(x)-A^1(x)\| &\le \Big[\frac{ta+c}{1-b}\Big]\|A(x)-x\| + \Big[\frac{\lambda(1-t)}{1-b}\Big]\|A(x)-x\| \\ &= \Big[\frac{ta+c}{1-b} + \frac{\lambda(1-t)}{1-b}\Big]\|A(x)-x\| = r\|A(x)-x\|,\end{aligned}$$

where

$$r = \frac{ta+c+\lambda(1-t)}{1-b}.$$

It is clear that $r<1$. Repeating the last step gives (3.2).
(2) For n and m, $n \le m$, using the standard triangular inequality technique for (3.2) gives

$$\begin{aligned}&\|A^m(x)-A^n(x)\| \\ \le& \|A^m(x)-A^{m-1}(x)\| + \|A^{m-1}(x)-A^{m-2}(x)\| + \cdots + \|A^{n+1}(x)-A^n(x)\| \\ \le& r^n\|A(x)-x\| + r^{n+1}\|A(x)-x\| + \cdots + r^{m-2}\|A(x)-x\| + r^{m-1}\|A(x)-x\| \\ \le& \frac{r^n}{1-r}\|A(x)-x\|.\end{aligned}$$

(3) For given natural number l and $x,y \in C$, we have

$$\begin{aligned}&\|A^l(x)-A^l(y)\| \\ \le& t\big[a\|A^{l-1}(x)-A^{l-1}(y)\| + b\|T(A^{l-1}(x))-A^{l-1}(x)\| + c\|T(A^{l-1}(y))-A^{l-1}(y)\|\big] \\ \le& (ta)\|A^{l-1}(x)-A^{l-1}(y)\| + b\|A^l(x)-A^{l-1}(x)\| + c\|A^l(y)-A^{l-1}(y)\| \\ &\quad + b(1-t)\|x_0-A^{l-1}(x)\| + c(1-t)\|x_0-A^{l-1}(y)\| \\ \le& (ta)\|A^{l-1}(x)-A^{l-1}(y)\| + r^l\big[b\|A(x)-x\| + c\|A(y)-y\|\big] \\ &\quad + (1-t)\lambda\|A^{l-1}(x)-A^{l-1}(y)\| \\ \le& [1-(b+c)]\|A^{l-1}(x)-A^{l-1}(y)\| + r^l\big[b\|A(x)-x\| + c\|A(y)-y\|\big].\end{aligned}$$

Repeating the last step with the term $\|A^{l-1}(x)-A^{l-1}(y)\|$ gives

$$\begin{aligned}&\|A^l(x)-A^l(y)\| \\ \le& [1-(b+c)]^2\|A^{l-2}(x)-A^{l-2}(y)\| + \big[(1-b-c)r^{l-1}+r^l\big]\big[b\|A(x)-x\| \\ &\quad + c\|A(y)-y\|\big].\end{aligned}$$

Repeating the last step $l-2$ times again gives

$$\begin{aligned}
&\|A^l(x)-A^l(y)\| \\
\le&[1-(b+c)]^l\|x-y\|+\big[[1-(b+c)]^l+[1-(b+c)]^{l-1}r+\cdots \\
&\quad+[1-(b+c)]r^{l-1}+r^l\big]\big[b\|A(x)-(x)\|+c\|A(y)-y\|\big] \\
=&[1-(b+c)]^l\|x-y\|+\frac{r^l\big[1-(\frac{1-(b+c)}{r})^l\big]}{1-(\frac{1-(b+c)}{r})}\big[b\|A(x)-(x)\|+c\|A(y)-y\|\big],
\end{aligned}$$

which gives (3.4).
(4) For given natural numbers l and m, $l\le m$, and $x,y\in C$, we have

$$\|A^m(x)-A^l(y)\|\le\|A^m(x)-A^{m-1}(x)\|+\|A^{m-1}(x)-A^{m-2}(x)\|+\cdots+\|A^l(x)-A^l(y)\|.$$

