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New Criterion for Starlike Integral Operators

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Abstract. In this paper, we introduce new sufficient conditions for certain integral operators to be starlike and p-valently starlike in the open unit disk.

Key Words: Analytic function, univalent function, p-valently starlike, integral operator.

AMS Subject Classifications: 30C45

1 Introduction

Let $\mathcal{U}=\{z\in\mathbf{C}:|z|<1\}$, the unit disk. We denote by $\mathcal{H}(\mathcal{U})$ the class of holomorphic functions defined on $\mathcal{U}$. Let $\mathcal{A}_p$ be the class of all p-valent analytic functions of the form

$$f(z)=z^p+a_{p+1}z^{p+1}+\cdots,\quad p\in\mathbf{N}=\{1,2,\cdots\}.$$

For $p=1$, we obtain $\mathcal{A}_1=\mathcal{A}$, the class of univalent analytic functions in the unit disk. Let $\mathcal{S}^*$ and $\mathcal{K}$ denote the subclasses of starlike and convex functions in $\mathcal{U}$ respectively. Recall that $f\in\mathcal{A}$ is convex if and only if

$$\mathrm{Re}\Big(\frac{zf''(z)}{f'(z)}+1\Big)>0,\quad z\in\mathcal{U},$$

and starlike if and only if

$$\Re\Big(\frac{zf'(z)}{f(z)}\Big)>0,\quad z\in\mathcal{U}.$$

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For $f_i(z)\in\mathcal{A}$ and $\alpha_i>0$, for all $i\in\{1,2,3,\cdots,n\}$, D. Breaz and N. Breaz [2] introduced the following integral operator:

$$F_n(z)=\int_0^z\Big(\frac{f_1(t)}{t}\Big)^{\alpha_1}\cdots\Big(\frac{f_n(t)}{t}\Big)^{\alpha_n}dt. \tag{1.1}$$

Recently Breaz et al. in [3] introduced the following integral operator:

$$F_{\alpha_1,\cdots,\alpha_n}(z)=\int_0^z[f_1'(t)]^{\alpha_1}\cdots[f_n'(t)]^{\alpha_1}dt. \tag{1.2}$$

The most recent, Frasin [1] introduced the following integral operators, for $\alpha_i>0$ and $f_i\in\mathcal{A}_p$,

$$F_p(z)=\int_0^z pt^{p-1}\Big(\frac{f_1(t)}{t^p}\Big)^{\alpha_1}\cdots\Big(\frac{f_n(t)}{t^p}\Big)^{\alpha_n}dt \tag{1.3}$$

and

$$G_p(z)=\int_0^z pt^{p-1}\Big(\frac{f_1'(t)}{pt^{p-1}}\Big)^{\alpha_1}\cdots\Big(\frac{f_1'(t)}{pt^{p-1}}\Big)^{\alpha_n}dt. \tag{1.4}$$

Remark 1.1. (i) For $p=1$, we get $F_1(z)=F_n(z)$, and $G_1(z)=F_{\alpha_1,\cdots,\alpha_n}(z)$.

(ii) For $p=n=1$, $\alpha_1=\alpha\in[0,1]$ in (1.3) we get the integral operator

$$F_\alpha(z)=\int_0^z\Big(\frac{f(t)}{t}\Big)^{\alpha}dt,$$

which is studied in [7].

(iii) For $p=n=1$, $\alpha=1$ in (1.3) we get the integral operator

$$G(z)=\int_0^z\frac{f(t)}{t}$$

introduced by Alexander [4].

(iv) For $p=n=1$, $\alpha_1=\alpha\in\mathbf{C}$, $|\alpha|\leq 1/4$ in (1.4) we get the integral operator

$$\int_0^z(f'(t))^{\alpha}dt,$$

which is studied in [5].

2 Main result

In order to prove our main results we shall need the following lemma due to S. S. Miller and P. T. Mocanu [6]:

Lemma 2.1. *Let a function* $\Phi:\mathbf{C}^2\to\mathbf{C}$ *satisfy*

$$\mathrm{Re}\Phi(ix,y)\leq 0$$

for all real x *and all real* y *with* $y\leq-(1+x^2)/2$. *If* $p(z)=1+p_1z+\cdots$ *is analytic in the unit disc* $\mathcal{U}=\{z:z\in\mathbf{C}|z|<1\}$ *and*

$$\mathrm{Re}\Phi(p(x),zp'(x))>0,\quad z\in\mathcal{U},$$

then

$$\mathrm{Re}p(z)>0,\quad z\in\mathcal{U}.$$

Firstly, we prove the following p-valent starlike result of the operator $F_p(z)$.

Theorem 2.1. *Let* $\alpha_i>0$ *for* $i=1,2,\cdots,n$, *and* $f_i\in\mathcal{A}_p$. *If*

$$\sum_{i=1}^{n}\alpha_i\Big(\mathrm{Re}\frac{zf_i'(z)}{f_i(z)}-p\Big)>1-p,$$

then F_p *is p-valent starlike. Here* F_p *is the integral operator define as in* (1.3).