This, together with (3.4), gives the desired result (3.5). □

Remark 3.2. Using Theorem 3.2, we have the following relations:

$$\lim_{n\to\infty}\|A^n(x)-A^{n-1}(x)\|=0, \qquad \forall x\in C, \tag{3.6a}$$

$$\lim_{n,m\to\infty}\|A^m(x)-A^n(x)\|=0, \qquad \forall x\in C, \tag{3.6b}$$

$$\lim_{l\to\infty}\|A^l(x)-A^l(y)\|=0, \qquad \forall x,y\in C, \tag{3.6c}$$

$$\lim_{m,l\to\infty}\|A^m(x)-A^l(x)\|=0, \qquad \forall x,y\in C. \tag{3.6d}$$

Theorem 3.3. *Let X be a weakly Cauchy normed space, C a closed convex subset of X, and T an $\{a,b,c\}$-generalized-nonexpansive mapping from C into C, $b,c\neq 1$. Assume in addition that x_0 is an abct-contracting point with respect to C. Then the mapping A from C into C defined by*

$$A(x)=(1-t)x_0+tT(x)$$

has a unique fixed point $y\in C$. Moreover, the sequence of iterates $\{A^n(x)\}_{n\in\mathcal{N}}$ is strongly convergent to y for every $x\in C$,

$$\lim_{n\to\infty}A^n(x)=y, \quad \forall x\in C.$$

Proof. Using the Eq. (3.6b), the sequence $\{A^n(x)\}_{n\in\mathcal{N}}$ is Cauchy in X. Since X is weakly Cauchy, $\{A^n(x)\}_{n\in\mathcal{N}}$ is weakly convergent to some element $y\in X$. Since C is convex closed, it is weakly closed, hence $y\in C$. We claim that y is the unique fixed point of A. To this end, let l be an arbitrarily natural number and define the proper lower-semi-continuous convex real valued function ϕ_l on C by the following formula:

$$\phi_l(x)=\|A^l(y)-x\|.$$

Using the same steps of the proof of Theorem 1 [6], the Eqs. (3.6c) and (3.6d) for the mapping A, we prove that

$$0 \leq \lim_{l\to\infty} \|A^l(y)-y\| < \varepsilon.$$

Since ε is arbitrarily, one gets

$$\lim_{l\to\infty} \|A^l(y)-y\| = 0.$$

Thus the sequence $\{A^n(x)\}_{n\in\mathcal{N}}$ converges strongly to y. Since

$$\begin{aligned}&\|A(x)-A(z)\| = t\|T(x)-T(z)\| \\ \leq &\big[ta+(1-t)\lambda\big]\,\|x-z\| + b\|A(x)-x\| + c\|A(z)-z\|, \quad \forall x,z\in C,\end{aligned}$$

we have

$$\begin{aligned}&\|A(y)-y\| = \lim_{n\to\infty} \|A(y)-A^n(y)\| \\ \leq &[ta+(1-t)\lambda] \lim_{n\to\infty} \|y-A^{n-1}(y)\| + b\|A(y)-y\| + c\lim_{n\to\infty} \|A^n(y)-A^{n-1}(y)\| \\ = &b\|A(y)-y\|.\end{aligned}$$

This proves that $A(y)=y$, and y is a fixed point of A.

Using the Eq. (3.6c) shows that $\lim_{n\to\infty} \|A^n(x)-y\| = \lim_{n\to\infty} \|A^n(x)-A^{n-1}(y)\| = 0$, and in turns shows that the weak limit of a sequence $\{A^n(x)\}_{n\in\mathcal{N}}$ is actually a strong limit.

Finally, to show that y is unique, let $z\neq y$ be another fixed point of A. Then we have the following contradiction,

$$\|y-z\| = \|A(y)-A(z)\| = t\|T(y)-T(z)\| \leq [ts+(1-t)\lambda]\,\|y-z\| < \|y-z\|.$$

This ends the proof. □

We have the following theorem:

Theorem 3.4. *Let X be a weakly Cauchy normed space and C a bounded closed convex subset of X. If T is an $\{a,b,c\}$-generalized nonexpansive operator from C into C, $b,c\neq 1$, assume in addition that C has a contracting point, then*

$$\inf\{\|x-T(x)\| : x\in C\} = 0.$$

Proof. Let $x_0\in C$ be the contracting point of C, $\{t_n(x)\}_{n\in\mathcal{N}}$ a sequence of non negative real numbers such that $\lim_{n\infty} t_n = 1$ and $0\leq t_n < 1$. Define the corresponding sequence of operators $\{A^n(x)\}_{n\in\mathcal{N}}$ from C into C as follows:

$$A_n(x) := (1-t_n)x_0 + t_n T(x). \tag{3.7}$$

Using Theorem 3.3, for each n, A_n has a unique fixed point $x_n \in C$, $A_n(x_n) = x_n$. Hence the sequence $\{x_n(x)\}_{n\in\mathcal{N}}$ in C satisfies

$$x_n = (1-t_n)x_0 + t_n T(x_n). \tag{3.8}$$

We have

$$\|x_n - T(x_n)\| = (1-t_n)\|x_0 - T(x_n)\| \leq (1-t_n)\mathrm{Diam}(C). \tag{3.9}$$

Taking the limit as $n \to \infty$, then $\|x_n - T(x_n)\| \to 0$. This limit insures that $\inf\{\|x - T(x)\| : x \in C\} = 0$. ☐

Using Corollary 3.1 and Theorem 3.4, we have the following Corollary:

Corollary 3.2. Let X be a weakly Cauchy normed space in which the parallelogram law holds and C a bounded closed convex subset of X. If T is a nonexpansive mapping from C into C, then T has fixed points.

For the nonexpansive mappings, we have the following interesting Theorem:

Theorem 3.5. *Let X be a weakly Cauchy normed space in which the parallelogram law holds, and C a bounded closed convex subset of X. If T is a nonexpansive mapping from C into C, $b,c \neq 1$, assume in addition that C has a contracting point, then T has fixed points.*

Proof. For a given $x_0 \in C$, and a given sequence of nonnegative real numbers $\{t_n(x)\}_{n\in\mathcal{N}}$, $0 \leq t_n < 1$, the asymptotic center of the sequence given by

$$x_n = t_n x_0 + (1-t_n)T(x_n) \tag{3.10}$$

is a fixed point of T. ☐

For another proof one can see Theorem 4 in [7]. We have the following two versions of Browder's strong convergence theorem.

Theorem 3.6. *Let X be a weakly Cauchy normed space in which every bounded sequence has a weakly convergent subsequence and the parallelogram law holds. If C is a bounded closed convex subset of X and T is a nonexpansive operator from C into C, then for every $y_0 \in C$ the sequence given in* (3.10) *converges strongly to the best approximation element of y_0 in $F(T)$.*

Proof. The theorem can be proved by using Lemma 3.1, Theorem 4 in [7], and the same steps of Theorem (3.2.1) in [8], together with the given assumptions. ☐

Theorem 3.7. *Let X be a weakly Cauchy normed space in which every bounded sequence has a weakly convergent subsequence and the parallelogram law holds. If C is a bounded closed convex subset of X that has an abor-contracting point, and T is an $\{a,b,0\}$-generalized nonexpansive operator from C into C, then for every $\{a,b,o,r\}$-contracting point of C, $x_0 \in C$, the sequence given in the Eq.* (3.8) *converges strongly to the best approximation element of x_0 in $F(T)$.*

Proof. Using Lemma 3.1, Corollary 3.2 and the same steps of Theorem (3.2.1) in [8], together with the given assumptions complete the proof of Theorem 3.7. ☐

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