Proof. From (1.3), we observe that $F_p\in\mathcal{A}_p$ and obtain

$$F_p{}'(z)=pz^{p-1}\Big(\frac{f_1(z)}{z^p}\Big)^{\alpha_1}\cdots\Big(\frac{f_n(z)}{z^p}\Big)^{\alpha_n}.$$

Differentiating the above expression logarithmically and multiply by z we obtain

$$\frac{zF_p''(z)}{F_p'(z)}=(p-1)+\sum_{i=1}^{n}\alpha_i\Big[\frac{zf_i'(z)}{f_i(z)}-p\Big]. \tag{2.1}$$

Let

$$h(z)=\frac{zF_p'(z)}{F_p(z)}$$

be a holomorphic function in $\mathcal{U}$ and $h(0)=1$. Differentiating $h(z)$ logarithmically, we obtain

$$h(z)-1+\frac{zh'(z)}{h(z)}=\frac{zF_p''(z)}{F_p'(z)}. \tag{2.2}$$

Substitute (2.2) in (2.1), we obtain

$$h(z)-1+\frac{zh'(z)}{h(z)}=(p-1)+\sum_{i=1}^{n}\alpha_i\Big[\frac{zf_i'(z)}{f_i(z)}-p\Big].$$

We define the function Ψ by

$$\Psi(u,v)=u-1+\frac{v}{u}.$$

In order to use Lemma 2.1, we must verify that $\Psi(ix,y)<0$ whenever x and y are real numbers with $y\le-(1+x^2)/2$, we have

$$\mathrm{Re}\Psi(ix,y)=\mathrm{Re}\Big(ix-1+\frac{y}{ix}\Big)=-1<0,$$

then

$$\mathrm{Re}\Phi(h(z),zh'(z))\ge0.$$

By Lemma 2.1, we deduce that $\mathrm{Re}h(z)>0$, $z\in\mathcal{U}$, and so

$$\mathrm{Re}\frac{zF_p'(z)}{F_p(z)}>0,$$

therefore the integral operator F_p is p-valent starlike. □

Our next result is the following:

Theorem 2.2. *Let $\alpha_i>0$ for $i=1,2,\cdots,n$, and $f_i\in\mathcal{A}_p$. If*

$$\sum_{i=1}^{n}\alpha_i\Big(\mathrm{Re}\frac{zf_i''(z)}{f_i'(z)}\Big)>(p-1)\Big(\sum_{i=1}^{n}\alpha_i-1\Big),$$

then G_p is p-valent starlike, where G_p is the integral operator define as in [4].

Proof. From (1.4), we observe that $G_p\in A_p$ and obtain

$$G_p'(z)=pz^{p-1}\Big(\frac{f_1'(z)}{pz^{p-1}}\Big)^{\alpha_1}\cdots\Big(\frac{f_1'(z)}{pz^{p-1}}\Big)^{\alpha_n}.$$

Differentiating the above expression logarithmically and multiply by z we obtain

$$\frac{zG_p''(z)}{G_p'(z)}=(p-1)+\sum_{i=1}^{n}\alpha_i\Big[\frac{zf_i''(z)}{f_i'(z)}-(p-1)\Big]. \tag{2.3}$$

Let

$$h(z)=\frac{zG_p'(z)}{G_p(z)}$$

be a holomorphic function in $\mathcal{U}$ and $h(0)=1$. Differentiating $h(z)$ logarithmically, we obtain

$$h(z)-1+\frac{zh'(z)}{h(z)}=\frac{zG_p''(z)}{G_p'(z)}. \tag{2.4}$$

Substitute (2.4) in (2.3), we obtain

$$h(z)-1+\frac{zh'(z)}{h(z)}=(p-1)+\sum_{i=1}^{n}\alpha_i\Big[\frac{zf_i''(z)}{f_i'(z)}-(p-1)\Big].$$

Following the same steps as in Theorem 2.1, we obtain that $\mathrm{Re}h(z)>0$, $z\in\mathcal{U}$ and so

$$\mathrm{Re}\frac{zG_p'(z)}{G_p(z)}>0,$$

therefore the integral operator G_p is p-valent starlike. □

Letting $p=1$ in Theorems 2.1 and 2.2 respectively we have

Theorem 2.3. *Let $\alpha_i>0$ for $i=1,2,\cdots,n$, and $f_i\in\mathcal{A}$. If*

$$\sum_{i=1}^{n}\alpha_i\Big(\mathrm{Re}\frac{zf_i'(z)}{f_i(z)}-1\Big)>0,$$

then F_n is starlike, where F_n is the integral operator defined as in (1.1).

Theorem 2.4. *Let $\alpha_i>0$ for $i=1,2,\cdots,n$, and $f_i\in\mathcal{A}$. If*

$$\sum_{i=1}^{n}\alpha_i\Big(\mathrm{Re}\frac{zf_i''(z)}{f_i'(z)}\Big)>0,$$

then $F_{\alpha_1,\cdots,\alpha_n}$ is starlike, where $F_{\alpha_1,\cdots,\alpha_n}$ is the integral operator defined as in (1.2).

We note that other works regarding the integral operators can be read in [8–10].

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